Sheet 2

Submission on Tuesday, 22.10.19 in class.

Exercise 1. (product rule)
Prove Theorem 2.5 from the lecture: Let $X$, $V$, $W$ and $Z$ be Banach spaces. For a continuous bilinear map $B : V \times W \to Z$ and differentiable functions $f : X \to V$, $g : X \to W$, the composition $B \circ (f,g) : X \to Z$ is differentiable and satisfies
\[ D[B \circ (f,g)](x)v = B(f(x), Dg(x)v) + B(Df(x)v, g(x)) \] (2.1)
for all $x,v \in X$.

(5 points)

Exercise 2. (Green’s second formula)
Let $\Omega$ be a smooth domain and $f,g : \Omega \to \mathbb{R}$ be two times continuously differentiable. Show that
\[ \int_{\Omega} (f \Delta g - g \Delta f) \, dx = \int_{\partial \Omega} \left( f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) \, dA. \] (2.2)

(5 points)

Exercise 3. (vector-valued divergence theorem)
Let $\Omega \subset \mathbb{R}^m$ be a smooth domain and $f : \Omega \to \mathbb{R}^n$, $A : \Omega \to \mathbb{R}^{n \times m}$ be continuously differentiable. Compute
\[ \int_{\Omega} \nabla f(x) \, dx \in \mathbb{R}^{m \times n} \quad \text{and} \quad \int_{\Omega} \nabla \cdot A(x) \, dx \in \mathbb{R}^n \] (2.3)
in terms of boundary integrals.

(5 points)

Exercise 4. (non-dimensionalization – depends on Thursday’s lecture)
Consider the Stokes equations for the velocity field $u : \Omega \to \mathbb{R}^d$ and pressure $p$,
\[ \nabla \cdot \left[ \mu \left( \nabla u + \nabla u^T \right) \right] - \nabla p = f, \]
\[ \nabla \cdot u = 0, \] (2.4a, 2.4b)
where $\mu(x)$ is the kinematic viscosity. They express the conservation of momentum and mass in a viscous incompressible fluid (boundary conditions omitted).

1. Determine the physical units of $\mu$ and the source term $f$.

2. Introduce a minimal set of scaling factors for the physical units involved, redefine variables as necessary and derive a non-dimensionalized, equivalent system.