Scientific Computing I

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Exercise sheet 2.

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Exercise 4. (Tensor divergence)

(6 Points)

Let $\Omega \subset \mathbb{R}^d$ be an open and bounded subset with smooth boundary $\partial\Omega$ and outer normal $\vec{n}$. Let $A : \Omega \to \mathbb{R}^{d\times d}$ be a second order tensor. The tensor divergence is a vector field, defined as

$$\operatorname{div} A = (\operatorname{div} A)_{i=1}^d = \left(\sum_{j=1}^d \partial_j A_{ij} \right)_{i=1}^d . \tag{1}$$

a) Use the Gauss theorem

$$\forall f : \Omega \to \mathbb{R}, \forall i = 1, \ldots, d : \quad \int_\Omega \partial_i f \, dx = \int_{\partial\Omega} \vec{n}_i f \, ds \tag{2}$$

to show the divergence theorem

$$\forall u : \Omega \to \mathbb{R}^d : \quad \int_\Omega \operatorname{div} u \, dx = \int_{\partial\Omega} \vec{n} \cdot u \, ds. \tag{3}$$

b) Using (3), you already showed the Green's identity

$$\int_\Omega \nabla f \cdot \nabla g \, dx = -\int_\Omega f \Delta g \, dx + \int_{\partial\Omega} \vec{n} \cdot f \nabla g \, ds \tag{4}$$

for all $f, g : \Omega \to \mathbb{R}$. This equation is fundamental to the discretization of Poisson's equation, as you will learn soon in the lecture. To discretize the Stokes equation or linear elasticity, one needs to go one step further. Show that for any $v : \Omega \to \mathbb{R}^d$ the divergence of Av can be related to the tensor divergence of A^T via

$$\operatorname{div}(Av) = v \cdot \operatorname{div} A^T + A : (\nabla v)^T, \tag{5}$$

with the Frobenius scalar product of second order tensors

$$\forall B, C : \Omega \to \mathbb{R}^{d\times d} : \quad B : C = \sum_{i,j=1}^d B_{ij} C_{ij}. \tag{6}$$

c) Finally, derive the relation

$$\forall v : \Omega \to \mathbb{R}^d : \quad \int_\Omega A : \nabla v \, dx = -\int_\Omega v \cdot \operatorname{div} A \, dx + \int_{\partial\Omega} \vec{n} \cdot Av \, ds. \tag{7}$$

d) Let $u, v : \Omega \to \mathbb{R}^d$ be vector fields. Show that

$$\int_\Omega \frac{1}{2}\left[\nabla u + (\nabla u)^T\right] : \nabla v \, dx = \int_\Omega \frac{1}{2}\left[\nabla u + (\nabla u)^T\right] : \frac{1}{2}\left[\nabla v + (\nabla v)^T\right] dx. \tag{8}$$

Exercise 5. (Material derivative)

(6 Points)

a) Let v be a constant velocity field. Consider the transformation

$$\tilde{x} = x - vt \qquad \text{and} \qquad \tilde{t} = t.$$

Show that under this change of coordinates the time derivative is given by the material derivative, i.e.,

$$Du(x,t) = \left(\frac{\partial}{\partial t} + v \cdot \nabla\right) u(x,t) = \frac{\partial}{\partial \tilde{t}} \tilde{u}(\tilde{x}, \tilde{t}).$$

b) Consider the rotation of a body about an axis in three dimensions. The rotation is described by the angular veloctiy vector ω which is directed along the rotation axis and the magnitude of the vector represents the angular speed, the rate at which the body rotates.

The tangential velocity at a point x is given by

$$v = \omega \times x.$$

- Show that v is divergence free, i.e.,

$$\operatorname{div}(v) = 0.$$

- Now consider the rotation around the z-axis and write ω as $\omega = |\omega| e_z$ where e_z is the unit vector in z direction.
  In this case the rotation of the body can be described using the rotation matrix

$$R(t) = \begin{bmatrix} \cos(|\omega|t) & -\sin(|\omega|t) & 0 \\ \sin(|\omega|t) & \cos(|\omega|t) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

  Show that if we consider the following transformation of coordinates

$$\tilde{x} = R^T(t)x \qquad \text{and} \qquad \tilde{t} = t,$$

  then again

$$\frac{\partial}{\partial \tilde{t}} \tilde{u}(\tilde{x}, \tilde{t}) = Du(x,t).$$

Exercise 6. (Fundamental solution of Poisson equation)

(0 Points)

a) Let $\varphi \in C^2(\mathbb{R}_{>0})$ and define $u \colon \mathbb{R}^d \to \mathbb{R}$, $u(x) := \varphi(|x|)$. Compute $-\Delta u(x)$.

b) (Fundamental solution.) Define $\Phi \colon \mathbb{R}^d \to \mathbb{R} \setminus \{0\}$,

$$\Phi(x) := \begin{cases} |x|^{2-d} & \text{for } d \geq 3, \\ \ln |x| & \text{for } d = 2. \end{cases}$$

Show $-\Delta\Phi(x) = 0$ for $x \neq 0$.

c) (Representation formula.) Let $\Omega \subset \mathbb{R}^3$ be a domain such that the divergence theorem holds and let $u \in C^2(\bar{\Omega})$ be a harmonic function. Show

$$u(x_0) = \frac{1}{4\pi} \int_{\partial\Omega} u(x) \frac{\partial}{\partial n} \frac{1}{|x - x_0|} - \frac{1}{|x - x_0|} \frac{\partial}{\partial n} u(x) \, \mathrm{d}s$$

Hint: Apply Green's second identity (Sheet 1 Ex. 2(c)) to $\Omega \setminus B_\varepsilon(x_0)$ and the pair (u, Φ).