Exercise 4

a) We write $r = |x|_2$. and use polar coordinates.

$$\int_{B_1(0)} \left(\log \log \frac{2}{|x|_2}\right)^p dx = \int_0^1 \int_{[0,2\pi]^{d-1}} \underbrace{\left(\log \log \frac{2}{r}\right)^p r^{d-1}}_{\text{bounded for } d \geq 2} d\varphi \prod_{j=1}^{d-2} (\sin \vartheta_j)^j d\vartheta_j \, dr$$

$$< \infty$$

(For $d=1$ also integrable)

Derivative:

$$\nabla u(x) = \frac{1}{\log \frac{2}{|x|_2}} \cdot \frac{|x|_2}{2} \cdot \left(-\frac{2}{|x|_2^2}\right) \cdot \frac{x}{2|x|_2}$$

i.e. $$|\nabla u(x)|_2 = \frac{1}{2\,|x|_2 \log \frac{2}{|x|_2}}$$

$$\int_{B_1(0)} |\nabla u(x)|_2^p \, dx = \int_0^1 \int_{[0,2\pi]^{d-1}} \left|\frac{1}{2r \log \frac{2}{r}}\right|^p \cdot r^{d-1} d\varphi \prod_{j=1}^{d-2} (\sin \vartheta_j)^j d\vartheta_j \, dr$$

$$= \text{const.} \int_0^1 \frac{1}{|\log \frac{2}{r}|^p} \cdot r^{d-1-p} \, dr$$

$< +\infty$ if and only if $d-1-p > -1$,

i.e. $d > p$

$$\Longrightarrow \quad u \in W^{1,p}(B_1(0)) \iff p < d$$

b) By definition of Sobolev spaces:

$(u_n) \in C^\infty(\Omega)$ s.t. $u_n \to u$ in L^p

$\nabla u_n \to \nabla u$ in $(L^p)^d$

$(v_n) \in C^\infty(\Omega)$ s.t. $v_n \to v$ in L^q

$\nabla v_n \to \nabla v$ in $(L^q)^d$

Let $j \in \{1, -, d\}$. It follows: (triangle inequality + Hölder ✓)

① $u_n v_n \longrightarrow uv$ in L^r

② $D_j(u_n v_n) = D_j u_n \cdot v_n + D_j v_n \cdot u_n \longrightarrow D_j u \cdot v + D_j v \cdot u$ in L^r

Fix $\varphi \in C_0^\infty(\Omega)$. It holds:

$$\int_\Omega uv\, D_j\varphi\, dx \overset{①}{=} \lim_{n\to\infty} \int_\Omega u_n v_n\, D_j\varphi\, dx$$

$$\overset{u_n, v_n \in C^\infty}{=} \lim_{n\to\infty} \left(- \int_\Omega D_j(u_n v_n)\, \varphi\, dx \right)$$

$$= \lim_{n\to\infty} \left(- \int \left(D_j u_n v_n + D_j v_n u_n \right) \varphi\, dx \right)$$

$$\overset{②}{=} - \int_\Omega \left(D_j u \cdot v + D_j v \cdot u \right) \varphi\, dx$$

$\Longrightarrow$ $D_j(uv) = D_j u \cdot v + D_j v \cdot u$ (weak sense)

[Regarding ①: $\|u_n v_n - uv\|_{L^r} = \|u_n(v_n - v) + (u_n - u)v\|_{L^r}$

$\leq \|u_n (v_n - v)\|_{L^r} + \|(u_n - u)v\|_{L^r}$

$\leq \|u_n\|_{L^p} \|v_n - v\|_{L^q} + \|u_n - u\|_{L^p} \|v\|_{L^q}$]

Exercise 5

a) Find $u \in H^1(\Omega)$ s.t.

$$\int_\Omega \nabla u \nabla v \, dx + \frac{1}{\varepsilon} \int_{\partial\Omega} u \cdot v \, ds = \frac{1}{\varepsilon} \int_{\partial\Omega} f v \, ds \quad \forall v \in H^1(\Omega).$$

Boundedness of the bilinear form: clear (but constant grows with $\frac{1}{\varepsilon}$)

Coercivity:

$$\int_\Omega |\nabla u|^2 dx + \frac{1}{\varepsilon} \int_{\partial\Omega} u^2 ds \overset{\frac{1}{\varepsilon} > 1,\ \text{Poincaré}}{\geq} c_\Omega \|u\|_{H^1}^2 \quad \forall u \in H^1(\Omega)$$

RHS is bounded linear form:

$$\left| \frac{1}{\varepsilon} \int_{\partial\Omega} f v \, ds \right| \leq \frac{1}{\varepsilon} \int_{\partial\Omega} |f v| ds \leq \frac{1}{\varepsilon} \|f\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)}$$

$$\overset{\text{trace Thm.}}{\leq} \frac{c}{\varepsilon} \cdot \|f\|_{L^2(\partial\Omega)} \|v\|_{H^1(\Omega)}$$

Lax-Milgram: $\exists$ unique solution $u \in H^1(\Omega)$

s.t. $$\|u\|_{H^1(\Omega)} \leq \frac{1}{c_\Omega} \left\| \frac{1}{\varepsilon} \cdot \int_{\partial\Omega} f \cdot ds \right\|_{H^1(\Omega)^*}$$

$$\leq \frac{c}{\varepsilon} \|f\|_{L^2(\partial\Omega)}$$

b) Due to $\varphi \in H^2(\Omega)$ we have $\partial_n \varphi \in L^2(\partial\Omega)$.

φ is the unique solution to $\begin{cases} -\Delta u = 0 \\ \varepsilon \cdot \partial_n u + u = f + \varepsilon \partial_n \varphi, \end{cases}$

i.e. we conclude with a):

$$\|u - \varphi\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon} \| f - (f + \varepsilon \partial_n \varphi) \|_{L^2(\partial\Omega)}$$

$$= c \cdot \|\partial_n \varphi\|_{L^2(\partial\Omega)} < \infty$$

independent of ε.