Book of Abstracts

10th International Conference on Computational Methods in Applied Mathematics CMAM-10

Funded by the TRA Modelling (University of Bonn) as part of the Excellence Strategy of the federal and state governments and the Hausdorff Center for Mathematics (HCM).

Conference Program

COMPUTATION OF PLATES

Carsten Carstensen

Department of Mathematics, Humboldt-Universität zu Berlin, 10099 Berlin, Germany

The most popular classic (piecewise) quadratic schemes for the fourth-order plate bending problems based on triangles are the nonconforming Morley finite element, two discontinuous Galerkin, the C^0 interior penalty, and the WOPSIP schemes. The first part of the presentation discusses recent applications to the linear bi-Laplacian and to semi-linear fourth-order problems like the stream function vorticity formulation of incompressible 2D Navier-Stokes problem and the von Karman plate bending problem. The role of a smoother is emphasised and reliable and efficient a posteriori error estimators give rise to adaptive mesh-refining strategies that recover optimal convergence rates in numerical experiments. The second part discusses adaptive Argyris finite element schemes and their high-order application to the biharmonic eigenvalue computation. We also mention the multigrid preconditioning with a local smoother in a V-cycle that is indeed optimal if the extended Argyris finite element scheme is employed.

The presentation is based on joint work with N. Nataraj from IITB in Powai, Mumbai, India and my PhD student B. Gräßle. Selected related references follow below.

REFERENCES

[1] C. Carstensen, B. Gräßle, and N. Nataraj. *Unifying a posteriori error analysis of five piecewise quadratic discretisations for the biharmonic equation*, J. Numer. Math. 2024. arXiv:2310.05648.

[2] C. Carstensen, B. Gräßle, and N. Nataraj. *A posteriori error control for fourth-order semi-linear problems with quadratic nonlinearity*, SINUM 62, 919-945, 2024. arXiv:2309.08427.

[3] C. Carstensen, Jun Hu. *Hierarchical Argyris finite element method for adaptive and multi-grid algorithms*, Comput. Methods Appl. Math., 21, 529-556, 2021.

[4] C. Carstensen, N. Nataraj. *A Priori and a Posteriori Error Analysis of the Crouzeix-Raviart and Morley FEM with Original and Modified Right-Hand Sides*, Comput. Methods Appl. Math., volume 21, pp. 289-315, 2021.

[5] C. Carstensen, N. Nataraj, G.C. Remesan, D. Shylaja. *Lowest-order FEM for fourth-order semi-linear problems with trilinear nonlinearity*, Numer. Math. 154, 323-368, 2023.

[6] C. Carstensen, N. Nataraj. *Lowest-order equivalent nonstandard finite element methods for biharmonic plates*, ESAIM: Math. Model. Numer. Anal., 56(1), 41-78, 2022.

[7] C. Carstensen, B. Gräßle. *Rate-optimal higher-order adaptive conforming FEM for biharmonic eigenvalue problems on polygonal domains*, Comput. Methods Appl. Mech. Engrg., volume 425, pp. 116931, 2024.

ON THE STABILITY AND CONVERGENCE OF A FICTITIOUS DOMAIN APPROACH FOR FLUID-STRUCTURE INTERACTION PROBLEMS

Daniele Boffi

King Abdullah University of Science and Technology (KAUST), Saudi Arabia
University of Pavia, Italy

We are developing a fictitious domain approach, based on a distributed Lagrange multiplier for the numerical simulation of fluid-structure interaction.

Starting from the immersed boundary method of Peskin (see [7] for a review), we introduced its finite element variant and then we showed how to use a distributed Lagrange multiplier for dealing with the kinematic interaction between solid and fluid [2]. Several theoretical results have been obtained, including unconditional time stability, existence and uniqueness for the continuous problem in a linearized setting [5, 6], as well as a discussion on the numerical approximation of the multiplier [1].

As opposed to other non fitted approaches, our method is uniformly stable with respect of the size of the cut cells.

In this talk I will recall the main features of the method and discuss how to deal with the coupling terms which involve the computation of integrals of shape functions defined on different meshes [3, 4].

REFERENCES

[1] N. Alshehri, D. Boffi, and L. Gastaldi, *Unfitted mixed finite element methods for elliptic interface problems*, Numer. Methods Partial Differential Eq., 40, e23063, 2024.

[2] D. Boffi, N. Cavallini, and L. Gastaldi, *The Finite Element Immersed Boundary Method with Distributed Lagrange multiplier*, SIAM J. Numer. Anal., 53(6), 2584–2604, 2015.

[3] D. Boffi, F. Credali, and L. Gastaldi, *On the interface matrix for fluid-structure interaction problems with fictitious domain approach*, Comp. Meth. Appl. Mech. Eng., 401(B), 115650, 2022.

[4] D. Boffi, F. Credali, and L. Gastaldi, *Quadrature error estimates on non-matching grids in a fictitious domain framework for fluid-structure interaction problems*, in preparation.

[5] D. Boffi and L. Gastaldi, *A fictitious domain approach with distributed Lagrange multiplier for fluid-structure interactions*, Numer. Math., 135(3), 711–732, 2017.

[6] D. Boffi and L. Gastaldi, *On the existence and the uniqueness of the solution to a fluid-structure interaction problem*, J. Diff. Equ., 279, 136–161, 2021.

[7] C.S. Peskin, *The immersed boundary method*, Acta Numerica, 11, 479–517, 2002.

MATHEMATICS FOR MACHINE LEARNING ALGORITHMS: A PDE AND NUMERICAL ANALYSIS PERSPECTIVE

Charalambos G. Makridakis

IACM-FORTH DMAM, University of Crete, Greece
MPS, University of Sussex, United Kingdom

In this talk we shall discuss problems arising in the mathematical description, understanding and advancement of machine learning algorithms. These algorithms find applications in various scientific and engineering domains, significantly impacting key aspects of research. Mathematical analysis is essential to address several crucial questions: a) the reliability of these algorithms, b) their advantages or potential limitations compared to conventional approaches, and c) the design novel and enhanced algorithms. Emphasis will be given to the connection of ML algorithms to notions and problems related to PDEs and to Numerical Analysis. In particular, we will discuss problems related to stability, convergence, a priori and a posteriori error control of algorithms designed to learn functions as well as solutions of differential equations.

ERROR IDENTITIES FOR PARABOLIC AND HYPERBOLIC EQUATIONS WITH MONOTONE SPATIAL OPERATORS

Sergey I. Repin

St.-Petersburg Department of Steklov Institute of Mathematics of Russian Academy of Sciences

We consider error identities that characterise distances between exact solutions of nonlinear evolutionary problems and functions considered as approximations. The restrictions imposed on such a function are minimal and actually come down to the condition that it belongs to the same functional class as the generalized solution of the problem under consideration. The identities reflect the most general relations between deviations from exact solutions of initial boundary value problems and those data that can be observed in a numerical experiment. They contain no mesh dependent constants and are valid for any function in the admissible (energy) class regardless of the method by which it was constructed. Therefore, they can be used as tools for deriving fully reliable a posteriori estimates of approximation errors and for analysis of modeling errors. Several examples related to both cases are discussed.

REFERENCES

[1] S. Repin, *Identities for Measures of Deviation from Solutions to Parabolic-Hyperbolic Equations*, Computational Mathematics and Mathematical Physics 64 (5), 2024 (in press).

NUMERICAL SOLUTION OF SPACE-FRACTIONAL PARABOLIC EQUATIONS

Svetoar Margenov

Institute of Information and Communication Technologies, Bulgarian Academy of Sciences, Sofia, Bulgaria

We consider the equation

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} + \mathcal{A}^\alpha u(x,t) &= f(x,t), \quad (x,t) \in \Omega \times [0,T], \\ u(x,t) &= 0 \qquad\quad x \in \partial\Omega \times [0,T], \\ u(x,0) &= u_0 \qquad\quad x \in \Omega. \end{aligned}$$

Here, Ω is a multidimensional bounded domain, $\mathcal{A}$ is a self-adjoint positive definite second-order elliptic operator, and $\alpha \in (0,1]$, which corresponds to the case of subdiffusion. Let FDM or FEM be used to approximate $\mathcal{A}$, thus obtaining the matrix $\mathbb{A} \in \mathbb{R}^N \times \mathbb{R}^N$. In both cases, continuous and discrete, the spectral definition of fractional power is used. Although $\mathbb{A}$ is sparse, the matrix $\mathbb{A}^\alpha$ is dense, which corresponds to the nonlocality of the operator $\mathcal{A}^\alpha$.

The considered first- and second-order finite-difference schemes in time require solving systems with the matrix $\mathbb{A}^\alpha + 1/\tau\mathbb{I}$, where $\tau = T/M$ is the time step, and multiplication of vectors by $\mathbb{A}^\alpha$. The contributions of this study concern the computationally efficient approximation of these dense linear algebra operations. For this purpose, BURA (Best Uniform Rational Approximation) methods [1, 2, 3] are applied. Sufficient conditions for balancing errors of different origins are obtained. In this way, consistent error estimates with respect to the discretization parameters h and τ are derived. In conclusion, near-optimal estimates of the computational complexity of the composite algorithms with respect to the number of degrees of freedom MN are obtained.

REFERENCES

[1] S. Harizanov, R. Lazarov, S. Margenov, *A survey on numerical methods for spectral space-fractional diffusion problems*, Fract. Calc. Appl. Anal., 23 (6), 1605–1646, 2020.

[2] S. Harizanov, R. Lazarov, S. Margenov, P. Marinov, J. Pasciak, *Analysis of numerical methods for spectral fractional elliptic equations based on the best uniform rational approximation*, J. Comput. Phys., 408, 109285, 2020.

[3] N. Kosturski, S. Margenov, *Analysis of BURA and BURA-based approximations of fractional powers of sparse SPD matrices*, Fract. Calc. Appl. Anal., to appear, 2024.

FINITE ELEMENT METHODS FOR LEAST-SQUARES PROBLEMS

Susanne C. Brenner

Department of Mathematics and Center for Computation & Technology, Louisiana State University, Baton Rouge, USA

Finite dimensional linear and nonlinear least-squares problems appear in data fitting and the solution of nonlinear equations. In this talk I will present some recent results for the infinite dimensional analogs of such problems. They include (i) a general framework for solving distributed elliptic optimal control problems with pointwise state constraints by finite element methods originally designed for fourth order elliptic boundary value problems, (ii) a multiscale finite element method for solving distributed elliptic optimal control problems with rough coefficients and pointwise control constraints, and (iii) a convexity enforcing nonlinear least-squares finite element method for solving the Monge-Ampere equation.

