Scientific Computing I

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Exercise sheet 3.

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Exercise 7. (Non-dimensionalization of the Stokes system)

(6 Points)

The Stokes equations describe the steady state of viscous thermal convection. An incompressible, viscous fluid moves via a time-constant velocity field $u(x)$ and experiences internal friction. It also expands by heating, which gives rise to a buoyancy force. The temperature in the system is both diffusive and advected by the velocity. We arrive at the system

$$\nabla \cdot u = 0, \tag{1a}$$

$$-\nabla \cdot \left(\mu(\nabla u + (\nabla u)^T)\right) + \nabla p = \alpha(T - T_0)\rho g, \tag{1b}$$

$$\rho C_v \left(\frac{\partial}{\partial t} + u \cdot \nabla\right) T = k \Delta T. \tag{1c}$$

Here α is the relative volume expansion ratio per unit of temperature, expressed in 1/K. The vector g with $|g| = 9.81\text{m/s}^2$ represents the gravity field. Another useful constant is $\kappa = k/\rho C_v$.

a) Non-dimensionalize the system by choosing a length unit L_0, a time unit t_0, a temperature transformation with given $\delta T = T_1 - T_0$ as

$$T = \tilde{T}(T_1 - T_0) + T_0, \tag{2}$$

and a given dynamic viscosity unit μ_0, and show that (1a) simplifies without parameters. Note that the velocity unit is derived, $u_0 = L_0/t_0$.

b) Find a time unit in terms of other units and constants that simplifies (1c), and do the same for the pressure p in (1b).

c) Show that (1b) simplifies with a single derived constant on the right hand side. We call it the Rayleigh number Ra. Express it in terms of known constants.

Exercise 8. (Representation theorem)

(0 Points)

Let V be a linear space and $a : V \times V \to \mathbb{R}$ a symmetric, bilinear form with $a[u,u] > 0$ for all $0 \neq u \in V$. Also, let $\ell : V \to \mathbb{R}$ be a linear functional. The quantity

$$J(v) := \frac{1}{2} a[v,v] - \ell(v) \tag{3}$$

attains its minimum at $u \in V$, if and only if

$$\forall v \in V : \quad a[u,v] = \ell(v). \tag{4}$$

Exercise 9. (Minimal principle)

(6 Points)

Let $\Omega \subset \mathbb{R}^d$ open, bounded, with a smooth boundary. Let $f \in C^0(\Omega)$ and $a_0, a_{i,k} \in C^1(\Omega)$ for all $i, k = 1, \dots, d$. Show that every solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying the boundary value problem

$$\begin{aligned} -\sum_{i,k=1}^{d} \partial_i \Big(a_{ik} \partial_k u \Big) + a_0 u &= f, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{9}$$

also solves the variational minimization problem

$$\min_{\substack{v \in C^2(\Omega) \cap C^0(\bar{\Omega}) \\ v_{|\partial\Omega = 0}}} \int_\Omega \frac{1}{2} \sum_{i,k=1}^{d} a_{ik}(x) \partial_i v \partial_k v(x) + \frac{1}{2} a_0(x) v(x)^2 - f(x) v(x)\, dx \tag{10}$$

Hint: 1) Use Green's formula

$$\int_\Omega v(x) \partial_i w(x)\, dx = -\int_\Omega \partial_i v(x) w(x)\, dx + \int_{\partial\Omega} v(x) w(x) \vec{n}_i(x)\, ds, \tag{11}$$

where $\vec{n}_i$ denotes the i-th component of the outer normal of Ω and v and w are C^1 functions.

2) Use the representation theorem.