Numerical Algorithms

Winter term 2019/20
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Sheet 4

Submission on **Tuesday, 5.11.19 in class**.

Exercise 1. (weak curl)

Let $\Omega \subset \mathbb{R}^3$ be open and bounded with smooth boundary and outer normal n. Define

$$H(\Omega, \operatorname{curl}) = \left\{u \in L_2(\Omega)^3 \colon \operatorname{curl} u \text{ exists weakly and } \operatorname{curl} u \in L_2(\Omega)^3\right\} \tag{4.1}$$

with scalar product $(u, v) := (u, v)_{L_2} + (\operatorname{curl} u \cdot \operatorname{curl} v)_{L_2}$. This is a Hilbert space, with $H_0(\Omega, \operatorname{curl})$ the modification with vanishing boundary $u \times n = 0$. For some $F \in L_2(\Omega)^3$, derive the weak formulation for the PDE

$$-\operatorname{curl}\operatorname{curl} u = F \quad \text{in } \Omega \tag{4.2}$$

$$u \times n = 0 \quad \text{on } \partial\Omega \tag{4.3}$$

in $H_0(\Omega, \operatorname{curl})$.

(5 points)

Exercise 2. (Helmholtz decomposition II)

Let $\Omega \subset \mathbb{R}^3$ be open and bounded with smooth boundary. Let $F \in L_2(\Omega)^3$. Derive formally that F can be decomposed into

$$F = -\nabla\Phi + \operatorname{curl} A\,, \tag{4.4}$$

with $\Phi \in H^1(\Omega)$ and $A \in H(\Omega, \operatorname{curl})$.
Hint: Φ and A are solutions to certain weak problems.

(5 points)

Programming Exercise 1. (Solving a PDE over a curved domain, part I)

Our goal is to compute a numerical solution for the scalar Laplace equation or the equations of linear elasticity in 2D. The domain Ω shall be the unit circle around the origin. To mesh it with quadrilateral elements, compose Ω from five mapped squares, one of them being a square centered at the origin. Each square shall be uniformly split into M^2 finite elements. For index sets, we will use the notation $[0, Q) \cap \mathbb{Z}$, omitting the intersection for brevity.

1. Propose five analytical mappings, one for each square. They will be run-time parameterized by the length of the center square. The domain of each mapping is the reference square $[-1, 1]^2$.

2. Propose an indexing scheme for all $E = 5 \times M^2$ elements and program expressions for $J_e : [-1, 1]^2 \to \Omega_e$, $e \in [0, E)$, which will be compositions with the mappings from 1.

3. You are free to choose either piecewise linear or piecewise quadratic, tensor-product finite element basis functions, with $Q = 4$ or $Q = 9$ degrees of freedom, respectively. They shall be continuous globally, which we will achieve by using Lagrange

polynomials with their outside node points on the element boundary. Identify a global numbering of $N = ?$ Lagrange nodes and program the subset of node points $\mathfrak{B} \subset \mathfrak{O} = [0, N)$ that lie on the boundary in your code. In addition, program a lookup table for the mapping $[0, E) \times [0, Q) \to [0, N)$ that assigns nodes in some consistent order to the elements. These lists shall be dynamically allocated and populated, since N must be a command line parameter for your program.

4. For a given mathematical expression $f : \Omega \to \mathbb{R}$, write a routine that computes its L_2 norm over Ω using tensor-product third-order Gauß quadrature. You will be using a loop over the elements and the transformation theorem from calculus. This will in turn require hardcoding $|\det \nabla J_e|$. Choose some example functions whose norm is known and plot the error against your quadrature code depending on N.

Do not worry if you are not using all the pieces yet. They will become useful in the subsequent parts of the exercise.

(20 points)

The programming exercise has to be presented during the exercise class on Wednesday, November 20, 2019 in the computer room 2.038. You may bring your own laptop to class or prepare a solution that can be executed on the INS computers.