Scientific Computing I
(Wissenschaftliches Rechnen I)

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12^{th} exercise sheet

Submission on January 16, before the lecture

Exercise 1. (2 + 3 = 5 points)

Given $A \in \mathbb{R}^{m\times m}$ symmetric and $B \in \mathbb{R}^{n\times m}$ we define

$$C := \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}.$$

a) Fix $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$. Show that the KKT-system for the equality-constrained optimization problem

$$\min_{x\in\mathbb{R}^m} f(x) := \frac{1}{2}x^T Ax - a^T x \qquad \text{such that } Bx = b.$$

can be written as

$$C \cdot \begin{pmatrix} x \\ \lambda \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} a \\ b \end{pmatrix}$$

with some $\lambda \in \mathbb{R}^n$.

b) Assume now that $\operatorname{rank}(B) = n$ and that A is positive definite on $\ker(B)$.

Show that C is invertible and that the corresponding solution $\begin{pmatrix} x \\ \lambda \end{pmatrix}$ of the linear system from a) is a minimizer of the optimization problem.

Exercise 2. (5 points)

Prove that also the following converse of the Fortin interpolation result from the lecture is true: Let X and M be Hilbert spaces and X_h and M_h be subspaces, respectively. If a bilinear form $b\colon X \times M \longrightarrow \mathbb{R}$ fulfills the inf-sup condition both on X, M and X_h, M_h with some constant $\beta > 0$, then there exists a bounded linear projection operator $\Pi_h\colon X \longrightarrow X_h$ such that

$$b(v - \Pi_h v, \mu_h) = 0 \qquad \forall \mu_h \in M_h$$

holds. Further, $\|\Pi_h\|_{\mathcal{L}(X)}$ does not depend on the concrete choice of X_h, M_h, but on β.

<u>Hint:</u> Construct a suitable saddle point problem. To do so choose the scalar product of X as the coercive bilinear form.

Exercise 3. (2 + 2 + 2 + 4 = 10 points)

Recall the Stokes equation and its formulation as saddle point problem from exercise 1 of Sheet 11. We further assume that the domain $\Omega \subset \mathbb{R}^2$ is convex and polygonal. In particular, the inf-sup condition is fulfilled in that case for the infinite dimensional spaces

$$X := H_0^1(\Omega)^2, \quad M := \left\{ p \in L^2(\Omega) \colon \int_\Omega p \,\mathrm{d}x = 0 \right\} \subset L^2(\Omega).$$

Now, let Ω be equipped with a family $\mathcal{T}_h$ of non-degenerate quasi-uniform triangulations. With $P_h^1 \subset H^1(\Omega)$ we denote the space of piecewise linear finite elements on $\mathcal{T}_h$ and by $P_{h,0}^1 := P_h^1 \cap H_0^1(\Omega)$ the subspace fulfilling homogeneous Dirichlet boundary conditions. A possible discretization of the Stokes equation could be given by the choice

$$X_h := (P_{h,0}^1)^2 = (P_h^1)^2 \cap X, \quad M_h := P_h^1 \cap M.$$

However, this discretization will be unstable, i.e. the inf-sup condition does not hold uniformly on the discrete level. A possible way to overcome this problem is to enrich the space X_h by adding an additional degree of freedom to the velocity approximation: Given the three ansatz functions

$$\phi_1(x, y) = 1 - x - y, \quad \phi_2(x, y) = x, \quad \phi_3(x, y) = y$$

of P_h^1 on the reference triangle we define on the reference triangle an additional ansatz function

$$b(x, y) := 27\phi_1(x, y)\phi_2(x, y)\phi_3(x, y),$$

the so called "bubble function". The bubble space $B_{3,h}$ is defined as

$$B_{3,h} := \{ v \in C(\overline{\Omega}) \colon v|_T \circ a_T \in \operatorname{span}\{b\} \quad \forall T \in \mathcal{T}_h \},$$

where $a_T \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ denote the affine linear transformation from the reference triangle to triangle T. Finally, we introduce the discrete ansatz spaces as follows:

$$X_h := \left(P_{h,0}^1 \oplus B_{3,h} \right)^2, \qquad M_h := P_h^1 \cap M.$$

We show that the inf-sup condition holds on the discrete level in several steps:

a) Prove that for any $v \in H_0^1(\Omega)$ there exists a unique $v_h \in P_{h,0}^1$ fulfilling

$$\langle v_h, w_h \rangle_{H^1} = \langle v, w_h \rangle_{H^1} \qquad \forall w_h \in P_{h,0}^1$$

and that the map $\pi_h^0 \colon H_0^1(\Omega) \longrightarrow P_{h,0}^1$ defined by $v \longmapsto v_h$ is continuous with respect to the $H^1(\Omega)$-norm.

b) Utilize the Aubin-Nietzsche trick and H^2-regularity to prove

$$\|\pi_h^0 v - v\|_{L^2(\Omega)} \le Ch\|v\|_{H^1(\Omega)}.$$

c) Show that the map $\pi_h^1 \colon L^2(\Omega) \longrightarrow B_3$, $v \longmapsto v_h$ defined by

$$\int_T (v_h - v) \,\mathrm{d}x = 0 \qquad \forall T \in \mathcal{T}_h$$

is well defined and continuous with respect to the L^2-norm.

Hint: Utilize the ansatz $v_h|_T = \alpha_T(b \circ a_T^{-1})$ with $\alpha_T \in \mathbb{R}$.

d) Define the Fortin interpolation as

$$\Pi_h v := \pi_h^0 v + \pi_h^1(v - \pi_h^0),$$

where application of π_h^0 resp. π_h^1 to $H^1(\Omega)^2$ has to be understood componentwise. Prove that this operator satisfies the required assumptions, i.e.

(i) $b(v - \Pi_h v, q_h) = \int_\Omega q_h \operatorname{div}(v - \Pi_h v)\,\mathrm{d}x = 0$ for $v \in X$, $q_h \in M_h$,

(ii) $\|\Pi_h v\|_{H^1(\Omega)^2} \leq C\|v\|_{H^1(\Omega)^2}$ for $v \in X$.

Hint: For (i) start with showing $\int_T(\Pi_h v - v)\,\mathrm{d}x = 0$ for all $T \in \mathcal{T}_h$ and $\Pi_h v - v = 0$ on $\partial\Omega$. Regarding (ii) recall the inverse estimate $\|v_h\|_{H^1(\Omega)} \leq ch^{-1}\|v_h\|_{L^2(\Omega)}$ for all $v_h \in P_h^1$.

This is the last exercise sheet that counts for admission to the final exam. In total, there are 213 theory points and 35 programming points to reach and therefore **107 theory points and 18 programming points will be *sufficient* for admission to the exam.**

The 13th (and final) exercise sheet will be submitted on January 23. The solutions will be discussed in the tutorials of the last week of the semester (January 27-31).