Scientific Computing I

Winter semester 2022/2023

Prof. Dr. Carsten Burstedde
Uta Seidler and Denis Düsseldorf

UNIVERSITÄT BONN

Exercise sheet 11.

Submission on **26.01.2023**.

Exercise 28. (1st Strang Lemma with error in the RHS)

(6 Points)

Let V denote a Hilbert space and for any $h > 0$ assume that $V_h \subset V$ denotes the approximation space resulting from a quasiuniform triangulation $\mathcal{T}_h$ of the domain. Moreover, suppose that for any $h > 0$ the bilinear form

$$a_h : V_h \times V_h \to \mathbb{R} \tag{1}$$

is uniformly elliptic and continuous, i.e. there exist constants $C_e, C_c > 0$ independent of h such that

$$\forall v_h \in V_h : \quad C_e \|v_h\|_V^2 \leq a_h(v_h, v_h). \tag{2}$$

and

$$\forall v_h, w_h \in V_h : \quad a_h(v_h, w_h) \leq C_c \|v_h\|_V \|w_h\|_V. \tag{3}$$

Suppose $\ell_h \in V_h^\star$ approximates $\ell \in V^\star$ by applying the midpoint quadrature method

$$\int_T w\,dx \approx Q_T(w) := |T| w(x_T)$$

for each triangle $T \in \mathcal{T}_h$, where $x_T = \frac{1}{3}(z_1 + z_2 + z_3)$ is the center of the triangle.
You already know that in this case there exists a constant $C > 0$ such that

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \left[\|u - v_h\|_V + \|a_h(v_h, \cdot) - a(v_h, \cdot)\|_{V_h^\star} + \|\ell_h - \ell\|_{V_h^\star} \right] \tag{4}$$

with the operator norm

$$\forall \mu \in V_h^\star : \quad \|\mu\|_{V_h^\star} := \sup_{v_h \in V_h \setminus \{0\}} \frac{\mu(v_h)}{\|v_h\|_V}. \tag{5}$$

Consider the finite element method with piecewise linear functions and show that for the true solution $u \in H^2$, and a right hand side function $f \in H^2(\Omega)$, the quadrature error $\|\ell_h - \ell\|_{V_h^\star}$ does not dominate the overall error.

Exercise 29. (Lemma of Berger, Scott & Strang (2nd Strang Lemma))

(0 Points)

Let $\Omega \subset \mathbb{R}^d$ be a sufficiently smooth domain and $H_0^m(\Omega) \subset V \subset H^m(\Omega)$. Similar to the approach performed in Strang's first Lemma (and Exercise 28), we replace the variational problem

$$\forall v \in V : \quad a(u, v) = \langle \ell, v \rangle \tag{11}$$

by a sequence of finite-dimensional problems: Find $u_h \in V_h$ (where $V_h \subset H^m$ must not necessarily be a subspace of V) such that

$$\forall v_h \in V_h : \quad a_h(u_h, v_h) = \langle \ell_h, v_h \rangle. \tag{12}$$

The bilinear forms a_h are assumed to be defined for functions from V as well as from V_h.
Moreover, assume that the bilinear forms a_h are uniformly elliptic and continuous, meaning that there exist constants $C_e, C_c > 0$ independent of h such that

$$\forall v_h \in V_h : \quad C_e \|v_h\|_m^2 \le a_h(v_h, v_h), \tag{13}$$

and

$$\forall u \in V + V_h,\, \forall v \in V_h : \quad |a_h(u, v) \le C_c \|u\|_m \|v\|_m. \tag{14}$$

Show that there exists a constant C independent of h satisfying

$$\|u - u_h\|_m \le C \left(\inf_{v_h \in V_h} \|u - v_h\|_m + \sup_{w_h \in V_h} \frac{|a_h(u, w_h) - \langle \ell_h, w_h \rangle|}{\|w_h\|_m} \right). \tag{15}$$

Note: The first term is called *approximation error*, whereas the second term is referred to as *consistency error*.

Exercise 30. (Duality)

(6 Points)

Let $(H, \|\cdot\|_H)$ and $(V, \|\cdot\|_V)$ be two Hilbert spaces with $V \subset H$. Additionally, assume that the inclusion $V \hookrightarrow H$ is continuous. Moreover, let $W_h \subset H$ for any $h > 0$. Replace the problem of finding $u \in V$ s.t.

$$\forall v \in V : \quad a(u, v) = \ell(v) \tag{21}$$

by a sequence of finite-dimensional problems: Find $u_h \in W_h$ such that

$$\forall v_h \in W_h : \quad a_h(u_h, v_h) = \ell_h(v_h). \tag{22}$$

The bilinear forms a_h are defined on $V \cup W_h$.
Show that

$$\begin{aligned} \|u - u_h\|_H \quad \le \quad \sup_{g \in H} \Big\{ & C \|u - u_h\|_H \|\varphi_g - \varphi_h\|_H \\ & + |a_h(u - u_h, \varphi_g) - \langle u - u_h, g \rangle_H| \\ & + |a_h(u, \varphi_g - \varphi_h) - \langle \ell, \varphi_g - \varphi_h \rangle| \Big\}, \end{aligned} \tag{23}$$

where for all $g \in H$, the elements $\varphi_g \in V$ and $\varphi_h \in W_h$ are the weak solutions of

$$\begin{aligned} \forall v \in V : \quad & a_h(v, \varphi_g) = \langle v, g \rangle_H \\ \forall w_h \in W_h : \quad & a_h(w_h, \varphi_h) = \langle w_h, g \rangle_H. \end{aligned} \tag{24}$$