Numerical Algorithms

Winter term 2019/20
Prof. Dr. Carsten Burstedde
Christopher Kacwin

UNIVERSITÄT BONN

Sheet 6

Submission on **Tuesday, 26.11.19 in class**.

Exercise 1. (mass matrix recursion in 1D)

Consider a set of basis functions $\{\phi_i(x)\}$ that satisfies the recursion formula

$$\phi_i(x) = (a_i x + b_i)\phi_{i-1}(x) + c_i\phi_{i-2}(x)\,. \tag{6.1}$$

Show that mass matrix entries $M_{ij} = \int g(x)\phi_i(x)\phi_j(x)\,\mathrm{d}x$ satisfy

$$M_{ij} = \frac{a_i}{a_{j+1}}M_{i-1,j+1} - \frac{a_i b_{j+1}}{a_{j+1}}M_{i-1,j} - \frac{a_i c_{j+1}}{a_{j+1}}M_{i-1,j-1} + b_i M_{i-1,j} + c_i M_{i-2,j}\,. \tag{6.2}$$

(5 points)

Exercise 2. (mass matrix implementation in 2D)

Let $\Omega = [0,1]^2$ and $\phi_{ij}(x,y) = \psi_i(x)\psi_j(y)$, $1 \le i,j \le p$ a tensorized basis set on Ω. For some weight function $G\colon \Omega \to \mathbb{R}$, we approximate mass matrix entries $M_{ij,kl} = \int_\Omega \phi_{ij}\phi_{kl}G$ via tensorized numerical integration

$$M_{ij,kl} \approx \sum_{\alpha,\beta} w_\alpha w_\beta \phi_{ij}(x_\alpha, x_\beta)\phi_{kl}(x_\alpha, x_\beta)G(x_\alpha, x_\beta) \tag{6.3}$$

with weights $w_1, \dots, w_p$ and nodes $x_1, \dots, x_p$. Derive an algorithm which computes the matrix-vector multiplication $y = Mu$ in $\mathcal{O}(p^3)$.

(5 points)

Exercise 3. (Friedrich's inequality)

Suppose Ω is an open and bounded domain. Show that for all $u \in H^1(\Omega)$ one has

$$\|u - [u]\|_{L^2(\Omega)} \le C\,\mathrm{diam}(\Omega)\,|u|_{H^1(\Omega)}\,, \tag{6.4}$$

where $[u] = \frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x$.
Hint: Use the Bramble-Hilbert-Lemma.

(5 points)

Exercise 4. (First Lemma of Strang)

Let Ω be an open and bounded domain in $\mathbb{R}^2$ and consider the PDE

$$-\Delta u = f \qquad \text{in } \Omega \tag{6.5}$$
$$u = 0 \qquad \text{on } \partial\Omega \tag{6.6}$$

with $f \in H^1(\Omega)$ and solution $u \in H^3(\Omega)$. Moreover, let $\mathcal{T}_h$ be a quasi-uniform, nondegenerate triangulation of Ω with $h = \max_{T\in\mathcal{T}}\mathrm{diam}(T)$. Define $[f](x) \in L^2(\Omega)$ as the piecewise constant function

$$[f]\big|_T = \frac{1}{|T|}\int_T f(y)\,\mathrm{d}y\,. \tag{6.7}$$

Let V_h be the linear finite element space with respect to $\mathcal{T}_h$ and replace f with $[f]$ in the above PDE, i.e. we consider:
Find $u_h \in V_h$ s.t. for all $v_h \in V_h$

$$\int_\Omega \nabla u_h(x) \cdot \nabla v_h(x)\,\mathrm{d}x = \int_\Omega [f](x) v_h(x)\,\mathrm{d}x\,. \tag{6.8}$$

Show that $\|u - u_h\|_{H^1(\Omega)} \in \mathcal{O}(h)$.

(5 points)