Scientific Computing I

(Wissenschaftliches Rechnen I)

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7^{th} exercise sheet

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Exercise 1. (2 + 2 + 1 + 2 + 2 = 9 points)

On the d-dimensional hypercube $Q_{\text{ref}} = [0,1]^d$ we consider for some $m \in \mathbb{N}$ the space

$$\mathcal{Q}_m := \operatorname{span}_{\mathbb{R}} \{(x_1, ..., x_d) \mapsto \psi_1(x_1) \cdot ... \cdot \psi_d(x_d) : \quad \psi_i \text{ polynomial of degree } \leq m\},$$

i.e. $\mathcal{Q}_m$ consists of all polynomials in the variables $x_1, ..., x_d$ such that each x_i appears only with exponents at most m.

a) Given arbitray $0 \leq z_0 < z_1 < ... < z_m \leq 1$, show that there is a nodal basis for $\mathcal{Q}_m$ with respect to the point set

$$Z = \left\{(z_{i_1}, ..., z_{i_d}) : \ i_1, ..., i_d = 0, ..., m\right\}.$$

b) Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygonal/polyhedral domain with a quadrilateral/hexahedral triangulation as described in exercise 1d of sheet 6 (affine multlinear transformations to the reference element Q_{ref}). Show that the finite element (Q_{ref}, $\mathcal{Q}_m$, point evaluations at Z) generates an $H^1(\Omega)$-conforming finite element space on Ω.

Remark: We call the finite element constructed above *tensor-product finite element of order m*, because $\mathcal{Q}_m$ is the d-fold tensor-product $P_m \otimes ... \otimes P_m$ of the space P_m of polynomials of degree $\leq m$ in one variable.

Now we consider only $d = 2$ and $m = 3$ and Ω being partitioned in a *rectangular* mesh, i.e. all cells of the mesh are rectangles w.l.o.g. with edges parallel to the coordinate axes. With $q_1, ..., q_4 \in Q_{\text{ref}}$ we denote the four corners of the reference square $Q_{\text{ref}} = [0,1]^2$.

c) Does the finite element (Q_{ref}, $\mathcal{Q}_3$, point evaluations at Z) generate an $H^2(\Omega)$-conforming finite element space on the rectangular mesh?

d) Show that

$$\mathcal{N} = \left\{p \mapsto p(q_i), p \mapsto \frac{\partial p}{\partial x_1}(q_i), p \mapsto \frac{\partial p}{\partial x_2}(q_i), p \mapsto \frac{\partial^2 p}{\partial x_1 \partial x_2}(q_i) : \ i = 1,2,3,4\right\}$$

is a basis for $(\mathcal{Q}_3)'$.

e) Does the finite element (Q_{ref}, $\mathcal{Q}_3$, $\mathcal{N}$) generate an H^2-conforming finite element space on the rectangular mesh?

Exercise 2. (2 points)

Let $\Omega \subset \mathbb{R}^d$ be a domain. For some $\lambda > 0$ we introduce the scaling operation

$$M_\lambda : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \\ x \longmapsto \lambda x.$$

Let $k \in \mathbb{N}_0$ and $p \in [1, +\infty]$. To a function $u \in W^{k,p}(\Omega)$, $k \in \mathbb{N}_0$, $p \in [1, \infty]$ we associate the scaled function $u_\lambda := u \circ M_\lambda^{-1} \in W^{k,p}(\Omega_\lambda)$ on the scaled domain $\Omega_\lambda := M_\lambda(\Omega)$.
How can you relate the Sobolev seminorms $|u_\lambda|_{W^{k,p}(\Omega_\lambda)}$ and $|u|_{W^{k,p}(\Omega)}$?

Exercise 3. (2 + 2 = 4 points)

Given $\alpha_0 \in \mathbb{N}^n$ define $A(\alpha_0) \subset \mathbb{N}_0^n$ by

$$A(\alpha_0) := \{(\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n : \ \alpha_i < \alpha_{0,i}, i = 1, ..., n\}.$$

For $\alpha_0 = (m, ..., m)$, $m \in \mathbb{N}$, the set $A_m := A(\alpha_0)$, e.g., corresponds to the set of all multiindices β defining polynomials $x^\beta = x_1^{\beta_1} \cdot ... \cdot x_n^{\beta_n} \in \mathcal{Q}_{m-1}$.
Given a domain $\Omega \subset \mathbb{R}^n$ that is starshaped with respect to a ball B around x_0 with radius $\rho > 0$ we introduce the averaged Taylorpolynomial

$$Q^{A(\alpha_0)} u(x) = \int_B \sum_{\alpha \in A(\alpha_0)} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \phi(y) dy,$$

where ϕ denotes a cutoff-function supported in $\overline{B}$ with $\int_{\mathbb{R}^n} \phi = 1$.

a) It is clear that $Q^{A(\alpha_0)} u$ is welldefined for $u \in W^{|\alpha_0|-n,p}(\Omega)$. Prove that there is a bounded linear extension

$$Q^{A(\alpha_0)} : L^1(B) \longrightarrow W^{k,\infty}(\Omega)$$

for each $k \in \mathbb{N}$ and that $Q^{A_m} u \in \mathcal{Q}_{m-1}$ for any $u \in L^1(B)$.

b) Let $\alpha \in A(\alpha_0)$. Show that

$$D^\alpha Q^{A(\alpha_0)} = Q^{A(\alpha_0 - \alpha)} D^\alpha u \qquad \forall u \in W^{|\alpha_0|-|\alpha|,p}(\Omega).$$

Exercise 4. (2 + 3 = 5 points)

Let $\mathcal{T}_h$ be a family of tetrahedral meshes on a polygonal domain $\Omega \subset \mathbb{R}^d$ and $\mathbf{M}_h$ the mass matrix associated with respect to the corresponding finite element space of piecewise linear finite elements, i.e.

$$(\mathbf{M}_h)_{ij} = \int_\Omega \varphi_i^h \varphi_j^h \, dx$$

with $(\varphi_i^h)_i$ being the nodal basis functions of the finite element space on the mesh $\mathcal{T}_h$.

a) Assume that $\mathcal{T}_h$ is non-degenerate. Show that there are constants $c_1, c_2 > 0$ such that

$$c_1 \operatorname{diam}(T)^d \leq |\det a_T| \leq c_2 \operatorname{diam}(T)^d$$

holds for any tetrahedron $T \in \mathcal{T}_h$ and the linear part $a_T \in \mathbb{R}^{d \times d}$ of the affine linear map that maps the reference tetrahedron to T.

b) Assume that $\mathcal{T}_h$ is non-degenerate and that there is $N \in \mathbb{N}$ such that each node of the triangulation belongs to at most N tetrahedra. Show that there is a constant $C > 0$ such that the condition number $\operatorname{cond}_2(\mathbf{M}_h)$ fulfills:

$$\operatorname{cond}_2(\mathbf{M}_h) \leq C \left(\frac{h}{\min_{T \in \mathcal{T}_h} \operatorname{diam}(T)} \right)^d.$$

What can you conclude for quasi-uniform $\mathcal{T}_h$?