Scientific Computing I
(Wissenschaftliches Rechnen I)

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Excercise sheet

for the tutorials of the week October 14-18

Excercise 1. (L^p-spaces)
Recall that on a Hilbert space H the parallelogramm identity

$$\|u+v\|_H^2 + \|u-v\|_H^2 = 2\left(\|u\|_H^2 + \|v\|_H^2\right) \qquad \forall u, v \in H$$

holds.

a) Conclude that $L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, is a Hilbert space if and only if $p = 2$, i.e. show that for $p \in [1, \infty] \setminus \{2\}$ the L^p-norm cannot be induced by a scalar product.

b) It is well known that for $1 \leq p \leq q \leq \infty$ there is a constant $C_{qp} > 0$ such that $\|\varphi\|_{L^p} \leq C_{pq}\|\varphi\|_{L^q}$ for all $\varphi \in L^q(\mathbb{R}^n)$, i.e. $L^q(\mathbb{R}^n)$ embeds continuously into $L^p(\mathbb{R}^n)$.

Now, find a bound for $\|\varphi\|_{L^q}$ in terms of $\|\varphi\|_{L^p}$ and M for those $\varphi \in L^\infty(\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty} \leq M$.[1]

Excercise 2. (H_0^1-product)
On the space of continuously differentiable function with homogenous boundary conditions $C_0^1([0,1]) := \{u \in C^1([0,1]) : u(0) = u(1) = 0\}$ we define the so called H_0^1-inner product

$$\langle u, v\rangle_{H_0^1} := \int_0^1 u'(t)v'(t)dt.$$

a) Verify that $\langle\cdot,\cdot\rangle_{H_0^1}$ indeed defines a scalar product, i.e. a symmetric, positive definite bilinearform on $C_0^1([0,1])$.

b) Show that $(C_0^1([0,1]), \langle\cdot,\cdot\rangle_{H_0^1})$ is not a Hilbert space nevertheless.

<u>Hint:</u> It may help to show that $\|\cdot\|_{C^0} \leq \|\cdot\|_{H_0^1}$, i.e. a Cauchy sequence w.r.t. the H_0^1-norm is in particular a Cauchy sequence in $C^0([0,1])$.

[1] Note the following useful consequence: Any uniformly bounded sequence of functions that converges in L^p for *some* p, already converges in L^p for *all* p!

Excercise 3. (Weak and strong convergence)

a) Let H be a Hilbert space and $(u_n)_n \subset H$, $(v_n)_n \subset H$ sequences such that $u_n \longrightarrow u$ (strongly) and $v_n \rightharpoonup v$ (weakly) as $n \longrightarrow \infty$. Show that it follows $\langle u_n, v_n \rangle_H \longrightarrow \langle u, v \rangle_H$ as $n \longrightarrow \infty$.

b) Does the statement of excercise 2a) stay true if $(u_n)_n$ converges only weakly to u?

c) Let $(\varphi_n)_n \subset L^p(\mathbb{R}^n)$, $p \in (1, \infty)$, such that $\varphi_n \geq 0$ almost everywhere on $\mathbb{R}^n$ for every $n \in \mathbb{N}$. Show that $\varphi_n \rightharpoonup \varphi$ as $n \longrightarrow \infty$ implies that $\varphi \geq 0$ almost everywhere on $\mathbb{R}^n$.

Excercise 4. (Spectral theory in infinite dimension)

a) Let $w \in L^\infty([0,1])$. Define $T_w : L^2([0,1]) \longrightarrow L^2([0,1])$ by $(T_w\varphi)(t) := w(t)\varphi(t)$ for $\varphi \in L^2([0,1])$. Show that T_w is a continuous linear operator and determine its operator norm.

b) Determine the spectrum of T_w for $w \in C([0,1])$. Which elements of the spectrum are eigenvalues?

<u>Recall:</u> If X is a Banach space and $T \in \mathcal{L}(X)$ a bounded linear operator on X, we define the spectrum $\sigma(T)$ of T as

$$\sigma(T) = \{\lambda \in \mathbb{C} : \quad \lambda I - T \text{ is } \underline{\text{not}} \text{ an isomorphism from } X \text{ to } X\},$$

and $\lambda \in \sigma(T)$ is called eigenvalue of T, if $\ker \lambda I - T \neq \{0\}$.

c) Let $T : H \longrightarrow H$ be a continuous linear operator on a Hilbert space H such that

$$\langle Tu, u \rangle_H > 0 \qquad \forall u \in H \setminus \{0\},$$

i.e. T is positive definite. Is it always true that there exists a $c > 0$ such that $\langle Tu, u \rangle_H \geq c\|u\|_H^2$ holds for all $u \in H$, i.e. T is coercive?