Scientific Computing I

Wintersemester 2018/2019
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Exercise Sheet 3.

Due date: **Tue, 6.11.2018**.

Exercise 1. (Maximum principle) (6 Points)

Let Ω a bounded domain in $\mathbb{R}^d$ and L a second order linear elliptic differential operator and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Assume that

$$Lu = f \leq 0 \quad \text{in } \Omega. \tag{1}$$

Show that u attains its maximum on the boundary of Ω.
Hint:

- First prove the case $f < 0$ assuming that there is an $x_0 \in \Omega$ with

$$u(x_0) = \sup_{\Omega} u > \sup_{\partial\Omega} u. \tag{2}$$

  Perform a suitable coordinate transformation to obtain a contradiction.

- The case $f \leq 0$ can be handled like the previous case by considering

$$w(x) = u(x) + \delta ||x - x_0||^2, \tag{3}$$

  for x_0 as in (2) and some $\delta > 0$ sufficiently small.

Exercise 2. (Corollaries of the maximum principle) (6 Points)

Let Ω a bounded domain in $\mathbb{R}^d$ and L a second order linear elliptic differential operator.

a) If $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ satisfy

$$Lu \leq Lv \quad \text{in } \Omega, \tag{4}$$
$$u \leq v \quad \text{on } \partial\Omega, \tag{5}$$

prove that $u \leq v$ in Ω.

b) For the differential operator

$$Lu := \sum_{i,k=1}^{d} a_{ik}(x) u_{x_i x_k} + c(x)u, \quad \text{with } c(x) \geq 0, \tag{6}$$

prove the following weaker form of the maximum principle: If $Lu \leq 0$, then

$$\sup_{x\in\Omega} u(x) \leq \max\{0, \sup_{x\in\partial\Omega} u(x)\}. \tag{7}$$

Hint: $Lu - cu$ is elliptic.

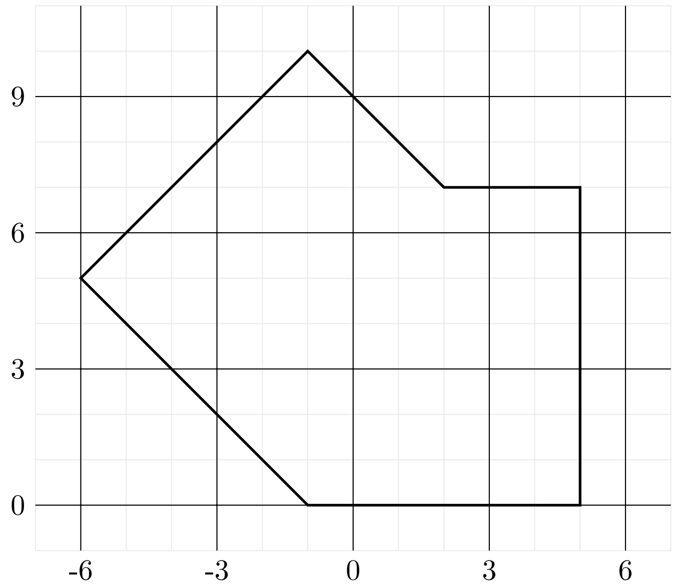

Figure 1: Grid and domain for Exercise 4.

Exercise 3. (6 Points)

Let $\Omega \subset \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ a sufficiently smooth function. For h_1, h_2 we consider $Tu : \Omega \to \mathbb{R}$ defined as

$$Tu := \alpha u(x - h_1) + \beta u(x) + \gamma u(x + h_2). \tag{8}$$

Determine the coefficients $\alpha = \alpha(h_1, h_2)$, $\beta = \beta(h_1, h_2)$, $\gamma = \gamma(h_1, h_2)$ such that

a) $Tu(x)$ approximates $u'(x)$ with order as high as possible.

b) $Tu(x)$ approximates $u''(x)$ with order as high as possible.

Hint: Determine the coefficients such that the formula is exact for polynomials with the degree as high as possible.

Exercise 4. (6 Points)

Let Ω be the domain depicted in Figure 4. Suppose we want to compute a finite difference approximation to the solution of Laplace's equation $\Delta u = 0$ with Dirichlet boundary conditions on Ω. To this end we employ a uniform mesh Ω_h with spacing $h = 3$, see Figure 4, and the five-point stencil approximation of the Laplace operator.

a) Give a suitable numbering for the points in Ω_h and $\partial\Omega_h$.

b) Write down the corresponding system of equations.