Scientific Computing I

Winter semester 2022/2023
Prof. Dr. Carsten Burstedde
Uta Seidler and Denis Düsseldorf

UNIVERSITÄT BONN

Exercise sheet 13.

Recap sheet — no submission

Exercise 35. (0 Points)

Let $\Omega := \{(x,y) \in \mathbb{R}^2\colon 0 < x < 1, 0 < y < x^2\}$ and $u(x,y) := x^{-1}$. Determine those $p \in [1,\infty)$ such that $u \in W^{1,p}(\Omega)$ holds.

Exercise 36. (0 Points)

On the reference interval $I_{ref} = (0,1)$ we consider the space of quadratic polynomials

$$\mathcal{P}_2 := \{p\colon I_{ref} \to \mathbb{R} \mid \quad p \text{ polynomial of degree } \leq 2\}.$$

a) Give an explicit nodal basis of $\mathcal{P}_2$ with respect to the point set

$$Z := \{z_0, z_1, z_2\}, \qquad z_0 = 0,\ z_1 = \frac{1}{2},\ z_2 = 1.$$

b) Let $I = [a,b]$ be an arbitrary interval equipped with a partition into n subintervals $[x_i, x_{i+1}]$, $i = 0, ..., n-1$, with

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

Show that there exists an invertible affine mapping F_i which maps I_{ref} to the subintervals $T_i = [x_i, x_{i+1}]$ for each $i = 0, ..., n-1$. Moreover, use F_i to construct finite elements (T_i, P_{T_i}, Σ_i) and show that the corresponding finite element space is contained in $H^1(I)$. Determine the dimension of this space.

c) Below, the matrix $\mathbf{M} \in \mathbb{R}^{7\times 7}$ is given. Is it possible that this matrix $\mathbf{M}$ is the mass matrix associated with the finite element space from b) equipped with its global nodal basis? Explain your answer.

$$\mathbf{M} = \begin{pmatrix} 0.01666667 & 0.00833333 & 0. & 0. & 0. & 0. & 0. \\ 0.00833333 & 0.06666667 & 0.025 & 0. & 0. & 0. & 0. \\ 0. & 0.025 & 0.11666667 & 0.03333333 & 0. & 0. & 0. \\ 0. & 0. & 0.03333333 & 0.2 & 0.06666667 & 0. & 0. \\ 0. & 0. & 0. & 0.06666667 & 0.16666667 & 0.01666667 & 0. \\ 0. & 0. & 0. & 0. & 0.01666667 & 0.06666667 & 0.01666667 \\ 0. & 0. & 0. & 0. & 0. & 0.01666667 & 0.03333333 \end{pmatrix}$$

[Hint: Look at the number of nonzero entries.]

Exercise 37. (0 Points)

If required you may use the following well-known identities without proof:

$$\int \sin(n\pi x)\sin(m\pi x)dx = \frac{\sin((m-n)\pi x)}{2(m-n)\pi} - \frac{\sin((m+n)\pi x)}{2(m+n)\pi} + c,$$

$$\int \sin(n\pi x)^2 dx = \frac{x}{2} - \frac{\sin(2n\pi x)}{4n\pi} + c,$$

$$\int \sin(n\pi x)^3 dx = \frac{\cos(3n\pi x)}{12n\pi} - \frac{3\cos(n\pi x)}{4n\pi} + c,$$

$$\int x\sin(n\pi x)dx = \frac{1}{n^2\pi^2}\left(\sin(n\pi x) - n\pi x\cos(n\pi x)\right) + c,$$

$$\int \cos(n\pi x)\cos(m\pi x)dx = \frac{\sin((m-n)\pi x)}{2(m-n)\pi} + \frac{\sin((m+n)\pi x)}{2(m+n)\pi} + c,$$

$$\int \cos(n\pi x)^2 dx = \frac{x}{2} + \frac{\sin(2n\pi x)}{4n\pi} + c,$$

$$\int \cos(n\pi x)^3 dx = \frac{9\sin(n\pi x) + \sin(3n\pi x)}{12n\pi} + c,$$

$$\int x\cos(n\pi x)dx = \frac{1}{n^2\pi^2}\left(\cos(n\pi x) + n\pi x\sin(n\pi x)\right) + c,$$

$$\int \sin(n\pi x)\cos(m\pi x)dx = \frac{n\cos(n\pi x)\cos(m\pi x) + m\sin(m\pi x)\sin(n\pi x)}{(m^2-n^2)\pi} + c,$$

$$\int \sin(n\pi x)\cos(n\pi x)dx = -\frac{\cos(2n\pi x)}{4n\pi} + c,$$

On the unit interval $\Omega = [0,1]$ we consider for some $n \in \mathbb{N}$ the n-dimensional subspace $V_n \subset H_0^1([0,1])$ spanned by the n basis functions

$$\phi_k(x) = \sin((2k+1)\pi x), \qquad k = 0, ..., n-1.$$

We consider the boundary value problem

$$\begin{aligned} -u''(x) &= 1 - 2x, \qquad \text{on } [0,1], \\ u(0) &= 0, \\ u(1) &= 0, \end{aligned} \tag{1}$$

with exact solution $u(x) = \frac{1}{3}x\left(x - \frac{1}{2}\right)(x-1)$, which you may use without verification. Compute the Galerkin approximate solution u_n for (1) in the ansatz space V_n.

Exercise 38. (0 Points)

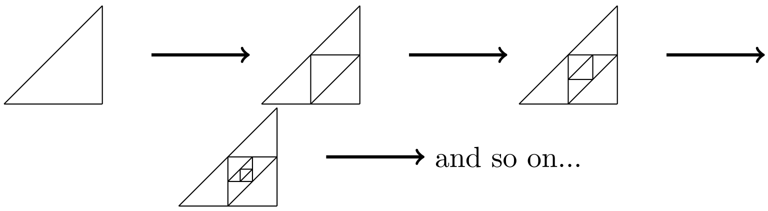
a) Show that a family of triangulations $\{\mathcal{T}_h\}$ is uniform iff it is quasi-uniform and there exist positive constants $c, C > 0$ such that

$$c\,\mathrm{diam}(T_1) \le \mathrm{diam}(T_2) \le C\,\mathrm{diam}(T_1) \quad \text{for all } T_1, T_2 \in \mathcal{T}_h.$$

b) We consider consider a family of triangular meshes on the polygonal domain

$$\Omega = \{(x,y) \in \mathbb{R}^2 \colon 0 < x < y, y < x\}.$$

that is constructed as follows: One starts with Ω and in every step *only the middle triangle* is divided into four new triangles. The new corners are given by the midpoints of the edges of the old triangle.



Decide whether the family of triangulations on Ω introduced above is quasi-uniform and if it is uniform.