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Sheet 9

Exercise 1

a) Weak formulation:

$$\int_0^1 u' \varphi' = \int_0^1 f \varphi \qquad \forall \varphi \in H_0^1(\Omega)$$

Due to $C_0^\infty(\Omega) \subset H_0^1(\Omega)$ this implies

$$-\int_0^1 u' \cdot \varphi' = -\int_0^1 f \varphi \qquad \forall \varphi \in C_0^\infty(\Omega),$$

i.e. $-f \in L^2(\Omega)$ is the weak derivative of u', i.e. $u \in H^2(\Omega)$.

Remark: This only works in 1D, because u' directly appears in the variational formulation in the given way.

b) Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$. It holds

$$\begin{aligned} -\operatorname{div}(a \nabla u) &= -a \cdot \Delta u - \nabla a \cdot \nabla u \\ &= a \cdot (-\Delta u) - \nabla a \cdot \nabla u \end{aligned}$$

Multiplication with a is an isomorphism on $L^2(\Omega)$ with inverse a^{-1}. Hence,

$$u \longmapsto a \cdot (-\Delta u)$$

is an isomorphism $H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$.

2.

Now, note that

$$u \longmapsto \nabla u$$

is bounded linear $H^2(\Omega) \cap H^1_0(\Omega) \to H^1(\Omega)^d$

and compact $H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)^d$

(due to compactness of embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$).

It follows that the first-order term

$$u \longmapsto -\nabla a \cdot \nabla u$$

$$H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$$

is linear and compact.

Therefore, the sum

$$u \longmapsto a \cdot (-\Delta u) - \nabla a \cdot \nabla u$$

is Fredholm of index 0. The kernel of the operator is trivial (Lax-Milgram, on $H^1_0(\Omega)$), hence it is an isomorphism again.

We know:

$$|u|_{H^1} \overset{\text{Lax-Milgram}}{\leq} \frac{1}{a_0} \cdot \|f\|_{H^{-1}} \leq \frac{c}{a_0} \|f\|_{L^2}$$

$$\|u\|_{H^1} \overset{\text{Poincaré}}{\leq} c\,|u|_{H^1} \leq \frac{c}{a_0} \|f\|_{L^2}$$

Now, rewrite the equation as

$$-\Delta u = \frac{1}{a}\left(f + \nabla a \cdot \nabla u\right), \quad u \in H^1_0(\Omega).$$

③ Estimate the rhs:

$$\left\| \frac{1}{a} (f + \nabla a \cdot \nabla u) \right\|_{L^2(\Omega)} \leq \frac{1}{a_0} \left(\|f\|_{L^2} + \|\nabla a \nabla u\|_{L^2} \right)$$

$$\leq \frac{1}{a_0} \left(\|f\|_{L^2} + |a|_{C^{1,0}} \underbrace{\|\nabla u\|_{L^2}}_{=|u|_{H^1}} \right)$$

$$\leq \frac{1}{a_0} \left(\|f\|_{L^2} + c \cdot \frac{|a|_{C^{1,0}}}{a_0} \|f\|_{L^2} \right)$$

$$= \frac{1}{a_0} \left(1 + \frac{c|a|_{C^{1,0}}}{a_0} \right) \|f\|_{L^2}$$

It follows from (1) being an isomorphism:

$$|u|_{H^2} \leq C_\Omega \left\| \frac{1}{a} (f + \nabla a \nabla u) \right\|_{L^2(\Omega)}$$

$$\leq c \cdot \frac{1}{a_0} \left(1 + \frac{|a|_{C^{1,0}}}{a_0} \right) \|f\|_{L^2}.$$

Alltogether:

$$\|u\|_{H^2 \cap H^1_0} \lesssim \frac{1}{a_0} \left(1 + \frac{|a|_{C^{1,0}}}{a_0} \right) \|f\|_{L^2}.$$

Bonus exercise 1c (Solution sketch)

- We "guess" a solution: For symmetry reasons we know $u'(0) = 0$, i.e. for $v(x) = a(x) \cdot \frac{\partial u}{\partial x}$ we know:

$$- \frac{\partial}{\partial x} v(x) \equiv 1, \qquad v(0) = 0$$

$$\Rightarrow \quad v(x) = -x$$

It follows:

$$\frac{\partial u}{\partial x} = -\frac{x}{a(x)} = \begin{cases} -\frac{x}{2} & x \leq \frac{1}{2} \\ -x & x > \frac{1}{2} \end{cases}$$

$$\Rightarrow \quad u(x) = \begin{cases} 7/16 - \frac{x^2}{4} & |x| \leq \frac{1}{2} \\ \frac{1}{2} - \frac{x^2}{2} & |x| > \frac{1}{2} \end{cases}$$

- Verify that this guess is correct by plugging in into weak formulation...

- $u \notin H^2((-\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon))$: If $u' \in H^1((-\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon))$, then $u' \in C((-\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon))$ by Sobolev embedding, which is clearly not the case.