Exercise 5

a) $F = (F_1, \dots, F_d) \in L^2(\Omega)^d$.

Define $$(\operatorname{div} F)(\varphi) := -\int_\Omega F \cdot \nabla\varphi \, dx$$
$$= -\int_\Omega \sum_{i=1}^d F_i \partial_i \varphi \, dx, \quad \varphi \in H_0^1(\Omega).$$

$$|\operatorname{div}(F)\cdot\varphi| \leq \sum_{i=1}^d \int |F_i \partial_i \varphi| \, dx$$
$$\leq \sum_{i=1}^d \|F_i\|_{L^2} \|\partial_i\varphi\|_{L^2} \leq \Big(\sum_i \|F_i\|_{L^2}^2\Big)^{1/2} \Big(\sum_i \|\partial_i\varphi\|_{L^2}^2\Big)^{1/2}$$
$$= \|F\|_{L^2(\Omega)^d} \|\varphi\|_{H_0^1(\Omega)},$$

i.e. $\operatorname{div}: L^2(\Omega)^d \to (H_0^1(\Omega))^* = H^{-1}(\Omega)$ is bounded (and obviously linear...)

For $F \in C^1(\Omega)^d$ and $\varphi \in C_0^\infty(\Omega)$ it holds:
$$\int_\Omega \operatorname{div} F \cdot \varphi = \int_\Omega \operatorname{div}(\varphi F) - \int_\Omega F \cdot \nabla\varphi$$
$$\overset{\text{Gauss}}{=} \int_{\partial\Omega} \underbrace{\varphi}_{=0} \cdot (n \cdot F) \, ds - \int_\Omega F \cdot \nabla\varphi$$
$$= -\int F \cdot \nabla\varphi$$

i.e. the definition above coincides with the classical divergence (strong form) for smooth F.

b) Similarly as in a): Let $F \in (L^p(\Omega))^d$ and $\varphi \in W_0^{1,p'}(\Omega)$. It follows:

$$|\operatorname{div} F \cdot \varphi| \leq \sum_{i=1}^d \int |F_i \, \partial_i \varphi| \, dx \leq \sum_{i=1}^d \|F_i\|_{L^p} \|\partial_i \varphi\|_{L^{p'}}$$

$$\leq \left(\sum_i \|F_i\|_{L^p}^p \right)^{1/p} \left(\sum_i \|\partial_i \varphi\|_{L^{p'}}^{p'} \right)^{1/p'}$$

$$= \|F\|_{L^p(\Omega)^d} \, \|\varphi\|_{W_0^{1,p'}}$$

$\Longrightarrow \operatorname{div} F \in W_0^{1,p'}(\Omega)^* = W^{-1,p}(\Omega)$ and $\|\operatorname{div} F\|_{W^{-1,p}} \leq \|F\|_{L^p(\Omega)^d}$

$\Longrightarrow \operatorname{div}: L^p(\Omega)^d \longrightarrow W^{-1,p}(\Omega)$ bounded linear

c) i) Let $\varphi \in W_0^{1,p'}(\Omega)$ and $u \in W_0^{1,p}(\Omega)$. It holds:

$$\underbrace{\left(-\operatorname{div}(a \nabla u)\right)}_{\in W^{-1,p}(\Omega)}(\varphi) := \int_\Omega a \nabla u \nabla \varphi$$

and

$$\left| \int_\Omega a \nabla u \cdot \nabla \varphi \right| \leq \|a\|_{L^\infty} \|u\|_{W_0^{1,p}} \|\varphi\|_{W_0^{1,p'}}$$

$$\Longrightarrow \quad \|-\operatorname{div}(a \nabla u)\|_{W^{-1,p}} \leq \|a\|_{L^\infty} \|u\|_{W_0^{1,p}}$$

$$\Longrightarrow \quad \|-\operatorname{div}(a \nabla \cdot)\|_{\mathcal{L}(W_0^{1,p}, W^{-1,p})} \leq \|a\|_{L^\infty}$$

ii) $\delta_{x_0} \in \mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})^*$ set of regular Borel measures on $\overline{\Omega}$.

In order to have $\mathcal{M}(\overline{\Omega}) \hookrightarrow W^{-1,p}(\Omega)$ it suffices to have $W_0^{1,p'}(\Omega) \hookrightarrow C(\overline{\Omega})$

Sobolev embedding:

$$W_0^{1,p'}(\Omega) \hookrightarrow C(\overline{\Omega}) \quad \text{if} \quad 1 - \frac{d}{p'} > 0$$

$$\iff 1 - d\left(1 - \frac{1}{p}\right) > 0$$

$$\iff 1 - d + \frac{d}{p} > 0$$

$$\iff \frac{d}{p} > d - 1$$

$$\iff p < \frac{d}{d-1}$$

$\Longrightarrow$ The weak formulation of $-\operatorname{div}(a\nabla u) = \delta_{x_0}$ is well-posed in $W_0^{1,p}(\Omega)$ for $p \in \left(1, \frac{d}{d-1}\right)$.