Scientific Computing I

Winter semester 2022/2023
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Exercise sheet 4.

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Exercise 10. (Heat Equation)

(6 Points)

a) Show formally that the function

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{|x-y|^2}{4t}\right) u_0(y) \mathrm{d}y \tag{1}$$

solves the heat equation $\frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u = 0$ in $(0,T) \times \mathbb{R}$ for every $T > 0$.

b) Explain why we can expect that $u(t,x) \to u_0(x)$ as $t \to 0$ pointwise, e.g., for continuous and bounded initial data u_0.

c) Let

$$u_0(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Show that $u(t,x)$ is positive for all $t \in (0,T)$ and $x \in \mathbb{R}$, and conclude that information is propagated with infinite speed.

d) How does the solution given in (1) have to be modified to provide a solution of the heat equation $\frac{\partial}{\partial t}u - D\frac{\partial^2}{\partial x^2}u = 0$ with a diffusion constant D.

Exercise 11. (Finite Differences)

(6 Points)

Let $u : \mathbb{R} \to \mathbb{R}$ be a function that is sufficiently smooth and consider a discretization of an interval $[a,b]$ of interest using $N+1$ points $\{x_i\}_{i=0}^N$ given by

$$a = x_0 < x_1 < \ldots < x_N = b, \tag{2}$$

for $x_i = a + ih$ and $h = \frac{b-a}{N}$. In order to shorten the notation, we write u_i for $u(x_i)$.

a) Use the Taylor expansion of u at a point x for distance h up to the $\mathcal{O}(h^3)$ error term to derive that

$$u'(x_0) = \frac{u_1 - u_0}{h} - \frac{1}{2}hu''(x_1) + \mathcal{O}(h^2) \tag{3}$$

b) Use a sufficiently accurate expression for $u''(x_1)$ to rewrite (3) only in terms of u_0, u_1, u_2 up to $\mathcal{O}(h^2)$.

c) Obtain a similar representation at the other boundary, i.e. express $u'(x_N)$ only in terms of u_{N-2}, u_{N-1}, u_N up to order $\mathcal{O}(h^2)$.

Exercise 12. (Finite Difference Scheme in 2D)

(0 Points)

Consider the Poisson equation in two dimensions

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega = (0,1)^2, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

We aim to discretize the equation on the uniform grid of Ω with $(n-1)^2$ interior nodes

$$(x_i,\, y_j) = \left(\frac{i}{n}, \frac{j}{n}\right), \quad i,j \in \{1,\dots,n-1\},$$

whose grid size we denote $h = 1/n$.

a) Replace the partial derivatives by central difference quotients and derive a finite difference scheme for the Possion equation.

b) Order the elements $u_{i,j} = u(x_i, y_j)$ and $f_{i,j} = f(x_i, y_j)$ by rows on the grid into vectors, i.e.

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}^{(1)} \\ \boldsymbol{u}^{(2)} \\ \vdots \\ \boldsymbol{u}^{(n-1)} \end{bmatrix} \in \mathbb{R}^{(n-1)^2} \qquad \text{with} \quad \boldsymbol{u}^{(j)} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n-1,j} \end{bmatrix}, \quad j = 1,\dots,n-1,$$

$$\boldsymbol{f} = \begin{bmatrix} \boldsymbol{f}^{(1)} \\ \boldsymbol{f}^{(2)} \\ \vdots \\ \boldsymbol{f}^{(n-1)} \end{bmatrix} \in \mathbb{R}^{(n-1)^2} \qquad \text{with} \quad \boldsymbol{f}^{(j)} = \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ \vdots \\ f_{n-1,j} \end{bmatrix}, \quad j = 1,\dots,n-1,$$

and state the arising linear system $\mathbf{A}\boldsymbol{u} = \boldsymbol{f}$.

c) The Kronecker product $\mathbf{A} \otimes \mathbf{B}$ of two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times m}$ is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{1,1}\mathbf{B} & \dots & a_{1,m}\mathbf{B} \\ \vdots & & \vdots \\ a_{m,1}\mathbf{B} & \dots & a_{m,m}\mathbf{B} \end{bmatrix} \in \mathbb{R}^{m^2\times m^2}.$$

Show that

$$\mathbf{A} = \mathbf{L} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{L} \in \mathbb{R}^{(n-1)^2\times(n-1)^2},$$

where $\mathbf{I} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the identity matrix and

$$\mathbf{L} = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$$

is the matrix arising form the discretization of the one-dimensional Laplace operator.