Scientific Computing I

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Prof. Dr. Carsten Burstedde
Uta Seidler and Denis Düsseldorf

UNIVERSITÄT BONN

Exercise sheet 9.

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Exercise 23. (Céa's Lemma)

(6 Points)

In the lecture, Céa's Lemma was shown: For a continuous and ellipitic bilinear form $a\colon H \times H \to \mathbb{R}$ and a continous linear functional $l \in H^*$ on the Hilbert space H we have

$$\|u - u_h\|_H \leq \frac{C_c}{c_e} \inf_{v_h \in V_h} \|u - v_h\|_H$$

where C_c is the continuity constant and c_e the ellipticity constant of a.
Show that if we additionally assume $a(\cdot, \cdot)$ to be *symmetric*, then

$$\|u - u_h\|_H \leq \sqrt{\frac{C_c}{c_e}} \inf_{v_h \in V_h} \|u - v_h\|_H.$$

Hint: Define a new scalar product on H and use Cauchy-Schwarz inequality.

Exercise 24. (Bilinear elements)

(6 Points)

Consider the unit square $\Omega = (0,1)^2 \subset \mathbb{R}^2$. We call a continuous function $f\colon \Omega \to \mathbb{R}$ affine bilinear if $f(\cdot, y)$ is affine linear for all $y \in \Omega$ and $f(x, \cdot)$ is affine linear for all $x \in \Omega$.

a) Let $Q(\Omega)$ be the space of affine bilinear functions on Ω. Show that $Q(\Omega)$ has dimension 4, and find a basis which is nodal with respect to the corners of Ω.

Let $n \in \mathbb{N}$. We define $a_i = i/n$ for $i = 1, \ldots, n$ and decompose Ω into a union of squares

$$\Omega_{ij} = \{(x, y)^\top \in \Omega \mid a_{i-1} \leq x \leq a_i,\, a_{j-1} \leq y \leq a_j\} \subset \Omega$$

for $i, j = 1, \ldots, n$.

b) Find the dimension of

$$V = \{f \in \mathcal{C}(\Omega) \mid f|_{\Omega_{ij}} \in Q(\Omega_{ij}) \text{ for } i, j = 1, \ldots, n\}$$

and determine whether a nodal basis with respect to the gridpoints $(a_i, a_j)^\top$, $i, j = 0, \ldots, n$ exists.