Scientific Computing I
(Wissenschaftliches Rechnen I)

Winter term 2019/20

Priv.-Doz. Dr. Christian Rieger
Fabian Hoppe

UNIVERSITÄT BONN

11^{th} exercise sheet

Submission on January 9, before the lecture

Exercise 1. (1 + 2 + 1 + 3 + 2 = 9 points)

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary. The motion of an incompressible viscous fluid with velocity field $u\colon \Omega \longrightarrow \mathbb{R}^d$ can be modeled with the following PDE ("Stokes equation"):

$$\begin{aligned} \Delta u + \nabla p &= -f \qquad \text{in } \Omega, \\ \operatorname{div} u &= 0 \qquad \text{in } \Omega, \\ u &= u_0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here, $p\colon \Omega \longrightarrow \mathbb{R}$ is the pressure density, $f\colon \Omega \subset \mathbb{R}^d$ is an external force and Δ denotes the componentwise Laplacian $\Delta u := (\Delta u_1, ..., \Delta u_d)^T$.

a) Assume that a strong solution (u, p) to (1) exists. Conclude that the boundary condition u_0 has to fulfill:

$$\int_{\partial\Omega} n \cdot u_0 \, \mathrm{d}s = 0,$$

where $n\colon \partial\Omega \longrightarrow \mathbb{R}^d$ is the outer normal vector field.

From now on wie consider only homogenous Dirichlet boundary conditions, i.e. $u_0 \equiv 0$. Moreover, since p is only determined up to a constant we additionally enforce

$$\int_\Omega p \, \mathrm{d}x = 0.$$

We introduce the spaces $V = H_0^1(\Omega)^d$ and $\Pi = \left\{ q \in L^2(\Omega) \colon \int_\Omega q \, \mathrm{d}x = 0 \right\}$ and the bilinear forms

$$\begin{aligned} a(u, v) &:= \int_\Omega \operatorname{trace}[(Du)^T Dv] \, \mathrm{d}x, \qquad u, v \in V, \\ b(v, q) &:= \int_\Omega q \operatorname{div} v \, \mathrm{d}x, \qquad v \in V, q \in \Pi, \end{aligned}$$

where $Du = (D_j u_i)_{i,j} \in L^2(\Omega)^{d\times d}$ denotes the weak Jacobian matrix of $u \in V$. Note that the space Π is equipped with the standard $L^2(\Omega)$-norm and V is equipped with norm

$$\|u\|_V^2 := \left(\sum_{i=1}^d \|u_i\|_{H^1(\Omega)}^2 \right)^{1/2}, \qquad u = (u_1, ..., u_d)^T \in V.$$

We formulate (1) in weak form as the following saddle point problem: Find $(u, p) \in V \times \Pi$ such that it holds

$$\begin{aligned} a(u, v) + b(v, p) &= F(v) \qquad \forall v \in V, \\ b(u, q) &= 0 \qquad \forall q \in \Pi, \end{aligned} \tag{2}$$

where $F \in V'$ is defined by $F(v) := \int_\Omega f \cdot v \, \mathrm{d}x$, $v \in V$.

b) Show that the bilinear form $a(\cdot,\cdot)$ is coercive on V.

c) The form $b(\cdot,\cdot)$ induces an operator $B' : \Pi \longrightarrow V'$ via

$$\langle v, B'q\rangle_{V,V'} := b(v,q) \qquad \forall v \in V, q \in \Pi.$$

What is the meaning of B' as (weak) differential operator?

d) Prove that the saddle point formulation (2) fulfills the inf-sup-condition if and only if it holds

$$\|q\|_{L^2(\Omega)} \leq c_\Omega \|\nabla q\|_{H^{-1}(\Omega)^d} \qquad \forall q \in \Pi. \tag{3}$$

Remark: The latter condition, the so called Necas inequality, indeed holds in a bounded domain with Lipschitz boundary. As the next exercise shows Lipschitz continuity of the boundary is essential for this result and hence for existence of solutions to (2).

e) Consider $\Omega = \{(x,y) \in \mathbb{R}^2 \colon\ 0 < x < 1, 0 < y < x^2\}$ and $q(x,y) := x^{-2}$. Show that q cannot not fulfill (3) .

