Sheet 7

Exercise 1

a) Monomials $\{x^\alpha \mid \alpha_i \leq m\}$ build a basis,
i.e. dimension of Q_m is m^d.
Nodal basis: There exist (Lagrange pol.)
$(\varphi_i)_i$ Polynomials of degree m, $i = 0, \dots, m$
s.t. $\varphi_i(z_j) = \delta_{ij}$

Nodal basis function w.r.t. $(z_{i_1}, \dots, z_{i_d})$:

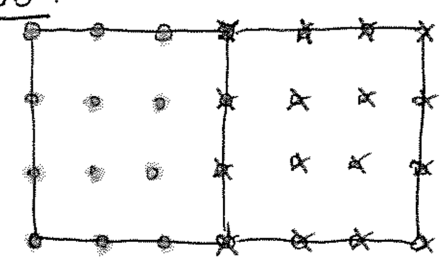
$(x_1, \dots, x_d) \mapsto \varphi_{i_1}(x_1) \cdot \ldots \cdot \varphi_{i_d}(x_d)$.

b) Apply Sheet 2, Ex. 3b: It suffices to show continuity of the global functions across the edges of the triangulation:

Affine multilinear transf. restricted to an edge/facet of the ref. cube are affine multilinear transformations. It follows that the restriction of a global FE function onto an edge/facet is composition of an $(d-1)$-dim. affine multilinear map with a function from Q_{m-1} on the $(d-1)$-dim. hypercube $[0,1]^{d-1}$. By induction this restriction is uniquely determined by its m^{d-1} nodal values...

$\longrightarrow$ continuity across edges/facets.

c) No.



Set all green nodes to zero, all red nodes to one and compute the derivatives across $\{x=1\}$.

d) Let $p \in \mathbb{Q}_3$ s.t. $f \cdot p = 0 \;\; \forall f \in \mathcal{N}$.
We have to show $p = 0$, from which follows that $\mathcal{N}$ generates $(\mathbb{Q}_3)'$. (Basis-property follows by reasons of dimensionality.)

$\frac{\partial p}{\partial x}$ is a polynomial of degree ≤ 3 in y
— " — ≤ 2 in x.

$y \mapsto \frac{\partial p}{\partial x}(0,y)$ is polynomial of degree ≤ 3 with zeros at $y=0$ and $y=1$ and zeros of the derivative at $y=0$ and $y=1$

$\Longrightarrow \quad \frac{\partial p}{\partial x}(0,\cdot) \equiv 0$

Similarly: $\frac{\partial p}{\partial x}(1,\cdot) \equiv 0$, $\frac{\partial p}{\partial y}(\cdot,0) \equiv 0$
$\frac{\partial p}{\partial y}(\cdot,1) \equiv 0$.

We conclude: ($\frac{\partial p}{\partial x}$ has degree ≤ 2 in x!)

$\frac{\partial p}{\partial x} = x(x-1) \cdot q_1(y)$ with $\deg q_1 \leq 3$

$\frac{\partial p}{\partial y} = y(y-1) \; q_2(x)$ with $\deg q_2 \leq 3$

Integration of $\frac{\partial p}{\partial x}$ yields:

$$p(x,y) = q_1(y)\left(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right) + q_3(y)$$

$$\deg q_3 \leq 3.$$

Differentiation w.r.t. y:

$$\frac{\partial p}{\partial y} = q_1'(y)\left(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right) + q_3'(y)$$

Apply results from above:

$$0 \equiv \frac{\partial p}{\partial y}(x,0) = q_1'(0)\left(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right) + q_3'(0)$$

$$0 \equiv \frac{\partial p}{\partial y}(x,1) = q_1'(1)\left(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right) + q_3'(1)$$

$$\Longrightarrow \quad q_1'(0) = q_1'(1) = 0$$
$$q_3'(0) = q_3'(1) = 0$$

furthermore:

$$0 = p(0,0) = q_3(0)$$
$$0 = p(1,0) = \tfrac{1}{6} q_1(0) + q_3(0)$$
$$0 = p(1,1) = 1/6\, q_1(1) + q_3(1)$$
$$0 = p(0,1) = q_3(1)$$

$$\Longrightarrow \quad q_3(0) = q_3(1) = 0 \quad \text{and} \quad q_1(0) = q_1(1) = 0$$

$$\xRightarrow{\deg q_1, \deg q_3 \leq 3} \quad q_1 \equiv q_3 \equiv 0$$

$$\Longrightarrow \quad p \equiv 0.$$

e) W.l.o.g. we check whether $\frac{\partial p}{\partial y}$, $\frac{\partial p}{\partial x}$ are uniquely determined on $\{y=0\}$ by the prescribed values on this edge.
(H^2-reg. follows from Ex. 36, Sheet 2).

