Scientific Computing I

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Exercise sheet 5.

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Exercise 13. (Convergence of differentiable functions)

(6 Points)

Show that there is a sequence $(f_n)_n \subset C^1([-1,1])$ such that

- $f_n(x) \to f(x)$ almost everywhere in $[-1,1]$
- $\|f_n - f\|_{L^2([-1,1])} \to 0$ and $\|f_n' - f'\|_{L^2([-1,1])} \to 0$,
- but $f \notin C^1((-1,1))$.

Hint: Choose e.g. $f_n(x) = \begin{cases} x & x < -\frac{1}{n}, \\ -\frac{n^3}{4}\left(x^4 + \frac{3}{n^4}\right) & |x| \leq \frac{1}{n}, \\ -x & x > \frac{1}{n}. \end{cases}$

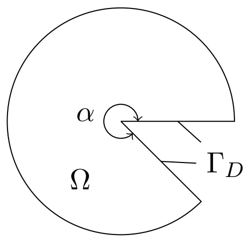
Exercise 14. (Poisson equation with singularity)

(6 Points)

For $\alpha \in (\pi, 2\pi)$, define

$$\Omega = \{r(\cos\varphi, \sin\varphi) \colon 0 < r < 1, 0 < \varphi < \alpha\}$$

and $\Gamma_D = \partial\Omega$.



a) For $u \in C^2(\Omega)$, let $\tilde{u}(r,\varphi) = u(r\cos\varphi, r\sin\varphi)$ and show that

$$\nabla u(r\cos\varphi, r\sin\varphi) = \left[\partial_r \tilde{u}(r,\varphi), r^{-1}\partial_\varphi \tilde{u}(r,\varphi)\right]^T$$

and

$$\Delta u(r\cos\varphi, r\sin\varphi) = \partial_r^2 \tilde{u}(r,\varphi) + r^{-1}\partial_r \tilde{u}(r,\varphi) + r^{-2}\partial_\varphi^2 \tilde{u}(r,\varphi).$$

b) Verify that the function $\tilde{u}(r,\varphi) = r^{\pi/\alpha}\sin(\varphi\pi/\alpha)$ solves the Poission equation with $f = 0$ in Ω and boundary condition

$$u_D(r,\varphi) = \begin{cases} 0 & \text{for } \varphi \in \{0, \alpha\} \\ \sin(\varphi\pi/\alpha) & \text{for } r = 1. \end{cases}$$

c) Show that the solution does not satisfy $u \in C^1(\bar{\Omega})$ and is therefore no strong solution to the Poisson equation.

d) Prove that for $\tilde{u}(r,\varphi) = r^{\pi/\alpha}\sin(\varphi\pi/\alpha)$ it holds

$$a(\tilde{u}, v) = l(v) \quad \text{for all } v \in C_c^\infty(\Omega)$$

where $a(u,v) = \int_\Omega \nabla u \nabla v \mathrm{d}x$ and $l(v) = \int_\Omega f v \mathrm{d}x$ and that $\tilde{u}$ and its first derivative are functions in $L^2(\Omega)$.

Exercise 15. (Representation theorem for convex sets)

(0 Points)

Let $V \subset H$ be a closed, convex set in a Hilbert space H and $a : H \times H \to \mathbb{R}$ symmetric and elliptic, i.e.

$$\forall v \in V : \quad a[v,v] \geq \alpha \|v\|_H^2, \quad \text{for an } \alpha > 0. \tag{1}$$

Let $\ell : H \to \mathbb{R}$ be a continuous linear form. Show that $u \in V$ solves the minimization problem

$$\min_{v \in V} J(v) := \frac{1}{2} a[v,v] - \ell(v) \tag{2}$$

if and only if the variational inequality

$$\forall v \in V : \quad a[u, v-u] \geq \ell(v-u) \tag{3}$$

is satisfied.