Exercise 2. (3 + 3 = 5 points)

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ we introduce the space $H(\mathrm{div},\Omega)$ as follows: A vector field $V \in L^2(\Omega)^d$ has weak divergence $w \in L^1_{\mathrm{loc}}(\Omega)$ if it holds:

$$\int_\Omega V \cdot \nabla\varphi \,\mathrm{d}x = -\int_\Omega w\varphi \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega).$$

In the following we will write $w = \mathrm{div}\, V$ and define

$$H(\mathrm{div},\Omega) := \{V \in L^2(\Omega)^d \colon\ \mathrm{div}\, V \in L^2(\Omega)\}$$

equipped with norm

$$\|V\|^2_{H(\mathrm{div})} := \sum_{i=1}^d \|V_i\|^d_{L^2(\Omega)} + \|\mathrm{div}\, V\|_{L^2(\Omega)}.$$

It turns out that $H(\mathrm{div},\Omega)$ is a Hilbert space and coincides with $\overline{C^\infty(\Omega)^d}^{\|\cdot\|_{H(\mathrm{div})}}$.
For $V = H(\mathrm{div},\Omega)$ and $\Pi = L^2(\Omega)$ we consider the saddle point problem: Find $(\sigma,u) \in V \times \Pi$ such that

$$\begin{aligned} \int_\Omega \sigma\cdot\tau \,\mathrm{d}x + \int_\Omega u \,\mathrm{div}\,\tau \,\mathrm{d}x &= 0 \qquad \forall \tau \in V, \\ \int_\Omega v \,\mathrm{div}\,\sigma = -\int_\Omega f v \,\mathrm{d}x & \qquad \forall v \in \Pi. \end{aligned} \tag{4}$$

a) Show that (4) admits a unique solution.

Hint: For $v \in \Pi$ choose $w \in C_0^\infty(\Omega)$ "close" to v and define $\tau \in H(\mathrm{div},\Omega)$ such that $\mathrm{div}\,\tau = w$ and $\|\tau\|_{L^2(\Omega)^d} \leq c\|w\|_{L^2(\Omega)^d}$.

b) Prove that for the solution $(\sigma,u) \in V \times \Pi$ of (4) it evens holds $u \in H^1(\Omega)$. Give the strong formulation of the PDE corresponding to (4).

Exercise 3. (3 + 2 = 5 points)

a) Let $\{\Omega_j\}_{j=1,\dots,J}$ be a partition of a domain Ω into piecewise smooth subdomains Ω_j, i.e. $\Omega \subset \cup_{j=1}^{J} \overline{\Omega_j}$ and $\Omega_j \cap \Omega_{j'} = \emptyset$ for $j \neq j'$. Show that every $V \in L^2(\Omega)^d$ such that $V_j := V|_{\Omega_j} \in C^1(\overline{\Omega}_j)$ for all $j = 1, \dots, J$ is an element of $H(\mathrm{div}, \Omega)$ if it holds $n_{\partial\Omega_i} \cdot V_i = n_{\partial\Omega_j} \cdot V_j$ on $\partial\Omega_i \cap \partial\Omega_j$.

b) Let $\mathcal{T}_h$ be a triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$. We define

$$\begin{aligned}\mathrm{RT}_0 := \{\varphi \in L^2(\Omega)^2 \colon \quad & \varphi|_T = a_T + c_T x, \quad a_T \in \mathbb{R}^2, c_T \in \mathbb{R}, \quad \forall T \in \mathcal{T}_h, \\ & n_e \cdot \varphi|_{T_1}(x_e) = n_e \cdot \varphi|_{T_2}(x_e) \quad \forall \text{ edges } e = T_1 \cap T_2\},\end{aligned}$$

where x_e denotes the midpoint of an edge e of $\mathcal{T}_h$. Show that RT_0 is a subspace of $H(\mathrm{div}, \Omega)$.

Recapitulation exercises

Exercise 4. (2 + 2 = 4 bonus points)

a) Consider the function

$$u \colon \mathbb{R}^d \longrightarrow \mathbb{R}, \qquad x \longmapsto \log\log\frac{2}{|x|_2}.$$

For which $d \in \mathbb{N}$, $p \in (1, \infty)$ does $u \in W^{1,p}(B_1(0))$ hold?

b) Let $p, q \in [1, \infty]$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $\Omega \subset \mathbb{R}^d$. Show that it holds $uv \in W^{1,r}(\Omega)$ if $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$, and give a formula for the weak derivatives of uv.

Exercise 5. (4 + 2 = 6 bonus points)

Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary and $f \in L^2(\partial\Omega)$ be given. For $\epsilon > 0$ we consider the following PDE in strong form:

$$\begin{aligned} -\Delta u &= 0 \qquad \text{on } \Omega, \\ \epsilon \cdot \partial_n u + u &= f \qquad \text{on } \partial\Omega. \end{aligned} \tag{5}$$

a) State the weak formulation of the PDE (5), including the spaces. Prove existence of a unique solution to (5) and give an estimate of this solution in terms of f.

b) Assume that there is $\varphi \in H^2(\Omega)$ such that $\Delta u = 0$ and $u|_{\partial\Omega} = f$ in the sense of traces. Show that $u = u(\epsilon)$ stays bounded in H^1 as $\epsilon \longrightarrow 0$.

Merry Christmas and a Happy New Year!