Numerical Algorithms

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Sheet 5

Submission on **Tuesday, 12.11.19 in class**.

Exercise 1. (block matrix)

Consider the block matrix

$$C = \begin{bmatrix} A & B \\ B^\top & 0 \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)} \tag{5.1}$$

with $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{m\times n}$. Assume that $\operatorname{rank}(B) = n$ and that A is positive definite on $\ker(B^\top)$. Show that C is invertible.

(5 points)

Exercise 2. (Stokes equation III)

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary $\partial\Omega$. Consider the function spaces $X = H_0^1(\Omega)^n$ and $M = \{p \in L_2(\Omega)\colon \int_\Omega p \,\mathrm{d}x = 0\}$ and the weak problem: *Find* $(u, p) \in X \times M$ *such that*

$$a(u,v) + b(v,p) = (f,v)_{L^2(\Omega)} \tag{5.2}$$

$$b(u,q) = 0 \tag{5.3}$$

is satisfied for all $v \in X$, $q \in M$.
with $a(u,v) = \int_\Omega \operatorname{Tr}[Du(x)Dv(x)^\top]\,\mathrm{d}x$, $b(v,q) = \int_\Omega q(x)\operatorname{div} v(x)\,\mathrm{d}x$ and $f \in L_2(\Omega)^n$.

a) Show that two solutions (u_1, p_1) and (u_2, p_2) are equal if additionally $p_1, p_2 \in H^1(\Omega)$.

b) Define suitable operators $A\colon X \to X'$ and $B\colon X \to M'$, $B'\colon M \to X'$ to rewrite the weak problem as an operator equation in $X' \times M'$. Show that B' restricted to $M \cap H^1(\Omega)$ is the negative weak gradient operator $B'(q) = -\nabla q \in L_2(\Omega)^n \subset X'$.

(5 points)