Scientific Computing I

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Exercise sheet 8.

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Exercise 20. (Biharmonic equation)

(6 Points)

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We consider the following PDE, the so called *biharmonic equation*:

$$\begin{aligned} \Delta^2 u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ \partial_n u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Here, $\partial_n u = \nabla u \cdot n$ denotes as usual the outer normal derivative of u on the boundary $\partial\Omega$ of Ω. With $\Delta^2 = \Delta \circ \Delta$ we denote double application of the Laplacian, which is defined in its strong form on $C^4(\Omega)$.

a) Derive the corresponding weak formulation of the biharmonic equation in the space

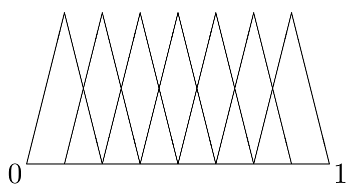
$$H_0^2(\Omega) := \{u \in H^2(\Omega) : u = 0 \text{ and } \partial_n u = 0 \text{ on } \partial\Omega\}.$$

b) Now we assume that Ω has a C^2-boundary or that Ω is convex. In this case there is some $c_\Omega > 0$ such that it holds $\|u\|_{H^2(\Omega)} \le c_\Omega \|\Delta u\|_{L^2(\Omega)}$ for every $u \in H_0^2(\Omega)$. Show that the biharmonic equation admits a unique solution $u \in H_0^2(\Omega)$ for any $f \in L^2(\Omega)$ and estimate the norm of u in terms of f.

Exercise 21. (1D Finite Elements)

(0 Points)

Let $I = (0, 1)$ be the unit interval. We divide I into n subintervals $[x_i, x_{i+1}]$ with $x_i = i/n$, $i = 0, \dots, n$. Consider the piecewise linear nodal basis $\{\psi_1, \dots, \psi_{n-1}\}$ which is uniquely defined by $\psi_k(x_j) = \delta_{kj}$ for $k = 1, \dots, n-1$ and $j = 0, \dots, n$.
The nodal basis sketched for $n = 8$:



a) For the bilinear form $a(u, v) = \int_I u'(x)v'(x)\,\mathrm{d}x$, compute $a(\psi_i, \psi_j)$ for $i, j = 1, \dots, n-1$.

b) Consider the weak formulation of the 1D Poisson problem with homogeneous Dirichlet boundary conditions:
Find $u \in H_0^1(I)$ s.t.

$$a(u, v) = \int_I f(x)v(x)\,\mathrm{d}x \quad \forall v \in H_0^1(I)$$

for some $f \in L^2(I)$. Define $V_n = \operatorname{span}\{\psi_1, \ldots, \psi_{n-1}\} \subset H_0^1(I)$. Use the Galerkin method to discretize the weak formulation with V_n and state the discrete system of equations in matrix form.

Exercise 22. (Transformation)

(0 Points)

Let $z_1, z_2, z_3 \in \mathbb{R}^2$ be the corners of an arbitrary non-degenerate triangle T and let $\xi_1 = (0, 0)$, $\xi_2 = (0, 1)$, $\xi_3 = (1, 0)$ be the corners of the reference triangle T_{ref}.

a) Compute the affine map $F_T\colon T_{\text{ref}} \to T$ such that $F_T(\xi_i) = z_i$, for $i = 1, 2, 3$, i.e. compute $B_T \in \mathbb{R}^{2\times 2}$, $c_T \in \mathbb{R}^2$, such that

$$B_T\xi_i + c_T = z_i, \qquad i = 1, 2, 3.$$

b) Show that it holds

$$|T| = \frac{1}{2}\,|\det B_T|$$

c) With

$$\varphi_1(\xi, \eta) = 1 - \xi - \eta, \qquad \varphi_2(\xi, \eta) = \xi, \qquad \varphi_3(\xi, \eta) = \eta$$

we denote the nodal basis on the reference element T_{ref}. and with $\psi_{i,T}$, $i = 1, 2, 3$, we refer to the corresponding nodal basis on T defined by

$$\psi_{i,T} = \varphi_i \circ F_T^{-1}.$$

Show that the local stiffness matrix $\mathbf{A}_T \in \mathbb{R}^{3\times 3}$ with

$$\mathbf{A}_{T,ij} := \int_T \nabla\psi_{i,T}\nabla\psi_{j,T}\,\mathrm{d}x\,\mathrm{d}y$$

is given by

$$\mathbf{A}_{T,ij} = |\det B_T| \int_{T_{\text{ref}}} \nabla\varphi_i^\top B_T^{-1}B_T^{-T}\nabla\varphi_j\,\mathrm{d}\xi\,\mathrm{d}\eta$$

Conclude that the local stiffness matrix $\mathbf{A}_T$ can be computed as

$$\mathbf{A}_T = \frac{1}{2}\,|\det B_T|\,\mathbf{C}\mathbf{C}^T, \qquad \text{with } \mathbf{C} := \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ are the coordinates of the vertices of T.