Scientific Computing I

Wintersemester 2018/2019
Prof. Dr. Carsten Burstedde
Jose A. Fonseca

Exercise Sheet 5.

Due date: **Tue, 20.11.2018**.

Exercise 1. (6 Points)

Let Ω_h be a two dimensional grid with the same mesh spacing $h > 0$ in both x and y coordinate directions. Consider the following finite difference stencil on Ω_h,

$$h^{-2}\begin{bmatrix} -1/4 & 0 & 1/4 \\ 0 & 0 & 0 \\ 1/4 & 0 & -1/4 \end{bmatrix}. \tag{1}$$

Indicate which differential operator this stencil discretizes and determine the corresponding order of consistency.

Exercise 2. (6 Points)

Let $\Omega = (-1, 1)$ and $f(x) := |x|$.

a) Compute the weak derivative of f on Ω.

b) Show that f does not have a second weak derivative on Ω.

Exercise 3. (6 Points)

Show the following statements.

a) If u has a weak derivative $D^\alpha u$ in Ω, then u is also weakly differentiable in each region $\Omega_0 \subset \Omega$, yielding the same derivative.

b) If $D^\alpha u$ has a weak derivative $D^\beta(D^\alpha u)$, then the derivative $D^{\alpha+\beta}u$ also exists and $D^{\alpha+\beta}u = D^\beta(D^\alpha u)$.

Definition 1. Let $m \in \mathbb{N}, m \geq 0$ and $1 \leq p \leq \infty$ and Ω a domain in $\mathbb{R}^d$. The Sobolev space $H^{m,p}(\Omega)$ is defined as the set of all functions $u \in L^p(\Omega)$ such that for every multiindex α with $|\alpha| \leq m$, $D^\alpha u$ exists and belongs to $L^p(\Omega)$. The spaces $H^{m,p}(\Omega)$ are endowed with the norms

$$\|u\|_{m,p;\Omega} = \|u\|_{m,p} := \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{1/p}, \quad 1 \leq p < \infty, \tag{2}$$

$$\|u\|_{m,\infty;\Omega} = \|u\|_{m,\infty} := \max_{|\alpha| \leq m} \|D^\alpha u\|_\infty, \quad p = \infty. \tag{3}$$

Exercise 4. (6 Points)

Let Ω a bounded domain in $\mathbb{R}^d$. Prove that $H^{m,p}(\Omega)$ is a Banach space (a complete vector space with a norm) for each $m \in \mathbb{N}, m \geq 0$ and $1 \leq p \leq \infty$.
Hint: $L^q(\Omega) \subseteq L^p(\Omega)$ for $1 \leq p \leq q \leq \infty$ and a bounded Ω.