Numerical Algorithms

Winter term 2019/20
Prof. Dr. Carsten Burstedde
Christopher Kacwin

UNIVERSITÄT BONN

Sheet 9

Submission on **Tuesday, 17.12.19 in class**.

Exercise 1. (1D heat equation)

We consider a metal rod and its temperature distribution

$$y\colon [0,1] \times [0,T] \longrightarrow \mathbb{R} \tag{9.1}$$

with initial condition $y(\cdot,0) = y^0$. Additionally,we assume that we are able to control the heat flux of the metal rod at the end points. More precisely, we model $y(x,t)$ to satisfy the partial differential equation

$$\begin{aligned} y_t - y_{xx} &= f \quad \text{in } [0,1]\times[0,T] && (9.2)\\ -y_x(0,\cdot) &= l \quad \text{in } [0,T] && (9.3)\\ y_x(1,\cdot) &= r \quad \text{in } [0,T] && (9.4)\\ y(\cdot,0) &= y^0 \quad \text{in } [0,1] && (9.5)\end{aligned}$$

with control parameters $l(t)$, $r(t)$ and additional enviromental influence $f(x,t)$.
We interpret $y(x,t) = y(t)(x) = y(t) \in V$ (f likewise), where y is now a function of time mapping into a function space V, which consists of functions defined on $[0,1]$ (for instance $C[0,1]$). Introducing the time steps $t_n = nT/N$ for $n = 0,\dots,N$ we define $y^n = y(t_n) \in V$, $f^n = f(t_n) \in V$, $l(t_n) = l^n \in \mathbb{R}$, $r(t_n) = r^n \in \mathbb{R}$.

a) Use the implicit Euler scheme to derive the time-discretized formulation

$$\begin{aligned} y^n &= y^{n-1} + \frac{T}{N}(f^n + y^n_{xx}), \quad n = 1,\dots,N && (9.6)\\ -y^n_x(0) &= l^n, \quad n = 1,\dots,N && (9.7)\\ y^n_x(1) &= r^n, \quad n = 1,\dots,N\,. && (9.8)\end{aligned}$$

This is a system of N Poisson-like differential equations, which from now on we consider in their weak form. Next, we do a spatial discretization $V_h \subset V$ with basis functions $\{\phi_0,\dots,\phi_m\}$.

b) Using the coefficient vector $\underline{y}^n \in \mathbb{R}^{m+1}$ with the Ansatz

$$y^n \approx \sum_{i=0}^{m} \underline{y}^n_i \phi_i\,, \tag{9.9}$$

derive the time-space discretized formulation

$$\left(M + \frac{T}{N}K\right)\underline{y}^n = \frac{T}{N}L^n + M\underline{y}^{n-1}\,, \quad n = 1,\dots,N\,. \tag{9.10}$$

Here, $M \in \mathbb{R}^{(m+1)\times(m+1)}$ is the mass matrix with $M_{ij} = \int \phi_i\phi_j$, $K \in \mathbb{R}^{(m+1)\times(m+1)}$ is the stiffness matrix with $K_{ij} = \int(\phi_i)_x(\phi_j)_x$, and $L^n \in \mathbb{R}^{m+1}$ is the load vector with $L^n_i = \int f^n\phi_i + \phi_i(0)l^n + \phi_i(1)r^n$.

c) Repeat the derivation until (9.10), using the Crank-Nicholson scheme in time.

(15 points)