Scientific Computing I
(Wissenschaftliches Rechnen I)

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3rd exercise sheet

Submission on October 31, before the lecture

Exercise 1. (3 + 3 = 6 points)

Let $\Omega = (0,1)$ and consider the bilinear form

$$a: H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}, \qquad (u,v) \mapsto \int_\Omega t^2 u'(t) v'(t) dt$$

resp. the linear form

$$\ell : H_0^1(\Omega) \longrightarrow \mathbb{R}, \qquad v \mapsto \int_\Omega v(t) dt.$$

a) Show that the functional

$$J : H_0^1(\Omega) \longrightarrow \mathbb{R}, \qquad u \mapsto \frac{1}{2} a(u,u) - \ell(u).$$

has infimum

$$\inf_{u \in H_0^1(\Omega)} J(u) = -\frac{1}{2}.$$

Hint: Show that $J(u) = \frac{1}{2}\int_0^1 \left((tu'(t)+1)^2 - 1\right) dt$ for all $u \in H_0^1(\Omega)$. For the construction of a infimizing sequence think about the differential equation given (in weak form) by $a(u,v) = \ell(v) \quad \forall v \in H_0^1(\Omega)$.

b) Show that J has no minimizer, i.e. the infimum from exercise a) is not a minimum. Explain why this does not contradict the Lax-Milgram theorem.

Exercise 2. (3 + 3 = 6 points)

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We consider the PDE

$$\begin{aligned} -\operatorname{div}(a\nabla u) + b \cdot \nabla u + u &= f \qquad &&\text{on } \Omega, \\ a\partial_n u &= g &&\text{on } \partial\Omega, \end{aligned}$$

with $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, $b \in (L^\infty(\Omega))^d$ and $a \in L^\infty(\Omega)$, $a(x) \geq a_0 > 0$ a.e. on Ω.

a) Derive a weak formulation of this PDE in the space $H^1(\Omega)$.

b) Give a suitable (non trivial) condition on b such that there exists a unique weak solution of the PDE. Estimate the H^1-norm of the solution in terms of the given data.

Hint: You may use that there is a continuous trace map $\operatorname{tr}: H^1(\Omega) \longrightarrow L^2(\partial\Omega)$ with operator norm $c_{\partial\Omega} > 0$.

Exercise 3. (4 points)

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $a, \hat{a} \in L^\infty(\Omega)$ such that $0 < a_0 \leq a(x), \hat{a}(x) \leq A < \infty$ a.e. on Ω. For some $f \in H^{-1}(\Omega)$ let $u \in H_0^1(\Omega)$ the solution of the PDE

$$-\operatorname{div}(a\nabla u) = f \text{ on } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

and $\hat{u} \in H_0^1(\Omega)$ the solution of

$$-\operatorname{div}(\hat{a}\nabla u) = f \text{ on } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

Show that it holds

$$\|u - \hat{u}\|_{H_0^1(\Omega)} \leq c\|a - \hat{a}\|_{L^\infty}$$

with some constant $c > 0$ and write explicitely down this constant c for the given data.

Exercise 4. (2 + 2 = 4 points)

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We consider the following PDE, the so called *biharmonic equation*:

$$\begin{aligned} \Delta^2 u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ \partial_n u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Here, $\partial_n u = \nabla u \cdot n$ denotes as usual the outer normal derivative of u on the boundary $\partial\Omega$ of Ω. With $\Delta^2 = \Delta \circ \Delta$ we denote double application of the Laplacian, which is defined in its strong form on $C^4(\Omega)$.

a) Derive the corresponding weak formulation of the biharmonic equation in the space

$$H_0^2(\Omega) := \{u \in H^2(\Omega) : u = 0 \text{ and } \partial_n u = 0 \text{ on } \partial\Omega\}.$$

b) Now we assume that Ω has a C^2-boundary or that Ω is convex. In this case there is some $c_\Omega > 0$ such that $\|u\|_{H^2(\Omega)} \leq c_\Omega\|\Delta u\|_{L^2(\Omega)}$ for every $u \in H_0^2(\Omega)$ holds.

Show that the biharmonic equation admits a unique solution $u \in H_0^2(\Omega)$ for any $f \in L^2(\Omega)$ and estimate the norm of u in terms of f.

Exercise 5. (2 + 1 + 1 + 2 = 6 bonus points)

Let $\Omega \subset \mathbb{R}^d$ be a domain.

a) Let $F \in (L^2(\Omega))^d$, i.e. F is a vector field on Ω with L^2-integrable components.

Define a weak divergence $\operatorname{div}(F) \in H^{-1}(\Omega)$ that coincides with the classical (i.e. strong) notion of divergence for a smooth vector field $F \in (C^\infty(\Omega))^d$.

For $p \in (1, \infty)$ we denote by $W^{-1,p}(\Omega) := (W_0^{1,p'}(\Omega))^*$ the dual space of $W_0^{1,p'}(\Omega)$. Here, p' is the Hölder conjugate exponent to p, i.e. $p^{-1} + (p')^{-1} = 1$.

b) Let $F \in (L^p(\Omega))^d$. Define again a weak divergence of F, $\operatorname{div}(F) \in W^{-1,p}$. Is the map $\operatorname{div} : (L^p(\Omega))^d \longrightarrow W^{-1,p}(\Omega)$ continuous?

c) Given $a \in L^\infty(\Omega)$, $a(x) \geq a_0 > 0$ a.e. on Ω, the weak form of the PDE

$$-\operatorname{div}(a\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

with $f \in H^{-1}(\Omega)$ gives rise to a bounded linear map $-\operatorname{div}(a\nabla\,\cdot\,) : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$.

(a) Show that this map is well defined and continuous from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$ as well for $p \in (1, \infty)$.

(b) In which $W_0^{1,p}(\Omega)$ do you therefore have to search for a weak solution of

$$-\operatorname{div}(a\nabla u) = \delta_{x_0} \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

if δ_{x_0} denotes the Dirac measure at some $x_0 \in \Omega$?

Hint: Use the Sobolev embeddings to determine those p for which $\delta_{x_0} \in W^{-1,p}(\Omega)$. To do so you can assume that Ω is bounded and Lipschitz.

Note: Existence of solutions in the $W_0^{1,p}$-$W^{-1,p}$-setting is much more difficult to prove than in the H_0^1-H^{-1}-setting because we cannot apply Lax-Milgram!