NONSTANDARD FINITE ELEMENT METHODS FOR BIHARMONIC PLATES AND ITS APPLICATIONS TO TIME DEPENDENT PROBLEMS

Neela Nataraj

Indian Institute of Technology Bombay, Mumbai, India, neela@math.iitb.ac.in

The popular (piecewise) quadratic schemes for the biharmonic equation based on triangles are the nonconforming Morley finite element, the discontinuous Galerkin, and the C^0 interior penalty schemes. Those methods are modified in their right-hand side $F \in H^{-2}(\Omega)$ replaced by $F \circ (JI_{\mathrm{M}})$ and then are quasi-optimal in their respective discrete norms [1]. The smoother JI_{M} is defined for a piecewise smooth input function by a (generalized) Morley interpolation I_{M} followed by a companion operator J. An abstract framework for the error analysis in the energy, weaker and piecewise Sobolev norms for the schemes applies to the biharmonic equation. Two modified Ritz projection operators using the smoother are employed for the time-dependent problems: the linear biharmonic wave equation and the semilinear extended Fisher-Kolmogorov model, both with clamped boundary conditions. The approach allows for both semidiscrete and fully discrete error analysis with minimal regularity assumptions on the exact solution. These are joint works with Carsten Carstensen, Avijit Das, Gopikrishnan Remesan, Ricardo Ruiz-Baier, and Aamir Yousuf.

REFERENCES

[1] Carsten Carstensen and Neela Nataraj, *Lowest-order equivalent nonstandard finite element methods for biharmonic plates*, ESAIM Math. Model. Numer. Anal., 56 (1):41–78, 2022.

[2] Avijit Das, Gopikrishnan C. Remesan, Neela Nataraj, *Fully-discrete analysis of extended Fisher–Kolmogorov equation with nonstandard FEMs for space discretization (preprint).*

[3] Neela Nataraj, Ricardo Ruiz-Baier, Aamir Yousuf, *Semi-discrete and fully-discrete analysis of lowest-order nonstandard finite element methods for the biharmonic wave problem (preprint).*

A UNIFIED FINITE ELEMENT APPROACH TO PDE-CONSTRAINED OPTIMAL CONTROL PROBLEMS

Richard Löscher[1], **Ulrich Langer**[2], **Olaf Steinbach**[1] **and Huidong Yang**[3]

[1] Institut für Angewandte Mathematik, Technische Universität Graz, Austria
[2] Institute of Numerical Mathematics, Johannes Kepler University Linz, Austria,
ulrich.langer@jku.at
[3] Faculty of Mathematics, University of Vienna, Austria

We propose, analyze, and test new iterative solvers for systems of linear algebraic equations arising from the finite element discretization of reduced optimality systems defining the finite element approximations to the solution of elliptic, parabolic, and hyperbolic distributed optimal control problems with both the standard L_2 and the more general energy regularizations. In contrast to the usual time-stepping approach, we discretize the optimality system arising from time-dependent optimal control problems by space-time continuous piecewise-linear finite element methods on fully unstructured simplicial meshes in the same fashion as in the case of elliptic problems.

If we aim at the best approximation of the given desired state y_d by the computed finite element state y_h, then the optimal choice of the regularization parameter ϱ is linked to the mesh-size h by the relations $\varrho = h^4$ and $\varrho = h^2$ for the L_2 and the energy regularization, respectively. For this setting, we can construct robust (parallel) iterative solvers for the reduced finite element optimality systems. These results can be generalized to variable regularization parameters adapted to the local behavior of the mesh-size that can heavily change in case of adaptive mesh refinement. In practice, the solver should be embedded in a nested iteration procedure that starts from some suitable coarse mesh and proceeds with finer and finer (adaptive) meshes until the desired accuracy of the computed state approximation to the desired state y_d is reached or the costs for the control exceed a prescribed threshold. The numerical results confirm the theoretical findings.

HYBRID METHODS AND PLATE BENDING

Norbert Heuer

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Santiago, Chile,
nheuer@uc.cl

We present an extended framework for hybrid finite element approximations of self-adjoint, positive definite operators. It covers the cases of primal, mixed, and ultraweak formulations both at the continuous and discrete levels, and gives rise to conforming discretizations. Our framework allows for flexible continuity restrictions across elements, and includes the extreme cases of conforming and discontinuous hybrid methods. We illustrate an application of the framework to the Kirchhoff–Love plate pending model. It generates conforming environments for (in the classical meaning) non-conforming elements of Morley, Zienkiewicz triangular, and Hellan–Herrmann–Johnson types.

Financial support by ANID-Chile through Fondecyt project 1230013 is gratefully acknowledged.

TWO EFFECTIVE NUMERICAL METHODOLOGIES FOR GENERAL INVERSE PROBLEMS OF PDES

Jun Zou

Department of Mathematics, The Chinese University of Hong Kong, Hong Kong SAR, China,
zou@math.cuhk.edu.hk

In this talk, we shall first review the general formulations of inverse problems of PDEs, then address the current developments of some most effective and robust numerical methodologies, especially the direct sampling-type methods and the adaptive-type methods, for solving general inverse problems of PDEs. The necessity, robustness and effectiveness of these methods will be discussed, along with the applications of the methods to some typical nonlinear highly ill-posed inverse problems.

This talk covers joint works with several coauthors, including Yat Tin Chow (UCR), Kazufumi Ito (NCSU), Bangti Jin (CUHK). The work was substantially supported by Hong Kong RGC General Research Fund (projects 14308322, 14306921 and 14306719).

ADAPTIVE MULTI-LEVEL ALGORITHM FOR A CLASS OF NONLINEAR PROBLEMS

Eun-Jae Park

Department of Computational Science and Engineering, School of Mathematics and Computing, Yonsei University, Seoul 03722, Korea, ejpark@yonsei.ac.kr

As a motivation, we present our polygonal staggered DG methods for linear Darcy equations [1] and non-Darcy flows [2, 3].

The main part of the talk consists of an adaptive mesh-refining based on the multi-level algorithm and derive a unified a posteriori error estimate for a class of nonlinear problems in the abstract framework of Brezzi, Rappaz, and Raviart. The multi-level algorithm on adaptive meshes retains quadratic convergence of Newton's method across different mesh levels both theoretically and numerically.

As applications of our theory, we consider the pseudostress-velocity formulation of Navier-Stokes equations and the standard Galerkin formulation of semilinear elliptic equations. Reliable and efficient a posteriori error estimators for both approximations are derived. Several numerical examples are presented to test the performance of the algorithm and validity of the theory developed. Lastly, ongoing work on Darcy-Forchheimer flows is presented.

REFERENCES

[1] L. Zhao and E.-J. Park, *A staggered DG method of minimal dimension on quadrilateral and polygonal meshes*, SIAM J. Sci. Computing 40(4) 2018, A2543-A2567.

[2] E.-J. Park, *Mixed finite element methods for generalized Forchheimer flow in porous media*, Numer. Methods. Partial. Differ. Equ., Vol. 21 (2005), pp. 213-228

[3] L. Zhao, E. Chung, E.-J. Park, and G. Zhou, *Staggered DG method for coupling of the Stokes and Darcy-Forchheimer problems*, SIAM J. Numer. Anal. 59 (1), 1–31, (2021).

[4] Carsten Carstensen, Dongho Kim, Eun-Jae Park, *A priori and a posteriori pseudostress-velocity mixed finite element error analysis for the Stokes problem*, SIAM J. Numer. Anal., Vol. 49 (2011), pp. 2501-2523

[5] Dongho Kim, Eun-Jae Park, Boyoon Seo, *A unified framework for two-grid methods for a class of nonlinear problems*, Calcolo, December 2018, 55:45

[6] Dongho Kim, Eun-Jae Park, Boyoon Seo, *Convergence of multi-level algorithms for a class of nonlinear problems*, J. Sci. Comput. 84 (2020), no. 2, Paper No. 34, 23 pp.

[7] Dongho Kim, Eun-Jae Park, Boyoon Seo, *Adaptive multi-level algorithms for a class of nonlinear problems*, Comput. Meth. Appl. Math. 2024.

A CONSTRUCTION OF C^r CONFORMING FINITE ELEMENT SPACES IN ANY DIMENSION

Jun Hu

Peking University, China

This talk proposes a construction of C^r conforming finite element spaces with arbitrary r in any dimension. It is shown that if $k \geq 2^d r + 1$ the space P_k of polynomials of degree $\leq k$ can be taken as the shape function space of C^r finite element spaces in d dimensions. This is the first work on constructing such C^r conforming finite elements in any dimension in a unified way.

OPTIMAL INTERPLAY OF ADAPTIVE MESH-REFINEMENT AND ITERATIVE SOLVERS FOR ELLIPTIC PDES

Dirk Praetorius

TU Wien, Institute of Analysis and Scientific Computing, Austria
dirk.praetorius@asc.tuwien.ac.at

The ultimate goal of any numerical scheme for partial differential equations (PDEs) is to compute an approximation of user-prescribed accuracy at quasi-minimal computational time. On the one hand, this requires adaptive mesh-refinement to resolve potential singularities. On the other hand, adaptive finite element methods (AFEMs) must integrate an inexact solver and nested iteration with discerning stopping criteria to balance the different error components. In our talk, we present recent advances of the AFEM analysis in this respect. Particular emphasis is on parameter-robust (full R-linear) convergence of AFEM (i.e., guaranteed convergence independently of the choice of the adaptivity parameters), while optimal complexity (i.e., optimal convergence rates with respect to the overall computational time) follows for sufficiently small parameters.

The talk is based on joint work with Philipp Bringmann (TU Wien), Gregor Gantner (University of Bonn), Ani Miraçi (TU Wien), and Julian Streitberger (TU Wien).

REFERENCES

[1] G. Gantner, A. Haberl, D. Praetorius, S. Schimanko, *Rate optimality of adaptive finite element methods with respect to overall computational costs*, Mathematics of Computation, 90:2011–2040, 2021.

[2] P. Bringmann, M. Feischl, A. Miraçi, D. Praetorius, J. Streitberger, *On full linear convergence and optimal complexity of adaptive FEM with inexact solver*, Preprint arXiv:2311.15738, 2023.

[3] M. Innerberger, A. Miraçi, D. Praetorius, J. Streitberger, *hp-robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs*, ESAIM: Mathematical Modelling and Numerical Analysis, 58:247–272, 2024.

[4] A. Miraçi, D. Praetorius, J. Streitberger, *Parameter-robust full linear convergence and optimal complexity of adaptive iteratively linearized FEM for nonlinear PDEs*, Preprint arXiv:2401.17778, 2024.

hp-FEM FOR THE INTEGRAL FRACTIONAL LAPLACIAN IN POLYGONS

B. Bahr[1], C. Marcati[2] M. Faustmann[1], J.M. Melenk[1], and C. Schwab[3]

[1] Institute of Analysis and Scientific Computing, TU Wien, Vienna, Austria
[2] Department of Mathematics, Unversità di Pavia, Pavia, Italy
[3] Seminar for Applied Mathematics, ETH Zürich, Zurich, Switzerland

For the Dirichlet problem of the integral fractional Laplacian in a polygon Ω and analytic right-hand side, we show exponential convergence of the *hp*-FEM based on suitably designed meshes, [2]. These meshes are geometrically refined towards the edges and corners of Ω. The geometric refinement towards the edges results in anisotropic meshes away from corners. The use of such anisotropic elements is crucial for the exponential convergence result. These mesh design principles are the same ones as those for *hp*-FEM discretizations of the Dirichlet spectral fractional Laplacian in polygons, for which [1] recently established exponential conver gence.

The *hp*-FEM convergence result relies on the recent [3], where weighted analytic regularity of the solution is shown in a way that captures both the analyticity of the solution in Ω and the singular behavior near the boundary. Near the boundary the solution has an anisotropic behavior: near edges but away from the corners, the solution is smooth in tangential direction and higher order derivatives in normal direction are singular at edges. At the corners, also higher order tangential derivatives are singular. This behavior is captured in terms of weights that are products of powers of the distances from edges and corners.

We will also discuss quadrature aspects of the *hp*-FEM with emphasis on the 1D fractional Laplacian, for which a full analysis is available, [4].

REFERENCES

[1] Lehel Banjai, Jens M. Melenk, and Christoph Schwab. Exponential convergence of *hp* FEM for spectral fractional diffusion in polygons. *Numer. Math.*, 153(1):1–47, 2023.

[2] M. Faustmann, C. Marcati, J.M. Melenk, and Ch. Schwab. Exponential convergence of *hp*-FEM for the integral fractional Laplacian in polygons. *SIAM J. Numer. Anal.*, 61:2601–2622, 2023.

[3] Markus Faustmann, Carlo Marcati, Jens Markus Melenk, and Christoph Schwab. Weighted Analytic Regularity for the Integral Fractional Laplacian in Polygons. *SIAM J. Math. Anal.*, 54(6):6323–6357, 2022.

[4] Björn Bahr, Markus Faustmann, Jens Markus Melenk. an Implementation of *hp*-FEM for the fractional Laplacian, 2023

ANALYTIC AND GEVREY CLASS REGULARITY FOR PARAMETRIC NONLINEAR PROBLEMS

Alexey Chernov and Tung Le

Institut für Mathematematik, Universität Oldenburg, Germany, alexey.chernov@uni-oldenburg.de

We investigate a class of parametric elliptic nonlinear problems, where the coefficients may depend on a high-dimensional parameter. The efficiency of various numerical approximations across the entire parameter space (generalized polynomial chaos, Quasi-Monte Carlo, etc) is closely related to the regularity of the solution with respect to the parameter, and hence the study of the precise estimation of the (mixed) derivatives of the solution is crucial. In this talk we demonstrate that in models with polynomial nonlinearity the analytic (or Gevrey class) regularity of the data translates to the parametric regularity of the solution of same type. In particular, this result is applicable to elliptic eigenvalue problems, semilinear reaction-diffusion problems and incompressible Navier-Stokes equations.

REFERENCES

[1] A. Chernov and T. Le, *Analytic and Gevrey class regularity for parametric elliptic eigenvalue problems and applications*, https://arxiv.org/abs/2306.07010

[2] A. Chernov and T. Le, *Analytic and Gevrey class regularity for parametric semilinear reaction-diffusion problems and applications in uncertainty quantification*, https://arxiv.org/abs/2309.17397

EXPONENTIAL CONVERGENCE OF MIXED hp-FEM FOR THE STATIONARY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH MIXED BOUNDARY CONDITIONS IN POLYGONS

Yanchen He[1] **and Christoph Schwab**[1]

[1] Seminar for Applied Mathematics, ETH Zürich, Switzerland

In this presentation, we discuss the mixed hp-FEM approximations of solutions to the following stationary incompressible Navier-Stokes equations(NSE) with mixed boundary conditions in a polygon Ω:

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} &\quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 &\quad \text{in } \Omega \\ \mathbf{u} = \mathbf{0} &\quad \text{on } \Gamma_D \quad \text{no-slip boundary condition} \\ \sigma(\mathbf{u},p)\mathbf{n} = \mathbf{0} &\quad \text{on } \Gamma_N \quad \text{open boundary condition} \\ (\sigma(\mathbf{u},p)\mathbf{n})\cdot\mathbf{t} = 0 \text{ and } \mathbf{u}\cdot\mathbf{n} = 0 &\quad \text{on } \Gamma_G \quad \text{slip boundary condition} \end{aligned} \tag{1}$$

where $\nu > 0$ is the kinematic velocity, $\Gamma_D, \Gamma_N, \Gamma_G$ are a disjoint partition of the boundary $\Gamma := \partial\Omega$ such that each of them consists of some complete edges of Ω or is an empty set. We assume also that $|\Gamma_D| > 0$. Moreover,

$$\sigma(\mathbf{u},p) := \nu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right) - p\,\mathrm{Id}_2\,.$$

Here Id_2 is the 2×2 identity matrix and $\nabla\mathbf{u}$ denotes the 2×2 matrix of the Cartesian partial derivatives of the components of $\mathbf{u}$.

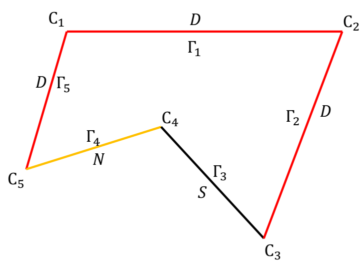

Figure 1: An example for Ω with five edges $\Gamma_i, i = 1,2,3,4,5$.

We denote the space of velocity fields $\mathbf{u}$ of variational solutions to the Navier-Stokes equations (1) as

$$\mathbf{W} = \left\{\mathbf{v}\in[H^1(\Omega)]^2 : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_G\right\}. \tag{2}$$

It can be shown (see [2]) that if $\|f\|_{\mathbf{W}^*}$ is sufficiently small, then there exists a variational solution pair $(\mathbf{u},p)\in\mathbf{W}\times L^2(\Omega)$ and $\mathbf{u}$ is unique in a bounded open ball $B(0,r):=\{\mathbf{u}\in\mathbf{W}:\|\mathbf{u}\|_{\mathbf{W}}<r\}$ with $r>0$ depending on Ω.

Due to the nonsmoothness of the data (e.g., corners on the boundary, changes of boundary conditions), the solutions to (1) may lose full regularity in the vicinity of the corners and thus only have limited regularity in standard Sobolev spaces, i.e., we may have $(\mathbf{u},p)\notin H^2(\Omega)^2\times H^1(\Omega)$ even if $\mathbf{f}$ is analytic in $\overline{\Omega}$. This would weaken standard h-type and p-type FEM regarding convergence rates. A powerful tool to remedy this lack of regularity is *corner-weighted Sobolev spaces*, which still allows higher-order regularity even with nonsmooth data. This type of spaces uses weight functions which vanish at the corners to compensate the loss of regularity. Such spaces have been used to study linear elliptic PDEs with nonsmooth data (see e.g. the pioneering work [3] and also [4, 5]). In this talk we will show based on our recent works [7, 6] that regarding the stationary incompressible NSE (1), with $\mathbf{f}$ that is weighted analytic in $\overline{\Omega}$, the solution pair belongs to certain *weighted analytic function classes*: $(\mathbf{u},p)\in(B^2_{\underline{\beta}}(\Omega))^2\times B^1_{\underline{\beta}}(\Omega)$.

This weighted analyticity implies that solutions to (1) can be approximated with exponential convergence by hp-Finite Element Spaces which uses a combination of geometrical mesh refinements towards the corners and polynomial degree[8]. This property allows us to implement mixed hp-Finite Element Methods (hp-FEM) to remedy the nonsmoothness of the solution.

The implementation of mixed hp-FEM on (1) would require an inf-sup pair of hp-Finite Element Spaces $\mathbf{V}_N\times Q_N$ (here $N\sim\dim(\mathbf{V}_N)\sim\dim(Q_N)$, $N\in\mathbb{N}$) such that its inf-sup constant is independent of mesh sizes h and polynomial degrees k or only depends polynomially or logarithmically on k. Existing works regarding this inf-sup pair $\mathbf{V}_N\times Q_N$ mostly deal with (1) with only no-slip boundary condition (For conforming FE pairs, see e.g., [13, 10, 9], for nonconforming FE pairs, see e.g., [15, 12, 11]).

We present a technique established in [1] which can enrich a stable hp-FE pair $\mathbf{V}_N\times Q_N$ for only no-slip boundary condition such that the enriched pair $\widetilde{\mathbf{V}}_N\times\widetilde{Q}_N\supset\mathbf{V}_N\times Q_N$ is suitable for mixed boundary conditions while its stability is kept. With the help of this technique, we introduce some hp-FE pairs for mixed boundary conditions as enrichment of some hp-FE pairs for no-slip condition listed in above paragraph and show that the corresponding hp-FE solutions $(\mathbf{u}_N,p_N)\in\widetilde{\mathbf{V}}_N\times\widetilde{Q}_N$ would exhibit exponential convergence: There exists $C,b>0$ independent of N such that

$$\|\mathbf{u}_N-\mathbf{u}\|_{\mathbf{V}}+\|p_N-p\|_{L^2(\Omega)}\leq C\exp(-bN^{\frac{1}{3}}).$$

Here $\|\cdot\|_{\mathbf{V}}$ is the H^1-norm if conforming FE pairs are considered and it is a mesh-dependent broken H^1-norm if nonconforming FE pairs are used.

REFERENCES

[1] Y. He and Ch. Schwab, *Exponential convergence of mixed hp-FEM for the stationary incompressible Navier-Stokes equations with mixed boundary conditions in polygons.* In preparation.

[2] M. Orlt and A.M. Sändig, *Regularity of viscous Navier-Stokes flows in nonsmooth domains.* Boundary Value Problems and Integral Equations in Nonsmooth Domains (M. Costabel,

M. Dauge, S. Nicaise, eds.). Marcel Dekker Inc (1995): 185-201.

[3] V. A. Kondrat'ev, *Boundary value problems for elliptic equations in domains with conical or angular points.* Trudy Moskovskogo Matematicheskogo Obshchestva, 16 (1967): 209–292.

[4] I. Babu˘ska and B. Q. Guo, *Regularity of the solution of elliptic problems with piecewise analytic data. I. Boundary value problems for linear elliptic equation of second order.* SIAM J. Math. Anal., 19 (1988): 172–203.

[5] M. Costabel, M. Dauge, and S. Nicaise, *Analytic regularity for linear elliptic systems in polygons and polyhedra.* Math. Models Methods Appl. Sci., 22 (2012): 1250015, 63.

[6] Y. He, C. Marcati, Ch. Schwab. *Analytic regularity for the incompressible Navier-Stokes equations in polygons with mixed boundary conditions.* SIAM Journal on Mathematical Analysis, in press, 2024

[7] C. Marcati, Ch. Schwab. *Analytic Regularity for the incompressible Navier-Stokes Equations in Polygons.* SIAM Journal on Mathematical Analysis 52(3) (2020):2945-2968.

[8] Ch. Schwab. *p- and hp- Finite Element Methods: Theory and Applications in Solid and Fluid Mechanics.* Oxford University Press (1998).

[9] M. Ainsworth and C. Parker. *Mass conserving mixed hp-fem approximations to stokes flow. Part I: Uniform stability.* SIAM Journal on Numerical Analysis, 59(3) (2021): 1218–1244..

[10] M. Ainsworth, P. Coggins. *A uniformly stable family of mixed hp-finite elements with continuous pressures for incompressible flow.* IMA Journal of Numerical Analysis, Volume 22, Issue 2 (2002): 307–327.

[11] S. Stefan. *The inf-sup constant for hp-Crouzeix-Raviart triangular elements.* Computers & Mathematics with Applications 149 (2023): 49-70.

[12] P.L. Lederer, J. Schöberl. *Polynomial robust stability analysis for (div)-conforming finite elements for the Stokes equations.* IMA Journal of Numerical Analysis 38.4 (2018): 1832-1860.

[13] R. Stenberg, M. Suri. *Mixed hp-finite element methods for problems in elasticity and Stokes flow.* Numer. Math., 72 (1996): 367–389.

[14] C. Bernardi, Y. Maday. *Uniform inf-sup conditions for the spectral discretization of the Stokes problem.* Math. Models Methods Appl. Sci., 9 (1999): 395–414.

[15] D. Schötzau, T. Wihler. *Exponential convergence of mixed hp-DGFEM for Stokes flow in polygons.* Numer. Math., 96 (2002): 339–361.

SMOOTHNESS ESTIMATION FOR *hp*-REFINEMENT OF VIRTUAL ELEMENT METHODS

Scott Congreve[1] **and Alice Hodson**[1]

[1] Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75, Prague, Czech Republic

In *hp*-adaptive mesh refinement it is useful to decide automatically, per element, whether to perform element subdivision (*h*-refinement) or whether to enrich the order of the approximation (*p*-refinement). Various algorithms have been developed for this task, cf., [1]. An often used method is based on comparing *a posteriori* error indicators to an expected error based on the expected convergence rate of the last refinement type for each element; cf. [2]. Another approach is based on attempting to estimate the smoothness of the analytical solution in each element based on the numerical solution; for example, for finite elements with Legendre basis it is possible to study the decay of the Legendre coefficients [3].

In this talk, we discuss various methods for attempting to estimate the smoothness of the analytical solution for conforming virtual element methods on second order elliptic partial differential equations. While the virtual element solution on each element is not necessarily polynomial, a projection of the solution and/or gradient into a polynomial space is required. Therefore, we can apply various smoothness estimates based on this projection; for example, if the basis of the polynomial space is selected as the Legendre polynomials we can apply [3] via the basis transformations from [4]. Additionally, more information is available for a virtual element solution than the polynomial projection, including potentially higher order terms, which can provide estimates for the smoothness of the solution.

REFERENCES

[1] W. F. Mitchell and M. A. McClain, *A comparison of hp-adaptive strategies for elliptic partial differential equations*, Technical Report NISTIR 7824, National Institute of Standards and Technology, 2011.

[2] J. M. Melenk and B. I. Wohlmuth, *On residual-based a posteriori error estimation in hp-FEM*, Adv. Comp. Math., 15(1–4):311–331, 2001

[3] P. Houston and E. Süli, *A note on the design of hp-adaptive finite element methods for elliptic partial differential equations*, Comput. Methods Appl. Mech. Engrg., 194(2–5):229–243, 2005

[4] A. Dedner, and A. Hodson, *A framework for implementing general virtual element spaces*, SIAM J. Sci. Comput., (accepted).

CONFORMING VIRTUAL ELEMENT METHOD FOR LINEAR ELLIPTIC EQUATIONS IN NONDIVERGENCE FORM

Guillaume Bonnet[1], **Andrea Cangiani**[2], **and Ricardo H. Nochetto**[3]

[1] Université Paris-Dauphine, Paris, France
[2] International School for Advanced Studies (SISSA), Trieste, Italy, andrea.cangiani@sissa.it
[3] University of Maryland, USA

The Virtual Element Method (VEM) offers a flexible framework for the construction of finite elements with the desired conformity over very general meshes [1]. In particular, H^2-conforming VEM of any order in any dimension are available in the leterature, see eg. [2, 3].

In this talk we present the H^2-conforming VEM for the solution of linear elliptic PDEs in nondivergence form with Cordes coefficients. The analysis relies on the continuous Miranda-Talenti estimate for convex domains and is rather elementary thanks to the conforming of the proposed discretisation. We prove stability and error estimates in H^2, including the effect of quadrature, under minimal regularity of the data. We will also report on a serie of numerical experiments illustrating the interplay of coefficient regularity and convergence rates.

REFERENCES

[1] Beirão da Veiga, L. and Brezzi, F. and Cangiani, A. and Manzini, G. and Marini, L. D. and Russo, A, *Basic Principles of Virtual Element Methods*, Math. Models Methods Appl. Sci., 23(01):199–214, 2013.

[2] Brezzi, F. and Marini, *Virtual Element Methods for Plate Bending Problems*, Comput. Methods in Appl. Mech. and Engrg., 253:455–462, 2013.

[3] Chen, C. and Huang, X. and Wei, H. *H^m-Conforming Virtual Elements in Arbitrary Dimension*, SIAM J. Numer. Anal., 60(6):3099–3123, 2022.

p-ROBUST GLOBAL–LOCAL EQUIVALENCE, p-STABLE LOCAL (COMMUTING) PROJECTORS, AND OPTIMAL ELEMENTWISE hp APPROXIMATION ESTIMATES IN H^1 AND $\boldsymbol{H}(\mathrm{div})$

Théophile Chaumont-Frelet[1], **Leszek Demkowicz**[2] and **Martin Vohralík**[3,4]

[1] Inria, Université de Lille, CNRS, UMR 8524 – Laboratoire Paul Painlevé, 59000 Lille, France
[2] Oden Institute for Computational and Engineering Sciences, 1 University Station, C0200, The University of Texas at Austin, Texas 78712, U.S.A.
[3] Inria, 2 rue Simone Iff, 75589 Paris, France, martin.vohralik@inria.fr
[4] CERMICS, Ecole des Ponts, 77455 Marne-la-Vallée, France

We present some recent advances of the results from [1, 2]. In [3, 4], we prove a polynomial-degree-robust equivalence of two piecewise (Lagrange or Raviart–Thomas) polynomial best approximations of a given target function from H^1 or $\boldsymbol{H}(\mathrm{div})$: 1) globally on the whole computational domain Ω, with the (normal) trace continuity requirement, and a divergence constraint in the $\boldsymbol{H}(\mathrm{div})$ case; 2) locally on each mesh element, without any interelement continuity requirement, and without any constraint on the divergence in the $\boldsymbol{H}(\mathrm{div})$ case. Consequently, we obtain fully h- and p- (mesh-size- and polynomial-degree-) optimal approximation estimates under the minimal Sobolev regularity only requested separately on each mesh element. These two results immediately follow by our construction of a p-stable local (commuting) projector from the entire infinite-dimensional Sobolev space H^1 or $\boldsymbol{H}(\mathrm{div})$ to its finite-dimensional finite element subspace with approximation property that is locally and p-robustly equivalent to that of the discontinuous (unconstrained) elementwise orthogonal projection.

REFERENCES

[1] T. Chaumont-Frelet and M. Vohralík, *p-robust equilibrated flux reconstruction in* $\boldsymbol{H}(\mathrm{curl})$ *based on local minimizations. Application to a posteriori analysis of the curl–curl problem*, SIAM J. Numer. Anal., 61(4):1783–1818, 2023.

[2] T. Chaumont-Frelet and M. Vohralík, *A stable local commuting projector and optimal hp approximation estimates in* $\boldsymbol{H}(\mathrm{curl})$, HAL Preprint 03817302, 2023.

[3] L. Demkowicz and M. Vohralík, *p-robust equivalence of global continuous constrained and local discontinuous unconstrained approximation, a p-stable local commuting projector, and optimal elementwise hp approximation estimates in* $\boldsymbol{H}(\mathrm{div})$, HAL Preprint 04503603, 2024.

[4] M. Vohralík, *p-robust equivalence of global continuous and local discontinuous approximation, a p-stable local projector, and optimal elementwise hp approximation estimates in* H^1, HAL Preprint 04436063, 2024.

POLYNOMIAL EXTENSION OPERATORS AND APPLICATIONS

Mark Ainsworth[1], Charles Parker[2], and Endre Süli[2]

[1] Division of Applied Mathematics, Brown University, USA
[2] Mathematical Institute, University of Oxford, UK, charles.parker@maths.ox.ac.uk

We present recent results on polynomial extension operators, a key theoretical tool in the analysis of p- and hp-finite element methods. These operators are right-inverses of the trace and higher-order trace operators on a reference element that preserve polynomials in the sense that if the datum is the trace (or higher-order trace) of a polynomial, then the extension is a polynomial of the same degree. We then discuss two applications with numerical examples: uniform stability of high-order mixed methods and uniform preconditioning of parameter-dependent problems.

REFERENCES

[1] M. Ainsworth and C. Parker, *H^2-stable Polynomial Liftings on Triangles*, SIAM J. Numer. Anal., 58(3):1867–1892, 2020.

[2] C. Parker and E. Süli, *Stable Lifting of Polynomial Traces on Triangles*, SIAM J. Numer. Anal., 62(2):692–717, 2024.

[3] C. Parker and E. Süli, *Stable Lifting of Polynomial Traces on Tetrahedra*, preprint, `https://arxiv.org/abs/2402.15789`, 2024.

HIGH-ORDER PROJECTION-BASED UPWIND METHOD FOR IMPLICIT LARGE EDDY SIMULATION

Philip L. Lederer[1], **Xaver Mooslechner**[2] **and Joachim Schöberl**[2]

[1] University of Twente, Department of Applied Mathematics, Enschede, Netherlands, p.l.lederer@utwente.nl
[2] TU Wien, Institute of Analysis and Scientific Computing, Austria, Germany

We assess the ability of high-order (hybrid) discontinuous Galerkin methods to simulate under-resolved turbulent flows. The capabilities of the mass conserving mixed stress-yielding method as structure resolving large eddy simulation (LES) solver are examined. A comparison of a variational multiscale model to no-model or an implicit model approach is presented via numerical results. In addition, we present a novel approach for turbulent modeling in wall-bounded flows which can be interpreted as an extension of the classical variational multiscale idea to implicit LES via discontinuous Galerkin methods. This new technique called high-order projection-based upwind (HOPU) technique provides a more accurate representation of the actual subgrid scales in the near wall region and gives promising results for highly under-resolved flow problems. We consider the turbulent channel flow and periodic hill flow problem as well as a flow over an Eppler airfoil.

REFERENCES

[1] P. L. Lederer, X. Mooslechner and J. Schöberl, *High-order projection-based upwind method for implicit large eddy simulation*, Journal of Computational Physics, Vol. 493, 2023, doi:10.1016/j.jcp.2023.112492

AN hp-ADAPTIVE STRATEGY BASED ON LOCALLY PREDICTED ERROR REDUCTIONS

Patrick Bammer[1], **Andreas Schröder**[1] **and Thomas P. Wihler**[2]

[1] Fachbereich Mathematik, Paris Lodron Universität Salzburg, Austria,
patrick.bammer@plus.ac.at
[2] Mathematisches Institut, Universität Bern, Sidlerstr. 5, Switzerland

In this talk, an hp-adaptive strategy for variational equations associated with self-adjoint elliptic boundary value problems is introduced, which neither relies on classical a posteriori error estimators nor on smoothness indicators to steer the adaptivity. Instead, the approach compares the predicted error reduction that can be expressed in terms of local modifications of the degrees of freedom in the underlying discrete approximation space. The computations related to the proposed prediction strategy involve low-dimensional linear problems that are computationally inexpensive and highly parallelizable. The concept is first presented in an abstract Hilbert space framework, before it is applied to hp-finite element discretizations. For the latter, an explicit construction of p- and hp-enrichment functions is given and a constraint coefficient technique allows a highly efficient computation of the predicted error reductions. The applicability and effectiveness of the resulting hp-adaptive strategy is finally illustrated with some numerical examples.

A P-VERSION OF CONVOLUTION QUADRATURE IN WAVE PROPAGATION

Alexander Rieder[1]

[1] Institute for Analysis and Scientific Computing, TU Wien, Austria

In this talk, we present a novel approach towards boundary element methods for wave propagation. It is based on the convolution quadrature idea by Lubich [Lub88], but instead of relying on reducing the timestep size in order to achieve higher accuracy, we use the p-refinement paradigm of increasing the order of the method while keeping the timestep size fixed. To get an easily computable and analyzable scheme, we rely on the ideas of discontinuous Galerkin timestepping [SS00]. This allows us to design a scheme which is root-exponentially convergent for certain very smooth initial conditions. We talk about possibilities to analyze this new scheme, as well its practical implementation and challenges.

REFERENCES

[Rie24] A. Rieder "A p-version of convolution quadrature in wave propagation", (2024), https://arxiv.org/abs/2402.17712

[Lub88] C. Lubich. "Convolution quadrature and discretized operational calculus. I." *Numer. Math.*, 52(2):129–145, 1988.

[SS00] D. Schötzau and C. Schwab. "An *hp* a priori error analysis of the DG time-stepping method for initial value problems." *Calcolo*, 37(4):207–232, 2000.

EXPONENTIAL CONVERGENCE OF *HP*-ILGFEM FOR SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS WITH MONOMIAL REACTION

Yanchen He[1], Paul Houston[2], Christoph Schwab[1], and Thomas P. Wihler[3]

[1] Seminar für Angewandte Mathematik, ETH Zürich, Switzerland
[2] School of Mathematical Sciences, University of Nottingham, UK
[3] Mathematisches Institut, Universität Bern, Switzerland

We study the fully explicit numerical approximation of a semilinear elliptic boundary value model problem, which features a monomial reaction and analytic forcing, in a bounded 2d polygon. In particular, we analyze the convergence of *hp*-type iterative linearized Galerkin (*hp*-ILG) solvers. Our convergence analysis is carried out for conforming *hp*-finite element discretizations on sequences of regular, simplicial meshes with geometric corner refinement, with polynomial degrees increasing in sync with the geometric mesh refinement towards the corners of the domain. For a sequence of discrete solutions generated by the ILG solver, with a stopping criterion that is consistent with the exponential convergence of the exact *hp*-FE solution, we prove exponential convergence in H^1 to the unique weak solution of the boundary value problem. Numerical experiments illustrate the exponential convergence of the numerical approximations obtained from the proposed scheme in terms of the number of degrees of freedom as well as of the computational complexity involved.

REFERENCES

[1] Y. He, P. Houston, Ch. Schwab, and T. P. Wihler, *Exponential Convergence of hp-ILGFEM for semilinear elliptic boundary value problems with monomial reaction*, in preparation.

[2] Y. He and Ch. Schwab, *Analytic regularity and solution approximation for a semilinear elliptic partial differential equation in a polygon*, Calcolo, 61:11, 2024.

ON THE REGULARITY ASSUMPTIONS IN THE ANALYSIS OF MSFEM

Rutger Biezemans[1], **Claude Le Bris**[2], **Frédéric Legoll**[2], **Alexei Lozinski**[3]

[1] CEA Paris-Saclay, France
[2] ENPC ParisTech & INRIA Paris, France
[3] Laboratoire de mathématiques de Besançon, France, alexei.lozinski@univ-fcomte.fr

The Multiscale Finite Element Method (MsFEM) is one of the well-established numerical approaches dedicated to multiscale problems. Its theoretical analysis, cf. [1], is usually performed supposing that the oscillating coefficients in the governing equations are periodic and sufficiently smooth (at least, Hölder continuous). Moreover, the homogenized solution is supposed to be in $W^{1,\infty}$. In a recent article [2], the regularity assumptions were significantly weaken in the case of a scalar diffusion equation. The oscillating coefficient can be now supposed in L^∞ without further assumptions, and the homogenized solution in H^2. However, some extra regularity should be still assumed in the case of systems of elliptic equations.

In this talk, we present another strategy for the error analysis, first introduced in [3]. It allows us to explore several variants of MsFEM (linear BC, Crouzeix-Raviart, mixed, ...) under the minimal regularity assumptions (the oscillating coefficient in L^∞, the homogenized solution in H^s, $s > 3/2$). Both the case of scalar equations and that of the systems of equations are covered.

REFERENCES

[1] Y. Efendiev and T. Y. Hou. *Multiscale finite element methods: theory and applications*, Springer, 2009.

[2] P. Ming and S. Song, *Error estimate of multiscale finite element method for periodic media revisited*, Multiscale Model. Simul., 22(1):106–124, 2024.

[3] R. Biezemans, *Multiscale methods: non-intrusive implementation, advection-dominated problems and related topics*, PhD thesis, ENPC ENPC ParisTech, 2023.

TRAINING AND ENRICHMENT BASED ON A RESIDUAL LOCALIZATION STRATEGY

Tim Keil[1], Mario Ohlberger[1], Felix Schindler[1], and Julia Schleuß[1]

[1] Institute for Analysis and Numerics, University of Münster, Germany

To tackle parametric partial differential equations with highly heterogeneous coefficients, we propose an adaptive localized basis construction procedure based on both offline training and online enrichment. First, in the offline phase, a set of problem-adpated local basis functions is precomputed. Next, in the online phase, we use a localized residual-based a posteriori error estimator to investigate the accuracy of the reduced solution for any given new parameter. As the error estimator is localized, we can exploit it to adaptively enrich the reduced solution space locally where needed. The approach thus guarantees the accuracy of reduced solutions given any possibly insufficient reduced basis that was constructed during the offline phase. Numerical experiments demonstrate the efficiency of the proposed method.

REFERENCES

[1] T. Keil, M. Ohlberger, F. Schindler, and J. Schleuß, *Training and enrichment based on a residual localization strategy*, In preparation, 2024.

LOCALIZED ORTHOGONAL DECOMPOSITION FOR NONLINEAR NONMONOTONE PDES

Maher Khrais[1], **Barbara Verfürth**[1]

[1] Institut für Numerische Simulation, Universität Bonn, Germany, khrais@ins.uni-bonn.de

In this talk we present a multiscale method in the framework of the localized orthogonal decomposition (LOD) for solving nonlinear nonmonotone elliptic equations of the following form

$$-\nabla \cdot (\alpha(x,u)\nabla u) = f.$$

In particular we present the construction of the problem-adapted multiscale space motivated by the ideas presented for monotone problems in [2]. We also describe some linearization techniques that are used to convert the corrector problem into a linear elliptic problem that can be solved efficiently and locally on the fine scale in order to define the basis of the problem-adapted space [1]. Then we apply Galerkin method using the new constructed multiscale space as trial and test spaces. We also discuss the elements of a priori error analysis of the method without any structural assumptions on the nonlinear coefficient, for example, periodicity, or scale separation. We then show some numerical experiments that elaborate the theoretical results and support the applicability of our numerical approach to nonmonotone PDEs e.g., stationary Richards equation.

REFERENCES

[1] D. Peterseim, *Variational multiscale stabilization and the exponential decay of fine-scale correctors*, Lect. Notes Comput. Sci. Eng., 114 :341–367, 2021.

[2] B. Verfürth, *Numerical homogenization for nonlinear strongly monotone problems*, IMA J. Numer. Anal., 42(2):1313–1338, 2022.

MULTISCALE FINITE ELEMENT METHODS FOR ADVECTION-DIFFUSION PROBLEMS

Rutger Biezemans[1], **Claude Le Bris**[1], **Frédéric Legoll**[1] **and Alexei Lozinski**[2]

[1] Ecole des Ponts ParisTech and Inria, France, frederic.legoll@enpc.fr
[2] Université de Franche-Comté, CNRS, LmB, Besançon, France

The Multiscale Finite Element Method (MsFEM) is a finite element (FE) approach that allows to solve partial differential equations (PDEs) with highly oscillatory coefficients on a coarse mesh, i.e. a mesh with elements of size much larger than the characteristic scale of the heterogeneities. To do so, MsFEMs use pre-computed basis functions, adapted to the differential operator, thereby taking into account the small scales of the problem [2].

When the PDE contains dominating advection terms, naive FE approximations lead to spurious oscillations, even in the absence of oscillatory coefficients. Stabilization techniques (such as SUPG) are to be adopted [3].

In this work (see [1]), we consider multiscale advection-diffusion problems in the convection-dominated regime. We discuss different ways to define the MsFEM basis functions, and how to combine the approach with stabilization-type methods. In particular, we show that methods using suitable bubble functions and Crouzeix-Raviart type boundary conditions for the local problems turn out to be very effective.

REFERENCES

[1] R. Biezemans, *Multiscale problems: non-intrusive implementation, advection-dominated problems and related issues*, PhD thesis, Ecole des Ponts ParisTech, 2023 (manuscript available at `https://pastel.hal.science/tel-04481733`).

[2] Y. Efendiev and T. Hou, *Multiscale Finite Element Methods*, Springer-Verlag, 2009.

[3] A. Quarteroni, *Numerical Models for Differential Problems*, Springer-Verlag, 2014.

RELIABLE COARSE SCALE APPROXIMATION OF SPATIAL NETWORK MODELS

Moritz Hauck[1], **Axel Målqvist**[1] **and Roland Maier**[2]

[1] Department of Mathematical Sciences, University of Gothenburg and Chalmers University of Technology, Sweden, hauck@chalmers.se
[2] Institute for Applied and Numerical Mathematics, Karlsruhe Institute of Technology, Germany

In this talk, we present a multiscale approach for the reliable coarse-scale approximation of spatial network models represented by a linear system of equations with respect to the nodes of a graph. The method is based on the ideas of the localized orthogonal decomposition (LOD) strategy and is constructed in a fully algebraic way. This allows to apply the method to geometrically challenging objects such as corrugated cardboard. In particular, the method can also be applied to finite difference or finite element discretizations of elliptic partial differential equations, yielding an approximation with similar properties as the LOD in the continuous setting. We present a rigorous error analysis of the proposed method under suitable assumptions on the network. Moreover, numerical examples are presented that underline our theoretical results.

REFERENCES

[1] M. Hauck, R. Maier, A. Målqvist, *An algebraic multiscale method for spatial network models*, arXiv preprint arXiv:2312.09752, 2023.

IMPROVING SUB-MESOSCALE RESOLVING OCEAN SIMULATIONS

Philip Freese[1], **Fabricio Lapolli**[4], **Yen-Sen Lu**[2], **Thibaut Lunet**[1], **Daniel Ruprecht**[1], **Maximilian Witte**[3]

[1] Institute of Mathematics, Hamburg University of Technology, Germany, philip.freese@tuhh.de
[2] Jülich Supercomputing Center, Germany
[3] Germany Climate Research Center, Germany
[4] Max-Planck Institute for Meteorology, Germany

We present our approach to improve the performance of the ICON-O model, an operational finite-volume based ocean code.
On the one hand it is based on Spectral Deferred Corrections (SDC), a time-integration method that iteratively computes the stages of a fully implicit collocation method using a preconditioned iteration. The proposed new SDC variant is based on diagonal preconditioning, which allows computations for SDC iterations to be performed in parallel-in-time.
On the other hand, we use machine learning techniques, also known as super-resolution, to enhance low-resolution outputs with high-resolution data. This correction is performed throughout the time stepping, allowing for additional performance gains due to the coarser grid structure while preserving small scale features.
The overall goal is to enable sub-mesoscale resolving simulations on climatologically relevant time scales. We present our recent results showing an improved throughput due to the combination of more efficient algorithms, reduced spatial resolution via ML correction and improved scalability.

A MULTISCALE GENERALIZED FEM BASED ON LOCALLY OPTIMAL SPECTRAL APPROXIMATIONS FOR HIGH-FREQUENCY WAVE PROBLEMS

Christian Alber[1], **Chupeng Ma**[2] **and Robert Scheichl**[1]

[1] Institute for Applied Mathematics and Interdisciplinary Center for Scientific Computing, Heidelberg University, Germany
[2] School of Sciences, Great Bay University, China, chupeng.ma@gbu.edu.cn

In this talk we present a generalized finite element method with optimal local approximation spaces for solving high-frequency wave problems with general L^∞ coefficients. The local spaces are built from selected eigenvectors of carefully designed local eigenvalue problems defined on local solution spaces. Wavenumber-explicit exponential convergence rates for Helmholtz, Maxwell, and elastic wave equations are established. Numerical results are provided to confirm the theoretical analysis and to validate the proposed method.

REFERENCES

[1] C. Ma, C. Alber and R. Scheichl, *Wavenumber explicit convergence of a multiscale generalized finite element method for heterogeneous Helmholtz problems*, SIAM J. Numer. Anal., 61(3):1546-1584, 2023.

[2] C. Ma, *A unified framework for multiscale spectral generalized FEMs and low-rank approximations to multiscale PDEs*, arXiv preprint arXiv:2311.08761, 2023.

SEQUENTIAL QUADRATIC PROGRAMMING FOR ACOUSTIC FULL WAVEFORM INVERSION

Luis Ammann[1] **and Irwin Yousept**[1]

[1] Fakultät für Mathematik, Universität Duisburg-Essen, Germany

In this talk, the SQP method applied to a hyperbolic PDE-constrained optimization problem is considered. The model arises from the acoustic full waveform inversion in the time domain [1]. The analysis is mainly challenging due to the involved hyperbolicity and second-order bilinear structure. This character leads to undesired effects of regularity loss in the SQP iteration calling for a substantial extension of developed parabolic techniques. We propose and analyze a strategy for the well-posedness and convergence analysis based on the use of a smooth-in-time initial condition, a tailored self-mapping operator, and a two-step estimation process along with Stampacchia's method for hyperbolic equations. Our final theoretical result is the R-superlinear convergence of the SQP method.

REFERENCES

[1] L. Ammann, I. Yousept *Acoustic full waveform inversion via optimal control: first- and second-order analysis*, Siam J. Control Optim., 61(4):2468 2496, 2023.

A HYBRID DISCONTINUOUS GALERKIN METHOD WITH STABILIZATIONS FOR LINEARIZED NAVIER–STOKES EQUATIONS

Dongwook Shin[1], **Youngmok Jeon**[1] **and Eun-Jae Park**[2]

[1] Department of Mathematics, Ajou University, Republic of Korea, dws@ajou.ac.kr
[2] School of Mathematics and Computing (Computational Science and Engineering), Yonsei University, Republic of Korea

In this talk, we focus on a hybrid discontinuous Galerkin (HDG) method and stabilization techniques for the linearized Navier-Stokes equations. The method allows for arbitrarily high-order approximations while preserving local conservation properties. The numerical solution of the problem is decomposed into linearized auxiliary problems of Oseen type. HDG methods offer high-order accuracy and significant reduction in the number of global degrees of freedom. In the previous study [1], the lifting operator associated with trace variables was analyzed for quadratic and cubic approximation on rectangular elements. Subsequent research [2] has generalized the result by proving that the lifting operator is injective for any polynomial degree. Furthermore, Optimal error estimates in the energy norm have been derived by introducing non-standard projection operators. However, since stabilization techniques were not introduced in [2], stability is guaranteed only under a specific condition. Therefore, we have investigated several stabilization techniques to ensure that the method is stable without the condition. Several numerical results support the theoretical results and demonstrate the performance of the algorithm.

REFERENCES

[1] Y. Jeon and E.-J. Park, *New locally conservative finite element methods on a rectangular mesh*, Numerische Mathematik, 123(1), pp. 97-119, 2013.

[2] D. Shin, Y. Jeon and E.-J. Park, *Analysis of hybrid discontinuous Galerkin methods for linearized Navier–Stokes equations*, Numerical Methods for Partial Differential Equations, 39, pp. 304-328, 2023.

EXPLICIT RK SCHEMES WITH HYBRID HIGH-ORDER METHOD FOR THE FIRST-ORDER FORMULATION OF THE WAVE EQUATION

Alexandre Ern[1] **and <u>Rekha Khot</u>**[2]

[1] CERMICS, Ecole des Ponts, 77455 Marne-la-Vallee Cedex 2, France; Inria, Paris 75012, France, email: alexandre.ern@enpc.fr
[2] Inria, Paris 75012, France, email: rekha.khot@inria.fr

In this talk we present the fully discrete theoretical analysis of the first-order formulation of the acoustic wave equation proposed and numerically investigated in [1]. We employ the explicit Runge-Kutta (ERK) schemes for the time discretization and the hybrid high-order (HHO) methods for the space discretization. The HHO design focuses on the mixed-order formulation, where the polynomial degree of the cell unknowns is one degree higher than that of the face unknowns. This benefits in the explicit nature of the method in addition to the simpler stabilization form, and consequently the consistency error bound can be dealt easily with the help of the new HDG+ projection developed in [2] for the flux component. We prove that under specific CFL conditions depending on the s-stage ERK scheme, the discrete error converges with expected rates.

REFERENCES

[1] E. Burman, O. Duran and A. Ern, *Hybrid high-order methods for the acoustic wave equation in the time domain*, Communications on Applied Mathematics and Computation, 4(2):597-633, 2022.

[2] S. Du and F. J. Sayas, *A note on devising HDG+ projections on polyhedral elements*, Math. Comp., 90(327):65-79, 2021.

QUASI-MONTE CARLO FINITE ELEMENT APPROXIMATION OF THE NAVIER-STOKES EQUATIONS WITH INITIAL DATA MODELED BY LOG-NORMAL RANDOM FIELDS

Seungchan Ko[1], **Guanglian Li**[2] **and Yi Yu**[3]

[1] Department of Mathematics, Inha University, Republic of Korea, scko@inha.ac.kr
[2] Department of Mathematics, The University of Hong Kong, Hong Kong, lotusli@maths.hku.hk
[3] School of Mathematics and Information Science, Guangxi University, PR China, yiyu@gxu.edu.cn

In this talk, I will analyze the numerical approximation of the Navier-Stokes problem over a bounded polygonal domain, where the initial condition is modeled by a log-normal random field. This problem usually arises in the area of uncertainty quantification. We aim to compute the expectation value of linear functionals of the solution to the Navier-Stokes equations and perform a rigorous error analysis for the problem. In particular, our method includes the finite element, fully-discrete discretizations, truncated Karhunen-Loeve expansion for the realizations of the initial condition, and lattice-based quasi-Monte Carlo (QMC) method to estimate the expected values over the parameter space. Our QMC analysis is based on randomly-shifted lattice rules for the integration over the domain in high-dimensional space, which guarantees faster error decays compared with the rate for the classical Monte Carlo method. To the best of our knowledge, this is the first theoretical QMC analysis for the nonlinear partial differential equation.

REFERENCES

[1] S. Ko, G. Li and Y. Yu, *Quasi-Monte Carlo finite element approximation of the Navier-Stokes equations with initial data modeled by log-normal random fields*, arXiv preprint arXiv:2210.15572, 2022.

OPTIMAL LONG-TIME DECAY RATE OF NUMERICAL SOLUTIONS FOR NONLINEAR TIME-FRACTIONAL EVOLUTIONARY EQUATIONS

Dongling Wang[1] **and Martin Stynes**[2]

[1] School of Mathematics and Computational Science, Xiangtan University, China, wdymath@xtu.edu.cn
[2] Applied and Computational Mathematics Division, Beijing Computational Science Research Center, China, m.stynes@csrc.ac.cn

The solution of the nonlinear initial-value problem $\mathcal{D}_t^\alpha y(t) = -\lambda y(t)^\gamma$ for $t > 0$ with $y(0) > 0$, where $\mathcal{D}_t^\alpha$ is the Caputo derivative of order $\alpha \in (0,1)$ and λ, γ are positive parameters, is known to exhibit $O(t^{-\alpha/\gamma})$ decay as $t \to \infty$. No corresponding result for any discretisation of this problem has previously been proved. In the present paper it is shown that for the class of complete monotonicity-preserving ($\mathcal{CM}$-preserving) schemes (which includes the L1 and Grünwald-Letnikov schemes) on uniform meshes $\{t_n := nh\}_{n=0}^\infty$, the discrete solution also has $O(t_n^{-\alpha/\gamma})$ decay as $t_n \to \infty$. This result is then extended to $\mathcal{CM}$-preserving discretisations of certain time-fractional nonlinear subdiffusion problems such as the time-fractional porous media and p-Laplace equations. For the L1 scheme, the $O(t_n^{-\alpha/\gamma})$ decay result is shown to remain valid on a very general class of nonuniform meshes. Our analysis uses a discrete comparison principle with discrete subsolutions and supersolutions that are carefully constructed to give tight bounds on the discrete solution. Numerical experiments are provided to confirm our theoretical analysis.

REFERENCES

[1] D. Wang, M. Stynes *Optimal long-time decay rate of numerical solutions for nonlinear time-fractional evolutionary equations*, SIAM J. Numer. Anal., 61(5):2011–2034, 2023.

FILTERED FINITE DIFFERENCE METHODS FOR HIGHLY OSCILLATORY SEMILINEAR HYPERBOLIC SYSTEMS

Christian Lubich[1] **and** **Yanyan Shi**[1]

[1] Mathematisches Institut, Univ. Tübingen, D-72076 Tübingen, Germany

Many wave phenomena are modelled by PDEs with fast oscillating solutions. We are interested in a class of semilinear hyperbolic systems with a trilinear nonlinearity. Both the differential equation and the initial data contain the inverse of a small parameter. Since the solution is highly oscillatory in space and time, traditional methods require very fine discretization and thus not feasible. This talk presents two filtered finite difference schemes. By introducing proper filter functions, we can prove error estimates for large step size and mesh width. The analysis is done by comparing modulated Fourier expansions of numerical approximation and exact solution. Numerical experiments illustrate the theoretical results.

A STABILIZATION-FREE MIXED DG METHOD FOR FLUID-STRUCTURE INTERACTION

Eric Chung[1] **and Lina Zhao**[2]

[1] Department of Mathematics, the Chinese University of Hong Kong, Hong Kong SAR
[2] Department of Mathematics, City University of Hong Kong, Hong Kong SAR,
linazha@cityu.edu.hk

In this talk we present a stabilization-free DG method in stress-velocity formulation for fluid-structure interaction problem. A unified mixed formulation is employed for the Stokes equations and the elastodynamic equations. We use the standard polynomial space with strong symmetry to define the stress space, and use the broken $H(\text{div};\Omega)$-conforming space of the same degree to define the vector space in a careful way such that the resulting scheme is stable without resorting to any stabilization. The transmission conditions can be incorporated naturally without resorting to additional variables or Nitsche-type stabilization owing to the bespoke construction of the discrete formulation. To show the optimal convergence, we establish a new projection operator for the stress space whose definition accounts for traces of the method. Several numerical experiments are presented to verify the proposed theories.

A UNIFORMLY ACCURATE METHOD FOR THE KLEIN-GORDON-DIRAC SYSTEM IN THE NONRELATIVISTIC REGIME

Yongyong Cai[1] **and Wenfan Yi**[2]

[1] Beijing Normal University, China, yongyong.cai@bnu.edu.cn
[2] Hunan University, China

In this talk, we present a multiscale time integrator Fourier pseudospectral (MTI-FP) method for discretizing the massive Klein-Gordon-Dirac (KGD) system which involves a small dimensionless parameter $0 < \varepsilon \leq 1$. In the nonrelativistic limit regime, the KGD system admits rapid oscillations in time as $\varepsilon \to 0^+$. In addition, the nonlinear Yukawa interaction and the indefinite Dirac operator bring other significant difficulties. The main idea of the MTI-FP method is to construct a precise multiscale decomposition by the frequency (MDF) to the solution of the KGD system at each time step and then employ the Fourier pseudospectral discretization for the spatial derivatives followed with the exponential wave integrator (EWI) for the time marching. This approach is explicit, easy to implement and preforms significantly better than the classical methods in the literature. More specifically, we rigorously establish the uniform error bounds at $O(\tau + h^{m_0-1})$ for all $\varepsilon \in (0, 1]$ and optimal quadratic temporal error bounds at $O(\tau^2)$ in the $\varepsilon = O(1)$ regime, where τ is the time step size, h is mesh size and m_0 depends on the regularity of the solution. Extensive numerical results demonstrate that our error bounds are optimal and sharp. Finally, we apply the MTI-FP method to numerically study the nonrelativistic limit behaviors of the KGD system when $\varepsilon \to 0^+$.

REFERENCES

[1] Y. Cai and W. Yi, *A uniformly accurate method for the Klein-Gordon-Dirac system in the nonrelativistic regime*, J. Comput. Phys., 486: 112105, 2023.

ANALYSIS OF AN INTERIOR PENALTY DG METHOD FOR THE QUAD-CURL PROBLEM

Gang Chen[1], **Weifeng Qiu**[2] **and Liwei Xu**[3]

[1] SIchuan University, China
[2] City University of Hong Kong, weifeqiu@cityu.edu.hk
[3] University of electronic science and technology, China

The quad-curl term is an essential part of the resistive magnetohydrodynamic equation and the fourth-order inverse electromagnetic scattering problem, which are both of great significance in science and engineering. It is desirable to develop efficient and practical numerical methods for the quad-curl problem. In this presentation we first present some new regularity results for the quad-curl problem on Lipschitz polyhedron domains, and then propose a mixed finite element method for solving the quad-curl problem. With a novel discrete Sobolev imbedding inequality for the piecewise polynomials, we obtain stability results and derive error estimates based on a relatively low-regularity assumption of the exact solution.

REFERENCES

[1] G. Chen, W. Qiu and L. Xu, *Analysis of an interior penalty DG method for the quad-curl problem*, IMA Journal of Numerical Analysis, Vol. 41, No. 4, 10., 2021.

NUMERICAL ANALYSIS OF QUANTITATIVE PHOTOACOUSTIC TOMOGRAPHY IN A DIFFUSIVE REGIME

Zhi Zhou

The Hong Kong Polytechnic University, Hong Kong SAR, China

In this talk, we consider the numerical analysis of quantitative photoacoustic tomography, which is modeled as an inverse problem to quantitatively reconstruct the diffusion and absorption coefficients in a second-order elliptic equation, utilizing multiple internal measurements. We show a conditional stability in L^2 norm, under some provable positive conditions, assuming randomly chosen boundary excitation data. Building upon this conditional stability, we propose and analyze a numerical reconstruction scheme based on an output least squares formulation, employing finite element discretization. We provide an *a priori* error estimate for the numerical reconstruction, serving as a valuable guideline for selecting computational parameters. Several numerical examples will presented to illustrate the theoretical results.

A MULTISCALE METHOD FOR THE WAVE EQUATIONS

Eric Chung[1]

[1] Department of Mathematics, The Chinese University of Hong Kong,
tschung@math.cuhk.edu.hk

In this talk, we consider a class of heterogeneous wave equations and employ the constraint energy minimizing generalized multiscale finite element method (CEM-GMsFEM) to solve this problem. The proposed method provides a flexible framework to construct crucial multiscale basis functions for approximating the pressure and velocity. These basis functions are constructed by solving a class of local auxiliary optimization problems over the eigenspaces that contain local information on the heterogeneity. Techniques of oversampling are adapted to enhance the computational performance. The convergence of the proposed method is proved and illustrated by several numerical tests. The research is partially supported by Hong Kong RGC General Research Fund (Projects: 14304021 and 14302620).

REFERENCES

[1] E. Chung and S.-M. Pun, *Computational multiscale methods for first-order wave equation using mixed CEM-GMsFEM*, J. Comput. Phys., 409:109359, 2020.

A HYBRID ITERATIVE METHOD BASED ON MIONET FOR PDES: THEORY AND NUMERICAL EXAMPLES

Jun Hu

Peking University, China

We propose a hybrid iterative method based on MIONet for PDEs, which combines the traditional numerical iterative solver and the recent powerful machine learning method of neural operator, and further systematically analyze its theoretical properties, including the convergence condition, the spectral behavior, as well as the convergence rate, in terms of the errors of the discretization and the model inference. We show the theoretical results for the frequently-used smoothers, i.e. Richardson (damped Jacobi) and Gauss-Seidel. We give an upper bound of the convergence rate of the hybrid method w.r.t. the model correction period, which indicates a minimum point to make the hybrid iteration converge fastest. Several numerical examples including the hybrid Richardson (Gauss-Seidel) iteration for the 1-d (2-d) Poisson equation are presented to verify our theoretical results, and also reflect an excellent acceleration effect. As a meshless acceleration method, it is provided with enormous potentials for practice applications.

SCATTERING AND UNIFORM IN TIME ERROR ESTIMATES FOR SPLITTING METHOD IN NLS

Réemi Carles[1], **Chunmei Su**[2]

[1] CNRS IRMAR-UMR6625,Univ Rennes, 35000 Rennes, France
[2] Yau Mathematical Sciences Center, Tsinghua University, 10084 Beijing, China,
sucm@tsinghua.edu.cn

We consider the nonlinear Schrödinger equation with a defocusing nonlinearity which is mass-supercritical and energy-subcritical. We prove uniform in time error estimates for the Lie-Trotter time splitting discretization. This uniformity in time is obtained thanks to a vectorfield which provides time decay estimates for the exact and numerical solutions. This vectorfield is classical in scattering theory and requires several technical modifications compared to previous error estimates for splitting methods.

REFERENCES

[1] R. Carles and C. Su, *Scattering and uniform in time error estimates for splitting method in NLS*, Found. Comput. Math., doi.org/10.1007/s10208-022-09600-9, 2022.

ASYMPTOTIC-PRESERVING HDG METHOD FOR THE WESTERVELT QUASILINEAR WAVE EQUATION

Sergio Gómez[1] and **Mostafa Meliani**[2]

[1] Department of Mathematics and Applications, University of Milano-Bicocca, Italy, sergio.gomezmacias@unimib.it
[2] Department of Mathematics, Radboud University, The Netherlands.

In this talk, we discuss the asymptotic-preserving properties of an hybridizable discontinuous Galerkin (HDG) method for the Westervelt model of ultrasound waves [1]:

$$\begin{cases} (1 + 2k\partial_t\psi)\partial_{tt}\psi - c^2\Delta\psi - \delta\Delta(\partial_t\psi) = 0 & \text{in } \Omega \times (0, T], \\ \psi = 0 & \text{on } \partial\Omega \times (0, T), \\ \psi(\mathbf{x}, 0) = \psi_0(\mathbf{x}), \qquad \psi_t(\mathbf{x}, 0) = \psi_1(\mathbf{x}) & \text{in } \Omega. \end{cases}$$

More precisely, we show that the proposed HDG method is robust with respect to small values of the sound diffusivity (damping) parameter δ. To do so, we first derive high-order energy stability estimates and *a priori* error bounds that are independent of δ. Such estimates are then used to show that, when $\delta \to 0^+$, the method remains stable and the discrete acoustic velocity potential $\psi_h^{(\delta)}$ converges to the singular vanishing dissipation limit $\psi_h^{(0)}$. Moreover, the method also provides an optimal convergent approximation of the acoustic particle velocity variable $\boldsymbol{v} = \nabla\psi$.

REFERENCES

[1] P. J. Westervelt. Parametric acoustic array. The Journal of the Acoustical Society of America, **35(4)**, (1963), pp. 535–537.

IMPROVED UNIFORM ERROR BOUNDS ON TIME-SPLITTING METHODS FOR LONG-TIME DYNAMICS OF THE NONLINEAR KLEIN-GORDON EQUATION WITH WEAK NONLINEARITY

Weizhu Bao[1], Yongyong Cai[2] and Yue Feng[3]

[1] Department of Mathematics, National University of Singapore, Singapore
[2] Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, China
[3] School of Mathematics and Statistics, Xi'an Jiaotong University, China, yue.feng@xjtu.edu.cn

In this talk, we establish improved uniform error bounds on time-splitting methods for the long-time dynamics of the nonlinear Klein–Gordon equation (NKGE) with weak cubic nonlinearity, whose strength is characterized by ε^2 with $0 < \varepsilon \leq 1$ a dimensionless parameter. Actually, when $0 < \varepsilon \ll 1$, the NKGE with $O(\varepsilon^2)$ nonlinearity and $O(1)$ initial data is equivalent to that with $O(1)$ nonlinearity and small initial data of which the amplitude is at $O(\varepsilon)$. We begin with a semi-discretization of the NKGE by the second-order time-splitting method, and followed by a full-discretization by the Fourier spectral method in space. Employing the regularity compensation oscillation (RCO) technique which controls the high frequency modes by the regularity of the exact solution and analyzes the low frequency modes by phase cancellation and energy method, we carry out the improved uniform error bounds at $O(\varepsilon^2\tau^2)$ and $O(h^m + \varepsilon^2\tau^2)$ for the second-order semi-discretization and full-discretization up to the long time $T_\varepsilon = T/\varepsilon^2$ with T fixed, respectively. Extensions to higher order time-splitting methods and the case of an oscillatory complex NKGE are also discussed. Finally, numerical results are provided to confirm the improved error bounds and to demonstrate that they are sharp.

REFERENCES

[1] W. Bao, Y. Cai and Y. Feng, *Improved uniform error bounds on time-splitting methods for long-time dynamics of the nonlinear Klein-Gordon equation with weak nonlinearity*, SIAM J. Numer. Anal., 60(4):1962–1984, 2022.

HIGH ORDER IN TIME, BGN-BASED PARAMETRIC FINITE ELEMENT METHODS FOR SOLVING GEOMETRIC FLOWS

Wei Jiang[1], **Chunmei Su**[2] **and Ganghui Zhang**[2]

[1] School of Mathematics and Statistics, Wuhan University, P.R. China,
jiangwei1007@whu.edu.cn
[2] Yau Mathematical Sciences Center, Tsinghua University, P.R. China.

Geometric flows have recently attracted lots of attention from scientific computing communities. One of the most popular schemes for solving geometric flows is the so-called BGN scheme, which was proposed by Barrett, Garcke, and Nurnberg (J. Comput. Phys., 222 (2007), pp. 441–467). However, the BGN scheme only can attain first-order accuracy in time, and how to design a temporal high-order numerical scheme is challenging. Recently, based on a novel approach, we have successfully proposed temporal high-order, BGN-based parametric finite element method for solving geometric flows of curves/surfaces. Furthermore, we point out that the shape metrics (i.e., manifold distance), instead of the function norms, should be used to measure numerical errors of the proposed schemes. Finally, ample numerical experiments demonstrate that the proposed BGN-based schemes are high-order in time in terms of the shape metric, and much more efficient than the classical BGN schemes.

REFERENCES

[1] W. Jiang, C. Su and G. Zhang, *Stable BDF time discretization of BGN-based parametric finite element methods for geometric flows*, arXiv preprint, arXiv:2402.03641, 2024.

[2] W. Jiang, C. Su and G. Zhang, *A second-order in time, BGN-based parametric finite element method for geometric flows of curves*, arXiv preprint, arXiv:2309.12875, 2023.

ADAPTIVE LEAST-SQUARES SPACE-TIME FINITE ELEMENT METHODS

Christian Köthe, Richard Löscher, Olaf Steinbach

Institut für Angewandte Mathematik, TU Graz, Austria

We consider the numerical solution of an abstract operator equation $Bu = f$ by using a least-squares approach. We assume that $B : X \to Y^*$ is an isomorphism, and that $A : Y \to Y^*$ implies a norm in Y, where X and Y are Hilbert spaces. The minimizer of the least-squares functional $\frac{1}{2}\|Bu - f\|^2_{A^{-1}}$, i.e., the solution of the operator equation, is then characterized by the gradient equation $Su = B^*A^{-1}f$ with an elliptic and self-adjoint operator $S := B^*A^{-1}B : X \to X^*$. When introducing the adjoint $p = A^{-1}(f - Bu)$ we end up with a saddle point formulation to be solved numerically by using a mixed finite element method. Based on a discrete inf-sup stability condition we derive related a priori error estimates. While the adjoint p is zero by construction, its approximation p_h serves as an a posteriori error indicator to drive an adaptive scheme when discretized appropriately. While this approach can be applied to rather general equations, here we consider second order linear partial differential equations, including the Poisson equation, the heat equation, and the wave equation, in order to demonstrate its potential, which allows to use almost arbitrary space-time finite element methods for the adaptive solution of time-dependent partial differential equations.

REFERENCES

[1] C. Köthe, R. Löscher, O. Steinbach: Adaptive least-squares space-time finite element methods. arXiv:2309.14300, 2023.

LEAST-SQUARES LINEAR ELASTICITY EIGENVALUE PROBLEM: THE TWO-FIELD FORMULATION AND ITS SPECTRUM OPERATOR

Linda Alzaben[1], **Fleurianne Bertrand**[2] **and Daniele Boffi**[1]

[1] King Abdullah University of Science and Technology (KAUST), Thuwal, Saudi Arabia, linda.alzaben@kaust.edu.sa
[2] Chemnitz University of Technology, Chemnitz, Germany.

In this talk, we present a detailed investigation on the spectral properties of the operator associated with the least-squares finite element method dealing with linear elasticity eigenvalue problem. We specifically study the two-field formulation and investigate the convergence analysis of its eigenmodes. We focus on how the value of the Lamé parameter plays a crucial role as we move the incompressible limit. Finally, numerical results confirm the theory presented and show how eigenvalues spread in the complex plane as the Lamé parameter becomes large.

REFERENCES

[1] L. Alzaben, F. Bertrand, and D. Boffi, *On the spectrum of an operator associated with least-squares finite elements for linear elasticity*, Computational Methods in Applied Mathematics 22, no. 3 (2022): 511-528.

MODEL REDUCTION FOR THE WAVE EQUATION BEYOND THE LIMITATIONS OF THE KOLMOGOROV N-WIDTH

Moritz Feuerle[1], Richard Löscher[2], Olaf Steinbach[2] and Karsten Urban[1]

[1] Institute for Numerical Mathematics, Ulm University, Germany, moritz.feuerle@uni-ulm.de
[2] Institute of Applied Mathematics, Graz University of Technology, Austria

The Reduced Basis Method (RBM) is a well-established model reduction technique to realize multi-query and/or realtime applications of Parameterized partial differential equations (PPDEs). The RBM relies on a well-posed variational formulation of the PPDE under consideration. Since the RBM is a linear approximation method, the best possible rate of convergence is given by the Kolmogorov N-width

$$d_N(\mathcal{P}) := \inf_{\substack{X_N \subseteq X,\\ \dim(X_N)=N}} \sup_{\mu\in\mathcal{P}} \inf_{v_N\in X_N} \|u_\mu - v_N\|_X, \qquad N\in\mathbb{N}, \tag{1}$$

where X is the function space in which the solution $u_\mu \in X$ is sought, $\mathcal{P} \subset \mathbb{R}^P$ is the set of parameters and N denotes the dimension of the reduced ansatz space X_N. It is well-known that the decay of $d_N(\mathcal{P})$ is exponentially fast for suitable elliptic and parabolic problems [1, 2], but is poor for transport- or wave-type problems [4, 6]. This motivates our goal of developing a well-posed variational formulation for the wave equation, which also allows for a *nonlinear* model reduction in order to overcome the limitations of a possibly poor Kolmogorov N-width.

To this end, we consider an abstract formulation of the parameterized wave equation of the form

$$B_\mu : X \to Y',\ f_\mu \in Y', \quad \text{seek } u_\mu \in X \text{ s.t.} \quad B_\mu u_\mu = f_\mu. \tag{2}$$

In order to avoid a linear approximation process, we consider a parameter-dependent norm on X defined by $\|\cdot\|_\mu := \|B_\mu \cdot\|_{Y'}$. As this norm might not be meaningful from an application point of view, we show, that $\|\cdot\|_{L_2} \lesssim \|\cdot\|_\mu$. Using a parameter dependent norm on X (and not on Y) is a key difference of our approach compared to existing ones in the literature (see e.g. [3] for the transport problem) and leads to a nonlinear approximation scheme.

We start by showing well-posedness for the wave equation by constructing appropriate spaces X and Y. Moreover, we introduce an unconditionally stable space-time Petrov-Galerkin discretization based upon a modified Hilbert type transformation as in [5]. This discretization is then used as a "truth" solver for an RBM. