Scientific Computing I

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Exercise sheet 12.

Recap sheet — no submission

Exercise 31. (Ellipticity of second order PDE)

(0 Points)

Let $\Omega \subset \mathbb{R}^d$ open and bounded with a smooth boundary. Consider the differential operator

$$\mathcal{L} : \mathcal{C}^k(\Omega) \to \mathcal{C}^0(\Omega) \tag{1}$$

with

$$\mathcal{L}u := -\partial_x \left(\partial_x + 2\partial_y\right) - 4\partial^2_{y^2}. \tag{2}$$

a) Show that $\mathcal{L}$ is an elliptic differential operator.

b) Consider the PDE

$$\begin{aligned} -\mathrm{div} A\nabla u &= f, && \text{in } \Omega \\ u &= g, && \text{on } \Gamma_D \subset \partial\Omega \\ A\nabla u \cdot \vec{n} &= h, && \text{on } \Gamma_N := \partial\Omega \setminus \Gamma_D. \end{aligned} \tag{3}$$

How do you choose the trial space $V^{\text{trial}} \ni u$ and the test space V^{test}?

c) Derive the weak formulation of the PDE

$$\begin{aligned} -\mathrm{div} A\nabla u &= 2, && \text{in } \Omega \\ u &= 5, && \text{on } \Gamma_D \subset \partial\Omega \\ A\nabla u \cdot \vec{n} &= 4, && \text{on } \Gamma_N := \partial\Omega \setminus \Gamma_D. \end{aligned} \tag{4}$$

Exercise 32. (Maximum principle)

(0 Points)

Let $\Omega \subset \mathbb{R}^d$ open and bounded, $\mathcal{L}$ a second order linear differential operator, and

$$f_1(x) = 1 + \left(\sum_{i=1}^{d} |x_i|^2\right)^{\frac{1}{2}}, \qquad f_2(x) = \left(\sum_{i=1}^{d} |x_i|^2\right)^{\frac{1}{2}} \tag{5}$$

Suppose that $u_1, u_2 \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ solve the differential equations

$$\forall x \in \Omega : \quad \mathcal{L}u_i(x) = f_i(x), \quad \text{for } i = 1, 2. \tag{6}$$

Show: Whenever there exists a point $x \in \Omega$ s.t. for all $\bar{x} \in \bar{\Omega}$ it holds that

$$(u_1 - u_2)(x) > (u_1 - u_2)(\bar{x}), \tag{7}$$

then $\mathcal{L}$ cannot be elliptic.

Exercise 33. (Helmholtz equation in 2D)

(0 Points)

Let $\Omega = (0,1)^2$ the unit square. Consider the 2D Helmholtz equation

$$\begin{aligned} -\Delta u(x) &= \lambda u(x), \quad \text{in } \Omega \\ u(x) &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{9}$$

for functions $u \in \mathcal{C}^2(\Omega)$.

a) Find a function u that solves the one dimensional Helmholtz equation

$$\begin{aligned} -\partial_{x^2}^2 u(x) &= \lambda u(x), \quad \text{in } (0,1) \\ u(0) = u(1) &= 0. \end{aligned} \tag{10}$$

for a given eigenvalue λ.

b) Let u_1 and u_2 be solutions to the Helmholtz equation for eigenvalue λ_1, resp. λ_2. Show that $u(x,y) = u_1(x)u_2(y)$ solves the 2D Helmholtz equation and compute the corresponding eigenvalue.

c) The 2D Helmholtz equation is discretized using a regular grid

$$\{x_{i,j} = (\frac{i}{n}, \frac{j}{n}) | \quad i,j = 1, \ldots, n-1\} \tag{11}$$

together with the 5-point stencil

$$-\Delta u(x_{i,j}) \approx n^2(4u(x_{i,j}) - u(x_{i-1,j}) - u(x_{i,j-1}) - u(x_{i+1,j}) - u(x_{i,j+1})). \tag{12}$$

State the resulting discrete system of $N = (n-1)^2$ equations. Here, use the ordering

$$U = (U_1, \ldots, U_N)^T, \quad U_{(n-1)(i-1)+j} := u(x_{i,j}). \tag{13}$$

Exercise 34. (Robin boundary conditions)

(0 Points)

Let $\Omega \subset \mathbb{R}^d$ open, bounded and with smooth boundary. For $f \in H^{-1}(\Omega)$ and $g \in L^2(\partial\Omega)$ consider the PDE

$$\begin{aligned} -\Delta u + \alpha u &= f, \quad \text{in } \Omega \\ \nabla u \cdot \vec{n} + \gamma u &= g, \quad \text{on } \partial\Omega \end{aligned} \tag{14}$$

for constants $\alpha, \gamma \in \mathbb{R}$.

a) Find the weak formulation of the PDE, i.e. find the bilinear form a and linear function ℓ such that

$$\forall v \in V : \quad a[u,v] = \ell(v). \tag{15}$$

b) Show that the linear function ℓ is continuous.
Tipp: For bounded domains Ω, there exists a constant C s.t. for all $v \in H^1(\Omega)$ the following estimate holds:

$$\|v\|_{L^2(\Gamma)} \leq C\|v\|_{H^1(\Omega)}. \tag{16}$$

c) Show that the weak problem has a unique solution for $\alpha = 1$ and $\gamma = 0$.

d) What happens for $\alpha = \gamma = 0$?