Numerical Algorithms

Winter term 2019/20
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Sheet 3

Submission on **Tuesday, 29.10.19 in class**.

Exercise 1. (higher regularity in 1D)

Let $I = [a, b] \subset \mathbb{R}$ and $f \in L_2(I)$. Let $u \in H_0^1(I)$ be the weak solution to the Poisson equation

$$-u'' = f \tag{3.1}$$

with Dirichlet boundary conditions. Show that u belongs to $H^2(I)$.

(5 points)

Exercise 2. (Stokes equations I)

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary $\partial\Omega$. The motion of an incompressible viscous fluid with velocity field $u\colon \Omega \to \mathbb{R}^n$ can be modeled with the PDE

$$-\Delta u + \nabla p = f \quad \text{in } \Omega\,, \tag{3.2}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega\,, \tag{3.3}$$

$$u = u_0 \quad \text{on } \partial\Omega\,. \tag{3.4}$$

Here, $p\colon \Omega \to \mathbb{R}$ is the pressure, $f\colon \Omega \to \mathbb{R}^n$ is an external force density field and Δ is the componentwise Laplacian. For the spaces $X = H_0^1(\Omega)^n$ and $M = L_2(\Omega)$, the weak formulation of this problem can be stated as follows:
Find $(u, p) \in X \times M$ *such that*

$$\begin{aligned} a(u, v) + b(v, p) &= (f, v)_{L^2(\Omega)} \\ b(u, q) &= 0 \end{aligned}$$

is satisfied for all $v \in X$, $q \in M$.
Determine the bilinear maps $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$.

(5 points)

Exercise 3. (Stokes equations II)

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary $\partial\Omega$. For smooth vector fields $u\colon \Omega \to \mathbb{R}^n$ consider the energy functional

$$I[u] = \frac{1}{2}\int_\Omega \operatorname{Tr}[Du(x)Du(x)^\top] - u(x) \cdot f(x)\,\mathrm{d}x \tag{3.5}$$

with smooth data $f\colon \Omega \to \mathbb{R}^n$ and $Du(x)_{ij} = \partial_j u_i(x)$. Show that the minimizer of this functional on $\{u \text{ smooth}\colon \operatorname{div} u = 0,\ u = u_0 \text{ on } \partial\Omega\}$ satisfies the Stokes equations from Exercise 2 for some appropiate p.
Hint: Introduce p as a Lagrange multiplier.

(5 points)

Exercise 4. (Helmholtz decomposition)

For smooth vector fields $A\colon \mathbb{R}^3 \to \mathbb{R}^3$, the *curl* operator is defined as

$$\operatorname{curl} A = \begin{pmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 \end{pmatrix} \in \mathbb{R}^3 . \tag{3.6}$$

Let $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$ be a smooth scalar field. Show that $\operatorname{div}[\operatorname{curl} A] = 0$ and $\operatorname{curl}[\nabla\Phi] = 0$.

(5 points)