Scientific Computing I
(Wissenschaftliches Rechnen I)

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4^{th} exercise sheet

Submission on November 7, before the lecture

Exercise 1. (3 points)

Let $a\colon V \times V \longrightarrow \mathbb{R}$ be a (not necessarily symmetric) bilinear form on a Hilbert space V such that it holds

$$|a(u,v)| \leq C\|u\|_V\|v\|_V \qquad \forall u,v \in V, \qquad \text{"boundedness"},$$
$$|a(u,u)| \geq \alpha\|u\|_V^2 \qquad \forall u \in V, \qquad \text{"coercivity"},$$

with some constants $\alpha, C > 0$. Given $F \in V^*$, let $u \in V$ be the solution of the variational problem $a(u,\cdot) = F$. For a subspace $V_h \subset V$ define $u_h \in V_h$ as solution of

$$a(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h.$$

In the lecture, Céa's Lemma was shown: It holds

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{v_h \in V_h} \|u - v_h\|_V.$$

Now, we assume that $a(\cdot,\cdot)$ is additionally *symmetric*. Show that it follows

$$\|u - u_h\|_V \leq \sqrt{\frac{C}{\alpha}} \min_{v_h \in V_h} \|u - v_h\|_V.$$

Hint: Define a new scalar product on V.

Exercise 2. (2 + 2 = 4 points)

Let H be a Hilbert space with scalar product $\langle \cdot,\cdot \rangle$, $C \subset H$ a convex subset and $f\colon D \longrightarrow \mathbb{R}$ be Fréchet differentiable, where $D \subset H$ is an open subset containing C.

[As an example you might think of $C = H$ and the quadratic functional $f(x) = \frac{1}{2}a(x,x) - F(x)$ where $a\colon H \times H \longrightarrow \mathbb{R}$ is a continuous bilinear form and F a bounded linear functional on H.]

a) Let $\bar{x} \in C$ be a minimizer of f over C, i.e. $f(\bar{x}) \leq f(x)$ for all $x \in C$. Show that the variational inequality

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0 \qquad \forall x \in C \tag{VI}$$

holds. Show that (VI) is equivalent to $\nabla f(\bar{x}) = 0$ if $\bar{x}$ is in the interior of C.

b) Let in addition f be convex. Show that $\bar{x}$ is a minimizer of f over C if (VI) holds. Does this stay true if f is not assumed to be convex?

Exercise 3. (4 points + 3 bonus points)

Every bounded domain $\Omega \subset \mathbb{R}^d$ that is *convex* or has *smooth boundary* has the following property: For $f \in L^2(\Omega)$ the solution $u \in H_0^1(\Omega)$ of

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

has additional regularity $u \in H^2(\Omega)$ and it holds

$$\|u\|_{H^2(\Omega)} \leq C_\Omega \|f\|_{L^2(\Omega)}$$

with a constant $C_\Omega > 0$ independent of f. Similiar results are obtained when replacing the Laplacian by a more general elliptic operator with sufficiently smooth (Lipschitz) coefficients.

In this exercise we will see that we cannot expect this rather "nice" behaviour, if the domain is neither smooth nor convex or boundary conditions change:

a) Given polar coordinates (r, ϕ) in $\mathbb{R}^2$ we consider for some $\omega \in (\pi, 2\pi]$ the domain

$$\Omega_\omega := \left\{ (r\cos\phi, r\sin\phi) \in \mathbb{R}^2 : \ r \in [0,1), \phi \in (0, \omega) \right\},$$

i.e. Ω_ω is a domain with a so-called *reentrent corner* with interior angle ω at 0.

We define

$$s\colon \Omega_\omega \longrightarrow \mathbb{R}, \qquad (r, \phi) \mapsto (1 - r^2) r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\phi\right).$$

Show that $\Delta s \in L^2(\Omega_\omega)$, but $s \notin H^2(\Omega_\omega)$.

