Scientific Computing I

Wintersemester 2018/2019
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Exercise Sheet 9.

Due date: **Tue, 18.12.2018**.

Exercise 1. (Bramble-Hilbert lemma in one space dimension) (6 Points)

For any function $v(x)$ defined on $[0,1]$, let Πv be the linear interpolant satisfying $\Pi v(0) = v(0)$ and $\Pi v(1) = v(1)$. Use Rolle's theorem to show that $e := v - \Pi v$ satisfies

$$\int_0^1 (e')^2 \, dx \le \int_0^1 (e'')^2 \, dx. \tag{1}$$

Exercise 2. (6 Points)

Let Ω be a square covered with a triangulation $\mathcal{T}$ with t triangles and v vertices. Find the dimension $d(t, v, n)$ of the continuous piecewise polynomials with total degree less than or equal than n. Consider the special case where the triangulation consists of $2m^2$ isosceles right triangles.

Hint: Count the Lagrange functions that belong to the triangulation that form a basis because of their interpolation property. Use the Euler formula $t - e + v = 1$ with e the number of edges.

Assumption 1. *Let $\Omega \subset \mathbb{R}^2$ a bounded domain and $V \subset H^1(\Omega)$. Furthermore let $V_h \subset V$ a finite dimensional subspace from V and $\mathcal{T}_h$ a regular triangulation of Ω. Consider the "broken" norm*

$$\|v_h\|_{0,h}^2 := \sum_{K \in \mathcal{T}_h} h_K^2 \sum_{a_i \in K} |v(a_i)|^2, \tag{2}$$

where $a_1, \ldots, a_N$ denote the degrees of freedom and h_K is the length of an element $K \in \mathcal{T}_h$. Then, we know that

1. *There exists positive constants C_1, C_2 independent from h, such that*

$$C_1 \|v\|_0 \le \|v\|_{0,h} \le C_2 \|v\|_0 \quad \text{for all } v \in V_h. \tag{3}$$

2. *There exists a positive constant C independent of h, such that*

$$\|v\|_1 \le C \left(\min_{K \in \mathcal{T}_h} h_K \right)^{-1} \|v\|_0 \quad \text{for all } v \in V_h. \tag{4}$$

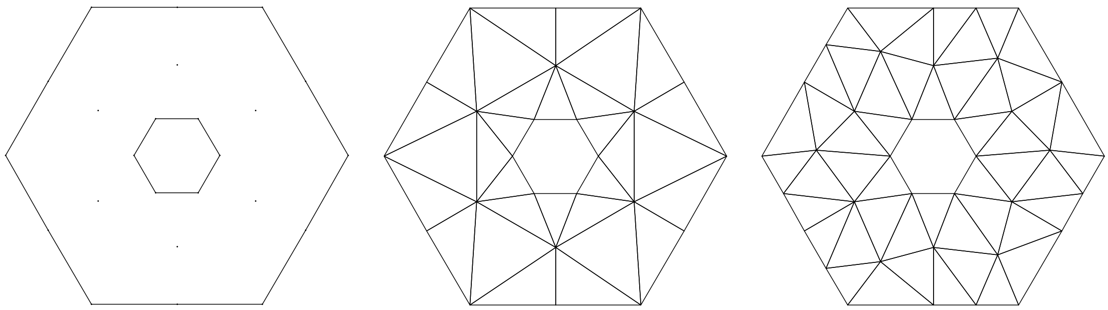
Exercise 3. (Condition number of the stiffness matrix) (4+2 Points)

With the same definitions as in Assumption 1, let $\{\varphi_1, \ldots, \varphi_N\}$ a basis of V_h and $a(\cdot,\cdot)$ a coercive and continuous bilinear form on $V \times V$. Additionally, assume that there exists a constant $C_A > 0$ such that for every node $p \in \mathcal{T}_h$

$$\#\{K \in \mathcal{T}_h \mid p \in K\} \le C_A. \tag{5}$$

Prove the following properties of the stiffness matrix $A = (a(\varphi_i, \varphi_j))_{i,j=1,\ldots,N}$ and mass matrix $B = ((\varphi_i, \varphi_j)_0)_{i,j=1,\ldots,N}$.

There exists a constant $C > 0$ independent of h such that the condition number $\kappa(\cdot)$ satisfies



(a) (b) (c)

Figure 1: Example meshes for the programming task

a) $\kappa(B^{-1}A) \le C \left(\min_{K \in \mathcal{T}_h} h_K \right)^{-2}$.

b) $\kappa(A) \le C \left(\min_{K \in \mathcal{T}_h} h_K \right)^{-2} \kappa(B)$.

Programming Exercise 1. (3+3+4 Points)

We want to solve the Poisson problem

$$\begin{aligned} -\Delta u &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{6}$$

using the finite element method. Ω is a polygon inscribed in the unit circle with a hole in the middle. See Figure (1a).

In order to produce the triangulations depicted in Figure (1b), we will employ the program `triangle`; see `http://www.cs.cmu.edu/~quake/triangle.html`. This program can take as input a `.poly` file describing the problem domain and produce high quality triangulations. In the web page of the lecture you can find the file `kreis.poly` and instructions to produce the meshes displayed in Figures (1b)–(1c).

a) Familiarize yourself with `triangle` and the `.poly` format. Write a program that takes the number of vertices of the polygon N and produces the corresponding `.poly` file that `triangle` needs as argument. For example the mesh from Figure (1b) corresponds to $N = 6$ and was produced with `triangle -pcq25 kreis.poly`.

   The parameter `q` controls the minimum angle constrain on the triangles. For visualization `triangle` provides the program `showme`.

b) `triangle`'s output includes two files `.node` and `.ele` that contain the list of nodes and elements in the mesh, respectively. Write a program that reads these files into memory. This data is then used for computing the matrix vector products with linear P_1 elements. Use the approach described in the lecture to perform all the calculations on a reference element.

c) Modify the CG method from the previous problem sheets such that it uses the above matrix vector product. Approximate the solution of the system of equations arising after discretization of (6) with P_1 linear elements for the above described Ω. Experiment with different values of N and the minimal angle condition. How is the run time of your program affected?

The programming task can be solved in groups of at most two students. Please present your solutions during the week of January 7–11, 2019 in either Mathematics CIP-Pool. Please make sure to pick a slot ahead of time by signing into the corresponding CIP-Pool list for this lecture.