Scientific Computing I

Wintersemester 2018/2019
Prof. Dr. Carsten Burstedde
Jose A. Fonseca

UNIVERSITÄT BONN

Exercise Sheet 4.

Due date: **Tue, 13.11.2018**.

Exercise 1. (Continuous dependence on the RHS) (6 Points)

Let Ω a bounded domain in $\mathbb{R}^d$ and L a second order linear differential operator. Additionally, assume that L is uniformly elliptic in Ω with ellipticity constant $\alpha > 0$. Prove that there is a constant $c = c(\alpha, \Omega)$ such that for every $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$

$$|u(x)| \leq \sup_{y\in\partial\Omega} |u(y)| + c \sup_{y\in\Omega} |Lu(y)|. \tag{1}$$

Hint: Suppose Ω is contained in a ball centered at the origin and radius $R > 0$. Consider the function $w(x) = R^2 - \sum_i x_i^2$, and part (a) of Exercise 2 from the previous problem sheet.

Exercise 2. (6 Points)

The first derivative of a real valued function $u(x)$ can be approximated on a mesh of step size $h > 0$ by using the following stencil

$$\frac{1}{12h}\begin{Bmatrix}1 & -8 & 0 & 8-1\end{Bmatrix} \tag{2}$$

a) For a given mesh point x_i, write down the approximation of $u'(x_i)$ defined by (2).

b) With the help of a suitable Taylor expansion of u at the point x_i, determine the consistency order of this discretization.

Exercise 3. (Non-dimensionalization) (6 Points)

Consider the stationary Stokes equations

$$\nabla \cdot \big(\mu(\nabla u + \nabla u^T)\big) - \nabla p = f, \tag{3a}$$
$$\nabla \cdot u = 0, \tag{3b}$$

with velocity vector field u $[\frac{\mathrm{m}}{\mathrm{s}}]$ and force density f $[\mathrm{N}/\mathrm{m}^3]$.

a) Confirm that the pressure p has the correct units force over area.

b) What is the physical unit of the dynamic viscosity μ (a material constant)?

c) Introduce characteristic units for length, time, etc. as needed and transform the equations into non-dimensional variables, moving the remaining scaling factors into μ in the first equation.

Programming Exercise 1. (10 Points)

For $u : (0, 1) \to \mathbb{R}$ we would like to approximate the solution of the Poisson problem

$$\begin{aligned} -u''(x) &= \sin(2\pi x), \\ u(0) &= u(1) = 0, \end{aligned} \tag{4}$$

using a finite difference discretization with N points, fixing $x_0 = x_{N-1} = 0$. This will lead to the following linear system of equations:

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \end{pmatrix} = \left(\frac{1}{N-1}\right)^2 \begin{pmatrix} \sin(2\pi x_1) \\ \sin(2\pi x_2) \\ \vdots \\ \sin(2\pi x_{N-2}) \end{pmatrix}, \tag{5}$$

where $x_k := \frac{k}{N-1}$ and u_k represents an approximation of the solution u at the location x_k.

a) Familiarize yourself with the BLAS software library. Among many others, BLAS provides a function implementing the product $A\vec{x}$, where A is a symmetric band matrix. Use this routine, as well as the dot (scalar) product also provided by BLAS, to write a program that solves (5) with the conjugate gradient method. Experiment with different stopping criteria and numbers of iterations.

b) Familiarize yourself with the Lapack software library. Lapack provides a solver for the equation $A\vec{x} = \vec{b}$, where A is a tridiagonal matrix. Use this function to write a program that solves (5). Compare your results with the solution obtained in (a).

c) The exact solution of (4) is given by

$$u(x) = \frac{1}{4\pi^2} \sin(2\pi x). \tag{6}$$

Evaluate it at the locations x_k defined above and compare with the approximations obtained in (a) and (b). Plot the errors (in the maximum norm) for several chosen numbers of CG iterations and the error of the Lapack direct solver in a diagram that depends on N. What can you say about the error drop of the CG method compared to the accuracy of the direct solver?

The programming task can be solved in groups of at most two students. Please present your solutions during the week of November 20 – 22, 2018 in either Mathematics CIP-Pool. Please make sure to pick a slot ahead of time by signing into the corresponding CIP-Pool list for this lecture. We will provide further information in class.