Hint: To show $s \notin H^2(\Omega_\omega)$ you may inspect $\frac{\partial^2 s}{\partial r^2}$.

b) (bonus exercise) For $\omega = \pi$ we partition the boundary of Ω_π into two parts:

$$\Gamma_D := \{(r\cos\phi, r\sin\phi) \in \partial\Omega_\pi : \ (r \in [0,1] \wedge \phi = 0) \vee (r = 1 \wedge \phi \in [0, \pi])\},$$
$$\Gamma_N := \partial\Omega \setminus \Gamma_D.$$

Construct $f \in L^2(\Omega_\pi)$ such that the solution u to

$$-\Delta u = f \text{ on } \Omega_\pi, \qquad u = 0 \text{ on } \Gamma_D, \qquad \partial_n u = 0 \text{ on } \Gamma_N$$

does not belong to $H^2(\Omega_\pi)$.

Hint: Use excercise 3a.

Exercise 4. (3 + 2 = 5 points)

On $\Omega = [0,1]$ we consider the Laplace equation with homogeneous Dirichlet boundary conditions

$$-u'' = f \quad \text{on } \Omega, \qquad u(0) = u(1) = 0.$$

The equation is understood in variational form on the Hilbert space $V = H_0^1(\Omega)$ and $f \in H^{-1}(\Omega)$, i.e.

$$u \in H_0^1(\Omega) \qquad \text{s.t.} \qquad \int_0^1 u'v' = f(v) \quad \forall v \in H_0^1(\Omega). \tag{1}$$

For some $N \in \mathbb{N}$ we consider the N-dimensional discrete subspace $V_N \subset V$ defined by

$$V_N := \operatorname{span}_{\mathbb{R}} \{b_1, ..., b_N\}$$

with the basis functions $b_n(x) := \sqrt{2}\sin(n\pi x)$.
Accordingly, let $u_N \in V_N$ be the solution of the Galerkin approximation of (1) with ansatz space V_N. Let $\mathbf{x} \in \mathbb{R}^N$ be its coefficient vector, i.e. $u_N = \sum_{n=1}^N \mathbf{x}_n b_n$, with respect to the basis functions $b_1, ..., b_N$.

a) Set up the linear system $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{N\times N}$, $\mathbf{b} \in \mathbb{R}^N$, that determines $\mathbf{x}$ for the following right hand side $f = f_i \in H^{-1}(\Omega)$:

  (i) $f_1(v) := \int_0^1 v(x)dx$ for $v \in H_0^1(\Omega)$.

  (ii) $f_2(v) := v\left(\frac{1}{2}\right)$ for $v \in H_0^1(\Omega)$.

  Show that u_N is the H_0^1-bestapproximation of the true solution in V_N.

b) Find the exact solution of (1) for right hand side $f = f_i$, $i = 1, 2$, respectively.

Programming exercise 1. (2 + 2 = 4 points)

We continue exercise 4 and consider the problem

$$-(au')' = f \quad \text{on } \Omega = [0,1], \qquad u(0) = u(1) = 0,$$

with some $a \in L^\infty([0,1])$, $a(x) \geq a_0 > 0$ a.e. on Ω. For some $N \in \mathbb{N}$, we define the N-dimensional finite element space V_N as in exercise 4.

a) Write a function that computes the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ for given coefficient function a, right hand side f and degrees of freedom N. For numerical integration you may use a scipy-routine.

  Test your implementation for $f(x) = \mathbf{1}_{[1/2,1]}(x)$ and $a(x) = 1 + 10 \cdot \mathbf{1}_{[0,1/3]}(x)$. Plot the solutions for different N.

b) Compute and plot the finite element approximation for the two problems from exercise 4a) for different N. Compute the L^2-difference to the true solutions (with help of a sufficiently fine numerical integration scheme) and determine the order of convergence of these erros with respect to the number of degrees of freedom N. Try to explain your observations.

Please submit the programming exercise til November 7, before the lecture, directly to your tutor via Email.