Scientific Computing I

Winter Semester 2018/2019
Prof. Dr. Carsten Burstedde
Jose A. Fonseca

UNIVERSITÄT BONN

Exercise Sheet 2.

Due date: **Tue, 30.10.2018**.

Exercise 1. (Compatibility condition) (6 Points)

Let Ω a bounded domain with Lipschitz boundary $\partial\Omega$ and $\vec{\eta}$ denote the outward pointing normal to $\partial\Omega$. Show that there is no solution of

$$\Delta u = f \quad \text{in } \Omega, \tag{1a}$$

$$\nabla u \cdot \vec{\eta} = g \quad \text{on } \partial\Omega \tag{1b}$$

unless

$$\int_\Omega f \, \mathrm{d}x = \int_{\partial\Omega} g \, \mathrm{d}s. \tag{2}$$

Exercise 2. (Explicit solution) (6 Points)

Let a, b positive real numbers, Find the harmonic function $u(x, y)$ in the square $D = (0, a) \times (0, b)$ satisfying the boundary conditions

$$u_x = -a \quad \text{for } x = 0 \qquad\qquad u_x = 0 \quad \text{for } x = a, \tag{3a}$$

$$u_y = b \quad \text{for } y = 0, \qquad\qquad u_y = 0 \quad \text{for } y = b. \tag{3b}$$

Hint: Note that the condition (2) holds. A shortcut is to guess that the solution might be a quadratic polynomial in x and y.

Exercise 3. (d'Alembert's formula) (6 Points)

a) Show that the general solution of $u_{xy} = 0$ is

$$u(x, y) = F(x) + G(y) \tag{4}$$

for arbitrary functions F, G.

b) Using the change of variables $\xi = x + t$, $\eta = x - t$, show that $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

c) Let g and h be given real valued functions. Use a) and b) to show that the solution of the initial value problem

$$u_{tt} - u_{xx} = 0, \quad \text{in } \mathbb{R} \times (0, \infty), \tag{5}$$

$$u = g, u_t = h, \quad \text{on } \mathbb{R} \times \{t = 0\} \tag{6}$$

is given by

$$u(x, y) = \frac{1}{2}\left[g(x + t) + g(x - t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y) \, dy. \tag{7}$$

Exercise 4. (Partition of unity) (6 Points)

Let $f \in \mathcal{C}^0(\mathbb{R})$ and φ a continuous, positive and compactly supported function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \varphi(x)\, dx = 1. \tag{8}$$

Define $\varphi_n(x) := n\varphi(nx)$ for $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} (\varphi_n * f)(x) = f(x)$ for all $x \in \mathbb{R}$.