- $\frac{\partial p}{\partial y}$ is polynomial of degree ≤ 2 in y
  — " — ≤ 3 in x

  $x \mapsto \frac{\partial p}{\partial y}(x,0)$ Pol. of degree ≤ 3 in x
  with prescribed values for $x = 0, 1$
  — " — of the derivative at $x = 0, 1$

  $\Longrightarrow$ $\frac{\partial p}{\partial y}$ uniquely determined

- $x \mapsto p(x,0)$ is Polynomial of degree ≤ 3
  with prescribed values at $x = 0, 1$ and
  — " — of the derivative at $x = 0, 1$

  $\Longrightarrow$ $x \mapsto p(x,0)$ uniquely determined

  $\Longrightarrow$ $x \mapsto \frac{\partial p}{\partial x}(x,0)$ uniquely determined.

Exercise 2

$$|u|_{W^{k,p}} = \begin{cases} \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^p \, dx \right)^{1/p} & p \in [1,\infty) \\ \sup_{|\alpha|=k} \|D^\alpha u\|_{L^\infty(\Omega)} & p = +\infty \end{cases}$$

Chain rule:

$$D^\alpha u_\lambda = D^\alpha (u \circ \lambda^{-1} id)$$
$$= \left((D^\alpha u) \circ \lambda^{-1} id \right) \cdot \lambda^{-|\alpha|}$$

hence:

$$\|D^\alpha u_\lambda\|_{L^\infty(\Omega_\lambda)} = \|D^\alpha u\|_{L^\infty(\Omega)} \cdot \lambda^{-|\alpha|}$$

and

$$\|D^\alpha u_\lambda\|^p_{L^p(\Omega_\lambda)} = \int_{\Omega_\lambda} |(D^\alpha u) \circ \lambda^{-1} id|^p \, \lambda^{-|\alpha| p}$$

$$= \lambda^{d-|\alpha|p} \int_{\Omega_\lambda} |D^\alpha u \circ \lambda^{-1} id|^p \, |\lambda^{-1}|^d$$

$$\overset{\text{transf. formula}}{=} \lambda^{d-|\alpha|p} \int_\Omega |D^\alpha u|^p \, dx = \lambda^{d-|\alpha|p} \|D^\alpha u\|^p_{L^p(\Omega)}$$

$$\Longrightarrow \quad |u_\lambda|_{W^{k,\infty}(\Omega_\lambda)} = \lambda^{-k} |u|_{W^{k,\infty}(\Omega)}$$

$$|u_\lambda|_{W^{k,p}(\Omega_\lambda)} = \lambda^{d/p - k} |u|_{W^{k,p}(\Omega)}$$

Exercise 3

a) The expressions are well-defined for $D^\alpha u \in L^1_{loc}(B)$, i.e. in particular if $u \in W^{|\alpha|,p}(\Omega)$ for all $\alpha \in A(\alpha_0)$, i.e. $u \in W^{|\alpha_0|-n,p}(\Omega)$.

$$Q^{A(\alpha_0)} u(x) \overset{def}{=} \int_B \sum_{\alpha \in A(\alpha_0)} \frac{1}{\alpha!} D^\alpha u(y) (x-y)^\alpha \phi(y) dy$$

$$\overset{\text{binomial formula}}{=} \sum_{\alpha \in A(\alpha_0)} \sum_{\beta \le \alpha} x^\beta \cdot \int_B \frac{(-1)^{|\alpha-\beta|}}{\alpha!} \binom{\alpha}{\beta} D^\alpha u(y)\, y^{\alpha-\beta} \phi(y) dy$$

$$\overset{\text{integr. by parts}}{=} \sum_{\beta \ge 0}' \left(\sum_{\substack{\alpha \in A(\alpha_0) \\ \beta \le \alpha}}' \int_B \underbrace{\frac{(-1)^{2|\alpha| - |\beta|}}{\alpha!} \binom{\alpha}{\beta} D_y^\alpha (y^{\alpha-\beta} \phi(y))}_{\in L^\infty} \cdot u(y) dy \right) \cdot x^\beta$$

which is even well-defined for $u \in L^1(B)$. Further $Q^{A(\alpha_0)} u$ is a Polynomial, because there are only finitely many multiindices β s.t. there is $\alpha \in A(\alpha_0)$ with $\beta \le \alpha$.

In case $A(\alpha_0) = A((m, \dots, m))$ this implies $\beta_i \le m-1 \;\; \forall i$ i.e.

$Q^{A((m,\dots,m))} u \in Q_{m-1}$.

b) For $u \in C^\infty(\Omega)$ we show:

$$D_x^\alpha \sum_{\beta < \alpha_0} \frac{1}{\beta!} D^\beta u(y) (x-y)^\beta$$

$$= \sum_{\beta < \alpha_0} \frac{1}{\beta!} D^\beta u(y) \, D_x^\alpha (x-y)^\beta$$

$$= \sum_{\alpha \le \beta < \alpha_0} \frac{1}{\beta!} D^\beta u(y) \frac{\beta!}{(\beta-\alpha)!} (x-y)^{\beta-\alpha}$$

$$\overset{\substack{\beta=\alpha+\eta \\ \eta \ge 0}}{=} \sum_{\substack{\eta \ge 0 \\ \alpha+\eta < \alpha_0}} \frac{1}{\cancel{(\alpha+\eta)!}} \frac{\cancel{(\alpha+\eta)!}}{\eta!} D^\eta (D^\alpha u(y)) (x-y)^\eta$$

$$= \sum_{\substack{\eta \ge 0 \\ \eta < \alpha_0 - \alpha}} \frac{1}{\eta!} D^\eta (D^\alpha u)(y) (x-y)^\eta$$

which is the claim in case of "classical" Taylor-polynomials.

