Scientific Computing I

Wintersemester 2018/2019
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Exercise Sheet 6.

Due date: **Tue, 27.11.2018**.

Exercise 1. (2+4 Points)

Let Ω a domain in $\mathbb{R}^2$ with Lipschitz boundary $\Gamma := \partial\Omega$ and $\vec{\eta}$ denote the outward pointing normal to Γ.

a) Consider the Poisson equation

$$-\Delta u = 0 \quad \text{in } \Omega, \tag{1}$$
$$u = 0 \quad \text{on } \Gamma. \tag{2}$$

Show that every classical solution $u \in \mathcal{C}^2(\Omega)$ satisfies the weak formulation

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = 0 \quad \text{for all } v \in \mathcal{C}_0^1(\Omega). \tag{3}$$

b) Derive the weak formulation in $H^1(\Omega)$ for the equation

$$-\Delta u + c(x)u = f \quad \text{in } \Omega, \tag{4}$$
$$\nabla u \cdot \vec{\eta} = g \quad \text{on } \Gamma. \tag{5}$$

That is, determine a bilinear form $a(\cdot,\cdot)$ and a linear functional $F(\cdot)$ for the following formulation: Find $u \in H^1(\Omega)$ such that

$$a(u, v) = F(v) \quad \text{for all } v \in H^1(\Omega). \tag{6}$$

Exercise 2. (3+2+1 Points)

Consider the bilinear form $a : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ defined by

$$a(u, v) := \int_0^1 x^2 u'v' \, \mathrm{d}x. \tag{7}$$

a) Show that the problem of finding a minimum of

$$\frac{1}{2}a(u, u) - \int_0^1 u \, \mathrm{d}x \tag{8}$$

does not have a solution in $H_0^1(0,1)$.

b) Show that $a(\cdot,\cdot)$ is not elliptic.

c) Write down the associated classical differential equation.

Exercise 3. (6 Points)

Let Ω be a bounded domain. With the help of Friedrichs' inequality, show that the constant function $u = 1$ is not contained in $H_0^1(\Omega)$, and hence $H_0^1(\Omega)$ is a proper subspace of $H^1(\Omega)$.

Exercise 4. (6 Points)

Let $\Omega \subset R^d$ be a ball with center at the origin. Show that $u(x) = ||x||^s$ possesses a weak derivative in $L^2(\Omega)$ if $2s > 2 - d$ or if $s = 0$.