Scientific Computing I

(Wissenschaftliches Rechnen I)

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10^{th} exercise sheet

Submission on December 19, before the lecture

Exercise 1. (3 + 2 = 5 points)

Let X and Y be Banach spaces and $K\colon X \longrightarrow Y$ be a linear operator. We call K a *compact* operator if $(Kx_n)_n \subset Y$ has a convergent subsequence whenever $(x_n) \subset X$ is a bounded sequence.

a) Prove the following statement: Let $T\colon X \longrightarrow Y$ be a bounded linear operator between Banach spaces such that there is a sequence of linear operators $(T_n)_n$ with finite dimensional range that converges to T with respect to the operator norm. Then, it follows that T is compact.

Remark: Whether the converse of this statement is true or not is related to an problem that was unsolved for almost 40 years. The mathematician Per Enflo gave a negative answer in 1972 and recieved a living goose as prize from Stanislaw Mazur, who had formulated the problem in 1936.

Source: `http://perenflo.com`

b) Now, let $\Omega \subset \mathbb{R}^d$ be a polygonal domain. Use the statement of b) to prove that the embeddings

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega), \qquad p > d,$$

are compact.

Remark: These are not the only compact Sobolev embeddings, of course, and they do not only hold on polygonal domains. You should remember the following Sobolev embeddings in dimension d, that hold at least for bounded Lipschitz domains:

$$W^{k,p} \hookrightarrow W^{\ell,q}, \qquad \text{as long as } k - \frac{d}{p} \geq \ell - \frac{d}{q} \text{ and } k \geq \ell,$$

with compactness if $k > \ell$ and $k - \frac{d}{p} > \ell - \frac{d}{q}$. Embedding into Hölder spaces is possible for $p > d$:

$$W^{k,p} \hookrightarrow C^{k-1,\alpha}, \qquad \text{with } 0 < \alpha \leq 1 - \frac{d}{p}.$$

which is compact for $\alpha < 1 - \frac{d}{p}$.

Exercise 2. (3 + 2 = 5 points)

In this exercise we want to illustrate that choosing the "right" mesh heavily influences how good a function can be approximated with respect to the number of degrees of freedom used for this approximation.

Let $\Omega = [0,1]$. For $n \in \mathbb{N}$, we denote by $\mathcal{T}_n$ the partition of $[0,1]$ into n subintervals of *equal length* with start- and endpoints

$$0 = x_0 < x_1 = h < \ldots < x_{n-1} = 1 - h < x_n = 1, \qquad h = \frac{1}{n}.$$

By $\mathcal{I}_{\mathcal{T}_n}$ we denote the interpolation operator with piecewise linear polynomials on $[0,1]$ with respect to the partition $\mathcal{T}_n$.

Given some $u \in H^1([0,1])$ we denote by $\hat{\mathcal{T}}_n(u)$ the partition of $[0,1]$ into n subintervals with start- and endpoints

$$0 = \hat{x}_0 < \hat{x}_1 < \ldots < \hat{x}_{n-1} < \hat{x}_n = 1$$

such that $\int_{\hat{x}_i}^{\hat{x}_{i+1}} |u'(x)|^2 \, dx = \frac{1}{n} |u|^2_{H^1([0,1])}$ for $i = 0, \ldots, n-1$. With $\mathcal{I}_{\hat{\mathcal{T}}_n(u)}$ we denote the corresponding interpolation operator with piecewise linear polynomials with respect to the partition $\hat{\mathcal{T}}_n(u)$.

a) Compute estimates for the interpolation errors $\|u - \mathcal{I}_{\mathcal{T}_n} u\|_{L^\infty([0,1])}$ and $\|u - \mathcal{I}_{\hat{\mathcal{T}}_n(u)} u\|_{L^\infty([0,1])}$.

b) Compute the mesh $\hat{\mathcal{T}}_n(u)$ for $u(x) := |x|^{\frac{1}{2}+\epsilon}$, $\epsilon > 0$. What do you observe?

Exercise 3. (3 + 3 + 2 = 8 points)

Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary. Early in this lecture the Poincaré inequality

$$\|u\|_{L^2(\Omega)} \le C(\Omega)|u|_{H^1} \qquad \forall u \in H_0^1(\Omega) \tag{1}$$

has been proven. Now we want to consider variants of this import theorem:

a) Let $E \subset \Omega$ have nonzero measure or let $D \subset \partial\Omega$ have nonzero measure with respect to the surface measure $\mathrm{d}s$ on $\partial\Omega$. Prove the following statement for $p \in [1, \infty)$: There exist constants $C_E = C_E(\Omega, E, p) > 0$ and $C_D = C_D(\Omega, D, p)$ such that

$$\|u\|^p_{W^{1,p}(\Omega)} \le C_E \left(|u|^p_{W^{1,p}} + \left| \int_E u \, \mathrm{d}x \right|^p \right) \qquad \forall u \in W^{1,p}(\Omega)$$

$$\text{and} \qquad \|u\|^p_{W^{1,p}(\Omega)} \le C_D \left(|u|^p_{W^{1,p}} + \left| \int_D u \, \mathrm{d}s \right|^p \right) \qquad \forall u \in W^{1,p}(\Omega).$$

Hint: Assume the contrary and use a compactness argument to obtain a contradiction.

