Scientific Computing I
(Wissenschaftliches Rechnen I)

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8^{th} exercise sheet

Submission on December 12, before the lecture

Exercise 1.
(2 + 3 = 5 points)

"Three appetizers on H^2-regularity"

a) Let $\Omega = [0,1]$ and $f \in L^2(\Omega)$. Prove that the weak solution $u \in H_0^1(\Omega)$ of the Laplace equation with homogeneous Dirichlet boundary conditions

$$-u'' = f \quad \text{on } \Omega, \qquad u(0) = u(1) = 0,$$

fulfills $u \in H^2(\Omega)$.

b) Let $\Omega \subset \mathbb{R}^d$ be a convex domain. It is a well-known fact that the Laplace equation with homogeneous Dirichlet boundary conditions

$$-\Delta u = f \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

admits a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $|u|_{H^2(\Omega)} \leq c_\Omega \|f\|_{L^2(\Omega)}$ with some constant $c_\Omega > 0$. In particular, the map

$$-\Delta\colon\ H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega), \tag{1}$$

is an isomorphism.

Now, let $a \in C^{1,0}(\overline{\Omega})$ such that $0 < a_0 \leq a(x) \leq a_1 < +\infty$ holds for all $x \in \overline{\Omega}$. Show that the PDE

$$-\operatorname{div}(a\nabla u) = f \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

admits a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ for every $f \in L^2(\Omega)$ and estimate its norm in terms of f and a.

Hint: Use the theory of Fredholm operators. To do so utilize the fact that $-\operatorname{div}(a\nabla u) = -a\Delta u - \nabla a \nabla u$ for $u \in H^2(\Omega)$ and the compactness of the embedding $H^1(\Omega) \hookrightarrow_c L^2(\Omega)$.

c) (2 bonus points) Let $\Omega = [-1, 1]$ and $K = [-1/2, 1/2] \subset \Omega$. We define $a(x) = 1 + \mathbf{1}_K(x)$ and consider the weak solution $u \in H_0^1(\Omega)$ to the equation

$$-\operatorname{div}(a\nabla u) = 1 \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Show that $u \notin H^2(\hat{\Omega})$ for an open neighbourhood $\hat{\Omega}$ of K.

Remark: This shows that even interior regularity might fail in case of non-Lipschitz coefficients. Utilizing polar coordinates and radially symmetric functions this example can be extended to higher dimensions as well.

Exercise 2. (2 + 2 + 3 = 7 points)

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a convex domain. Given a regular Borel measure $\mu \in C(\overline{\Omega})^*$ we consider the equation

$$-\Delta u = \mu \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

that has to be understood in the following so-called *very weak* sense: Find $u \in L^2(\Omega)$ such that

$$-\int_\Omega u \Delta v \, \mathrm{d}x = \int_\Omega v \, \mathrm{d}\mu \qquad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3}$$

This very weak formulation is necessary, because $\mu \notin H^{-1}(\Omega)$, in general, and hence we cannot apply Lax-Milgram.

a) Show that there exists a unique solution $u \in L^2(\Omega)$ of (2) in the very weak sense (3) and estimate the norm of u in terms of μ.

Hint: As in exercise 1b you may use the H^2-regularity of the Laplace problem. Consider the adjoint of the inverse of the map (1) and use the embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$.

Now, let Ω be additionally polygonal and equipped with a family of non-degenerate quasi-uniform triangulations $\mathcal{T}_h$. Let $V_h \subset H_0^1(\Omega)$ be the space of piecewise linear finite elements on this triangulation with homogeneous Dirichlet boundary condition.

b) Let $z \in H^2(\Omega) \cap H_0^1(\Omega)$ and let $z_h \in V_h$ be the solution of the discrete variational problem

$$\int_\Omega \nabla v_h \nabla z_h \, \mathrm{d}x = \int_\Omega \nabla v_h \nabla z \, \mathrm{d}x \qquad \forall v_h \in V_h.$$

Show that there is $c > 0$ independent of h and z such that

$$\|z - z_h\|_{L^\infty(\Omega)} \leq c h^{2-\frac{d}{2}} \|z\|_{H^2(\Omega)}.$$

Hint: Exercise 4 from sheet 8.

c) Finally, conclude that the solution $u_h \in V_h$ of the discrete variational problem

$$\int_\Omega \nabla u_h \nabla v_h \, \mathrm{d}x = \int_\Omega v_h \, \mathrm{d}\mu \qquad \forall v_h \in V_h$$

satisfies the error estimate

$$\|u - u_h\|_{L^2(\Omega)} \leq c h^{2-\frac{d}{2}} \|\mu\|_{C(\overline{\Omega})^*}$$

with some $c > 0$ independent of μ and h.

Hint: Apply the Aubin-Nietzsche trick. Try to estimate $\int_\Omega (u - u_h) p \, \mathrm{d}x$ for an arbitrary $p \in L^2(\Omega)$.

Exercise 3. (2 + 2 + 2 = 6 points)

Let $\Omega \subset \mathbb{R}^d$ be a convex domain with Lipschitz boundary. We consider the subspace

$$V := \left\{ u \in H^1(\Omega) : \int_\Omega u \, \mathrm{d}x = 0 \right\} \subset H^1(\Omega).$$

equipped with the H^1-norm.

a) Show that the bilinear form $a(u, v) := \int_\Omega \nabla u \nabla v \, \mathrm{d}x$ is coercive on V (w.r.t. H^1-norm).

b) Show that for any $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, there exists a unique $u \in V$ that fulfills

$$a(u, v) \stackrel{!}{=} F(v) \qquad \forall v \in V,$$

with $F(v) := \int_\Omega f v \, \mathrm{d}x + \int_{\partial\Omega} g v \, \mathrm{d}s$. Estimate the norm of u in terms of f

c) We call $u \in H^1(\Omega)$ a weak solution of the so-called *pure Neumann problem*

$$-\Delta u = f \quad \text{on } \Omega, \qquad \partial_n u = g \quad \text{on } \partial\Omega, \tag{4}$$

if it holds

$$a(u, v) = F(v) \qquad \forall v \in H^1(\Omega).$$

[Make sure that you understand why this is the correct weak formulation!]

Show that the $u \in V$ from b) is a weak solution of (4) if and only if the following compatibility condition is fulfilled:

$$\int_\Omega f \, \mathrm{d}x + \int_{\partial\Omega} g \, \mathrm{d}s = 0.$$

Exercise 4. (2 points)

Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary and $A \in L^\infty(\Omega)^{d \times d}$, $b \in L^\infty(\Omega)^d$, $c \in L^\infty(\Omega)$. We consider the linear operator $T : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ defined (in weak form) by

$$\langle Tu, v \rangle_{H^{-1}, H_0^1} := \int_\Omega \left((A\nabla u)^T \nabla v + b^T \nabla u v + c u v \right) \, \mathrm{d}x.$$

The strong formulation of this operator reads as

$$Tu = -\operatorname{div}(A\nabla u) + b\nabla u + cu,$$

which is of course completely formal (unless the entries of A are sufficient regular) and therefore has to be understood in the weak sense in general. When applying duality arguments, e.g. the Aubin-Nietzsche trick, one has to consider a PDE associated with the adjoint operator $T^* : H_0^1(\Omega) = (H^{-1}(\Omega))^* \longrightarrow H^{-1}(\Omega) = (H_0^1(\Omega))^*$.

Find a strong formulation for this adjoint operator T^* (i.e. in terms of ∇, div etc.) similiar to the strong form given for T above.