Scientific Computing I

Wintersemester 2018/2019
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Exercise Sheet 7.

Due date: **Tue, 3.12.2018**.

Assumption 1 (Fixed point theorem). *Let V be a Banach space, $\rho \in (0,1)$ and $T : V \to V$ an operator with the property*

$$\|Tv - Tw\|_V \leq \rho \|v - w\|_V \quad \textit{for all } v, w \in V. \tag{1}$$

Then, we know there exists a unique $u \in V$ such that $Tu = u$.

Assumption 2 (Riesz representation theorem). *Let H denote a Hilbert space with inner product $(\cdot,\cdot)_H$ and induced norm $\|\cdot\|_H$. Then, for any continuous linear functional $F \in H'$ there exists a unique element $u \in H$ such that $F(v) = (u,v)_H$. Furthermore,*

$$\|F\|_{H'} := \sup_{v \in H, v \neq 0} \frac{|F(v)|}{\|v\|_H} = \|u\|_H. \tag{2}$$

The map $\mathcal{R} : H' \to H$ that identifies F with u is called Riesz isometry.

Exercise 1. (1+1+1+3 Points)

Let V be a Hilbert space with inner product $(\cdot,\cdot)_V$. We consider a continuous, coercive bilinear form $a(\cdot,\cdot)$ over V and $F \in V'$ a continuous linear functional.

a) Given $u \in V$ define the functional $Au(v) := a(u,v)$ for all $v \in V$. Verify that Au is linear and continuous for each $u \in V$.

b) Furthermore, show that the map $u \to Au$ is a continuous linear map between V and V'. That is, $Au \in L(V,V')$ and $\|Au\|_{V'} \leq C\|u\|_V$ for some constant $C > 0$.

c) Let $\mathcal{R}$ be the corresponding Riesz isometry identifying V' and V. Show that finding $u \in V$ such that

$$a(u,v) = F(v) \quad \text{for all } v \in V \tag{3}$$

is equivalent to finding $u \in V$ such that

$$\mathcal{R}Au = \mathcal{R}F. \tag{4}$$

d) Show that it is possible to choose $\kappa > 0$ such that the operator $T : V \to V$ defined as $Tu := u - \kappa(\mathcal{R}Au - \mathcal{R}F)$ satisfies (1). Conclude that there is a unique $u \in V$ solving (3).

Exercise 2. (2+4 Points)

Let $\Omega = (0,1)$ and k a positive constant. Consider the boundary value problem

$$\begin{aligned} -u'' + ku' + u &= f \quad \text{in } \Omega, \\ u' &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where $f \in L^2(\Omega)$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $u' \in \mathcal{C}(\bar{\Omega})$. A variational formulation of (5) can be obtained by taking $V = H^1(\Omega)$, $a : V \times V \to \mathbb{R}$ and $F : V \to \mathbb{R}$ defined by

$$a(u,v) := \int_0^1 \big(u'v' + ku'v + uv\big)\,\mathrm{d}x \quad \text{and} \quad F(v) := \int_0^1 fv\,\mathrm{d}x, \tag{6}$$

respectively.

- Show that $a(\cdot,\cdot)$ is continuous.
- Find a value of k such that $a(v,v) = 0$ but $v \neq 0$ for some $v \in H^1(\Omega)$. That is, show that $a(\cdot,\cdot)$ is not coercive on V.

Exercise 3. (3+3 Points)

Let $\Omega = (0,1)$ and consider the boundary value problem

$$\begin{aligned} -u''(x) &= f, \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{7}$$

where $f \in L^2(\Omega)$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Let $N \in \mathbb{N}$, $N > 0$, the boundary value problem is now discretized by a finite element method on an equidistant mesh $x_i = i/N$ for $i = 0, \dots, N$. The finite element basis is given by linear functions $\lambda_i(x)$, such that $\lambda_i(x_j) = \delta_{ij}$.

a) Indicate explicitly the system of equations corresponding to the discrete weak formulation of (7).

b) In the special case that $f \in \mathcal{C}(\bar{\Omega})$, we can replace the right hand side by the ansatz $f_h(x) = \sum_{i=0}^{N} f(x_i)\lambda_i(x)$. Repeat part a) for this case.

Programming Exercise 1. (1+2+1+4+2 Points)

Let Ω be the rectangle $(0,2) \times (0,1)$. We want to solve the Poisson problem

$$\begin{aligned} -\Delta u &= 1, \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{8}$$

using the finite element method. To this end, we decompose Ω into $2N^2$ squares of edge length $h := 1/N$, with $N \in \mathbb{N}$. Then, each one of the squares is further divided into the two triangles resulting from connecting its lower left with its upper right corner. The grid points form the discrete space Ω_h. The finite element basis functions ϕ_{ij} for $i = 0, \dots, 2N$, $j = 0, \dots, N$ are defined over Ω_h as

$$\phi_{ij}(x,y) := \begin{cases} 1 & \text{if } x = ih \text{ and } y = jh, \\ 0 & \text{if } x \neq ih \text{ or } y \neq jh. \end{cases} \tag{9}$$

and linear in the x and y coordinate directions. Such discretization leads to solve a linear system of equations of the form

$$A\vec{z} = \vec{b}. \tag{10}$$

a) Write down the entries of the stiffness matrix A in (10).

b) Implement a function that computes the action of the stiffness matrix on a given vector without assembling the full matrix. The loop used should run over the elements of the mesh and in each case add the contributions of the degrees of freedom.

c) Use a middle point rule quadrature to compute the entries of the right hand side in (10).

d) With the matrix vector routine coded in part a) and write a program that solves (10) with the CG method.

e) Familiarize yourself with the PETSc software library. Investigate how to use the PETSc implementation of the CG method (KSPCG) and employ it to solve (10). Compare the results with those from part d).

Hint: For part e) the example

`www.mcs.anl.gov/petsc/petsc-3.9/src/ksp/pc/examples/tutorials/index.html`

might be useful.

The programming task can be solved in groups of at most two students. Please present your solutions during the week of December 10–15, 2018 in either Mathematics CIP-Pool. Please make sure to pick a slot ahead of time by signing into the corresponding CIP-Pool list for this lecture.