Scientific Computing I

Wintersemester 2018/2019
Prof. Dr. Carsten Burstedde
Jose A. Fonseca

UNIVERSITÄT BONN

Exercise Sheet 8.

Due date: **Tue, 11.12.2018**.

Exercise 1. (Tensor Lagrange elements) (2+4 Points)

Let $k \in \mathbb{N}$ and $\mathcal{P}_k$ denote the set of polynomials of degree less or equal than k in one variable. We further define

$$\mathcal{Q}_k := \left\{ \sum_j c_j p_j(x) q_j(y) \mid p_j, q_j \in \mathcal{P}_k \right\} \tag{1}$$

a) Show that $\dim \mathcal{Q}_k = (\dim \mathcal{P}_k)^2$ and that $\{x^i y^j \mid 0 \leq i, j \leq k\}$ is a basis for $\mathcal{Q}_k$.

b) Let T be the unit square, $\Pi = \mathcal{Q}_k$ and Σ denote point evaluations at the points $\{(t_i, t_j) \mid i, j = 0, 1, \ldots, k\}$ where $\{0 = t_0 < t_1 \ldots < t_k = 1\}$. Prove that (T, Π, Σ) is a finite element.

Exercise 2. (Isoparametric elements) (2+1+3 Points)

Consider the following basis functions defined over the square $[-1, 1]^2$,

$$\chi_1(\xi, \eta) = (\xi - 1)(\eta - 1)/4, \tag{2a}$$
$$\chi_2(\xi, \eta) = -(\xi + 1)(\eta - 1)/4, \tag{2b}$$
$$\chi_3(\xi, \eta) = (\xi + 1)(\eta + 1)/4, \tag{2c}$$
$$\chi_4(\xi, \eta) = -(\xi - 1)(\eta + 1)/4. \tag{2d}$$

These basis functions may be mapped to a quadrilateral with vertices (x_ν, y_ν), for $\nu = 1, 2, 3, 4$, by the change of variables

$$x(\xi, \eta) = \sum_{\nu=1}^{4} x_\nu \chi_\nu(\xi, \eta), \quad y(\xi, \eta) = \sum_{\nu=1}^{4} y_\nu \chi_\nu(\xi, \eta). \tag{3}$$

Compute the Jacobian matrix J of the transformation (3) and verify the following statements

a) J is a constant matrix if the mapped element is a parallelogram with vertices $(x_0, y_0), (x_0 + h_x, y_1), (x_1 + h_x, y_1 + h_y)$ and $(x_1, y_0 + h_y)$.

b) J is a diagonal matrix if the mapped element is a rectangle aligned with the coordinate axes.

c) The determinant of J is a linear function of the coordinates (ξ, η).

Definition 1. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. The space $H(\mathrm{div}, \Omega)$ is defined as the completion of the space of vector valued functions $(\mathcal{C}^\infty(\Omega))^d$ with respect to the norm

$$\|\vec{v}\|^2 := \|\vec{v}\|^2_{0,\Omega} + \|\mathrm{div}\, \vec{v}\|^2_{0,\Omega}. \tag{4}$$

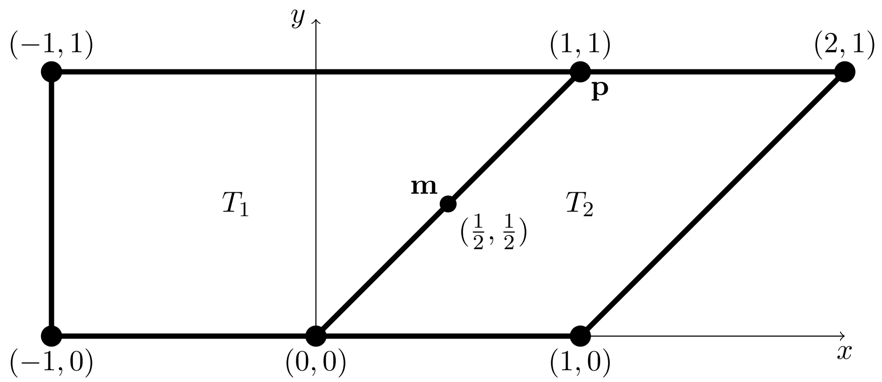

Figure 1: Illustration for exercise 4

Exercise 3. (6 Points)

Prove that a piecewise polynomial $\vec{v}$ is an element of $H(\mathrm{div}, \Omega)$ if and only if the components $\vec{v} \cdot \vec{\eta}$ in the direction of the normals are continuous on inter-element boudaries. *Hint: Theorem 2.32 from the lecture and an appropriate Green formula.*

Exercise 4. (6 Points)

a) For the pair of elements illustrated in Figure 1, show that the respective bilinear function that takes the value 1 at the vertex $\mathbf{p}$ and zero at the other vertices gives different values at the midpoint $\mathbf{m}$ on the common edge.

b) Show that the isoparametrically mapped bilinear function defined via (2) and (3) is continuous along the common edge.