For aveaged Taylor-Polynomials the claim follows by integrating this and using a density argument.

Exercise 4

a) Note that for any $T \in \mathcal{T}_h$:

$$\mathrm{vol}(T) = \int_T 1\,dx = \int_{T_{ref}} |\det a_T|\,dx = \mathrm{vol}(T_{ref}) \cdot |\det a_T|$$

Obviously:

$$\mathrm{vol}(T) \leq c_d \cdot (\mathrm{diam}\,T)^d$$

and further for any inner ball $B_T \subset T$:

$$\underbrace{\tilde{c}_d\,(\mathrm{diam}\,B_T)^d}_{\overset{\boxed{\text{non-deg. } \mathcal{T}_h}}{\geq}\ \tilde{c}_d\,\rho^d\,(\mathrm{diam}\,T)^d} = \mathrm{vol}(B_T) \leq \mathrm{vol}(T)$$

$$\overset{\text{together}}{\Longrightarrow} \underbrace{\frac{\tilde{c}_d\,\rho^d}{\mathrm{vol}(T_{ref})}}_{=:\,c_1} \cdot (\mathrm{diam}\,T)^d \leq |\det a_T| \leq \underbrace{\frac{c_d}{\mathrm{vol}(T_{ref})}}_{=:\,c_2} \cdot (\mathrm{diam}\,T)^d$$

with c_1, c_2 depending on dimension and parameter $\rho > 0$ of non-degeneracy.

b) On the reference element there is a mass matrix M_{ref} that is symm. pos. def., in particular

(*) $$\lambda \|v\|_2^2 \leq v^T M v \leq \Lambda \|v\|_2^2 \quad \forall v \in \mathbb{R}^{dim}.$$

Given a global vector x we denote by x_T the local vector on triangle $T \in \mathcal{T}_h$.

It follows:

$$x^T M_h x = \sum_{T \in \mathcal{T}_h} x_T^T M_{h,T} x_T$$

$$\overset{\text{Sheet 5, ex. 2}}{=} \sum_{T \in \mathcal{T}_h} |\det a_T| \, x_T^T M_{ref} \, x_T$$

$$\geq \sum_{T \in \mathcal{T}_h} |\det a_T| \, \lambda \|x_T\|_2^2$$

$$\geq \min_{T \in \mathcal{T}_h} \underbrace{|\det a_T|}_{\geq c_1 (\operatorname{diam} T)^d} \cdot \lambda \cdot \underbrace{\sum_{T \in \mathcal{T}_h} \|x_T\|_2^2}_{\geq \|x\|_2^2}$$

$$\geq c_1 \cdot \lambda \cdot \min_{T \in \mathcal{T}_h} (\operatorname{diam} T)^d \cdot \|x\|_2^2$$

and $$x^T M_h x = \sum_{T \in \mathcal{T}_h} x_T^T M_{h,T} x_T$$

$$\leq \sum_{T \in \mathcal{T}_h} c_2 (\operatorname{diam} T)^d \Lambda \|x_T\|_2^2$$

$$\leq c_3 h^d \Lambda \cdot \underbrace{\sum_{T \in \mathcal{T}_h} \|x_T\|_2^2}_{\text{in this sum, each entry of } x \text{ is contained at most } N \text{ times...}} \leq c_3 h^d \Lambda N \|x\|_2^2$$

$\operatorname{diam} T \leq h \cdot \operatorname{diam} \Omega$

Rayleigh quotients:

$$\lambda_{min}(M_h) \geqslant c_1 \lambda \min_{T \in \mathcal{T}_h} (\operatorname{diam} T)^d$$

$$\lambda_{max}(M_h) \leqslant c_3 \Lambda N \cdot h^d$$

$$\Longrightarrow \quad \operatorname{cond}_2(M_h) = \frac{\lambda_{max}(M_h)}{\lambda_{min}(M_h)} \leqslant C \cdot \left(\frac{h}{\min_{T \in \mathcal{T}_h} (\operatorname{diam} T)} \right)^d.$$

Quasi-uniform triangulation:

$$\min_{T \in \mathcal{T}_h} (\operatorname{diam} T) \geq \min_{T \in \mathcal{T}_h} (\operatorname{diam} B_T) \geq c \cdot h$$

$\Longrightarrow$ $\operatorname{cond}_2(M_h)$ <u>bounded</u>!