Remark: The second inequality yields a generalization of (1) for $H_D^1(\Omega)$ with homogeneous Dirichlet boundary conditions on an arbitrary subset D of $\partial\Omega$ with nonzero surface measure.

b) Prove the following further variant of the Poincaré inequality: There exists a constant $C = C(\Omega, p) > 0$ such that

$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u \, \mathrm{d}x \right\|_{L^p(\Omega)} \le C|u|_{W^{1,p}} \qquad \forall u \in W^{1,p}(\Omega).$$

c) Use the results obtained above in order to prove that the following PDEs admit unique weak solutions in $H^1(\Omega)$:

$$\begin{aligned} -\Delta u + cu &= f \qquad \text{on } \Omega, \\ \partial_n u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

with $f \in L^2(\Omega)$, $c \in L^\infty(\Omega)$, $c \ge 0$ a.e. on Ω, but $c \not\equiv 0$, and

$$\begin{aligned} -\Delta u &= f \qquad \text{on } \Omega, \\ \partial_n u + bu &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

with $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, $b \in L^\infty(\partial\Omega)$, $b \ge 0$ $\mathrm{d}s$-a.e. on $\partial\Omega$, but $b \not\equiv 0$.

(Note that we do not assume positive lower bounds for c and b, as we always did before!)

Programming exercise 1. (3 + 7 = 10 points)

A numerical walk trough the zoo of elliptic equations

In this last regular[1] programming exercise we want to illustrate the convergence rates of piecewise linear finite elements for different elliptic model problems.

a) Consider the 1D-example from Exercise 2 on Sheet 8. When implementing the `Expression`-object for the oscillating coefficient a_n choose `degree=3` to ensure that the oscillating behaviour gets captured correctly during integration.

Plot the finite element approximation errors measured in L^2-, H^1- and L^∞-norms with respect to the mesh size $h \in \{5^{-1}, 10^{-1}, 15^{-1}, 20^{-1}, 25^{-1}, 30^{-1}, 40^{-1}, 50^{-1}, 75^{-1}, 150^{-1}, 300^{-1}\}$ for different choice of the oscillation frequency $n \in \{10, 15, 30\}$. Use the `errornorm`-function of `FEniCS` that allows you to compute the error between a true solution (that might by given as an `Expression`-object) and a finite element solution. How can you explain your observations?

b) For the following three PDEs we do not know an analytical solution, so we have to compute a *reference solution* u_{ref} on a fine mesh and compute error estimates of solutions on coarser meshes with respect to this reference solution acting as exact solution. Note that this can be done with the help of `errornorm` as well by specifying the fine mesh of the reference solution as mesh on which the norms are to compute. In order to interpolate the coarse mesh solution to the fine mesh of the reference solution it might be necessary to use the following command at the beginning of your `FEniCS` code:

```
parameters["allow_extrapolation"] = True
```

This allows extrapolation of the coarse mesh solutions when interpolation onto the fine mesh requires evaluation of the coarse mesh solution outside the domain described by the coarse mesh.

For the following PDEs plot L^2- and H^1-errors with respect to the mesh size h and the number of degrees of freedom:

- Domain with and without reentrent corner

  Let $\Omega_\omega := \{(r\cos\phi, r\sin\phi) \in \mathbb{R}^2 \colon\ r \in [0,1], \phi \in [\omega, 2\pi]\}$ be the domain given by the unit circle with a circular sector of angle $\omega \in (0, 2\pi)$ cut out. We consider the PDE

$$\begin{aligned} -\Delta u &= 1 \quad \text{on } \Omega_\omega, \\ u &= 0 \quad \text{on } \partial\Omega_\omega, \end{aligned}$$

  first, for $\omega = \pi$, and second, for $\omega = \frac{\pi}{4}$. Can you observe a difference in the convergence rates? Why?

  Repeat the experiment with piecewise quadratic instead of piecewise linear finite elements. What can you observe?

- Discontinuous coefficient

  On $\Omega = [0,1]^2$ we consider the PDE

$$-\operatorname{div}(a_\epsilon \nabla u) = 1 \quad \text{on } \Omega, \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3}$$

  with coefficient $a_\epsilon(x) := \epsilon + \epsilon^{-1}\mathbf{1}_K$ for some $\epsilon > 0$ and $K := B_{1/3}((1/2, 1/2))$.

  Which convergence rates do you observe? How does convergence react on changing from $\epsilon = 1$ to $\epsilon = 10^{-1}$ or $\epsilon = 10^{-2}$?

[1] Therefore, there are 35 programming points to reach in total and 18 programming points are sufficient for admittance to the exam. If required, there will be a further bonus programming exercise in January.

- Mixed boundary condition
  On $\Omega = [0,1]^2$ we consider the PDE

$$\begin{aligned} -\Delta u &= 1 \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D := \{x \in \partial\Omega\colon\ x_1 \geq 1/2\}, \\ \partial_n u &= 1 \quad \text{on } \Gamma_{N,1} := \{x \in \partial\Omega\colon\ x_1 < 1/2, x_2 < 1/2\}, \\ \partial_n u &= -1 \quad \text{on } \Gamma_{N,2} := \{x \in \partial\Omega\colon\ x_1 < 1/2, x_2 \geq 1/2\}. \end{aligned}$$

  Which convergence rates do you observe?

c) (3 bonus points) On the unit circle $\Omega = B_1(0) \subset \mathbb{R}^2$ we consider the elliptic problem

$$-\Delta u = \delta \quad \text{on } \Omega, \tag{4}$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{5}$$

where δ denotes the Dirac measure centered at 0. Solvability and finite element approximation of this equation has been considered in Exercise 2 on Sheet 9. The exact solution is given by

$$u(x) := -\frac{1}{2\pi} \log\|x\|_2.$$

Plot L^2- and H^1-errors of the finite element approximation with respect to the mesh size. Does the behaviour of L^2-errors confirm our theoretical analysis?

Utilize the `PointSource`-construction provided by `FEniCS`. When implementing an `Expression` for logarithmic terms you might be required to use `std::log` (instead of just `log`, which might produce an error) in the `C++`-code part.

Please submit the programming exercise til **January 9**, before the lecture, **directly to your tutor** via Email.