Numerical Algorithms

Winter term 2019/20
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Sheet 2

Submission on **Tuesday, 22.10.19 in class**.

Exercise 1. (product rule)

Prove Theorem 2.5 from the lecture: Let X, V, W and Z be Banach spaces. For a continuous bilinear map $B\colon V \times W \to Z$ and differentiable functions $f\colon X \to V$, $g\colon X \to W$, the composition $B \circ (f, g)\colon X \to Z$ is differentiable and satisfies

$$D[B \circ (f, g)](x)v = B(f(x), Dg(x)v) + B(Df(x)v, g(x)) \tag{2.1}$$

for all $x, v \in X$.

(5 points)

Exercise 2. (Green's second formula)

Let Ω be a smooth domain and $f, g\colon \Omega \to \mathbb{R}$ be two times continuously differentiable. Show that

$$\int_\Omega (f\Delta g - g\Delta f)\,\mathrm{d}x = \int_{\partial\Omega} \left(f\frac{\partial g}{\partial \nu} - g\frac{\partial f}{\partial \nu} \right) \mathrm{d}A. \tag{2.2}$$

(5 points)

Exercise 3. (vector-valued divergence theorem)

Let $\Omega \subset \mathbb{R}^m$ be a smooth domain and $f\colon \Omega \to \mathbb{R}^n$, $A\colon \Omega \to \mathbb{R}^{n\times m}$ be continuously differentiable. Compute

$$\int_\Omega \nabla f(x)\,\mathrm{d}x \in \mathbb{R}^{m\times n} \quad \text{and} \quad \int_\Omega \nabla \cdot A(x)\,\mathrm{d}x \in \mathbb{R}^n \tag{2.3}$$

in terms of boundary integrals.

(5 points)

Exercise 4. (non-dimensionalization – depends on Thursday's lecture)

Consider the Stokes equations for the velocity field $u : \Omega \to \mathbb{R}^d$ and pressure p,

$$\nabla \cdot \left[\mu \left(\nabla u + \nabla u^T\right)\right] - \nabla p = f, \tag{2.4a}$$
$$\nabla \cdot u = 0, \tag{2.4b}$$

where $\mu(x)$ is the kinematic viscosity. They express the conservation of momentum and mass in a viscous incompressible fluid (boundary conditions omitted).

1. Determine the physical units of μ and the source term f.
2. Introduce a minimal set of scaling factors for the physical units involved, redefine variables as necessary and derive a non-dimensionalized, equivalent system.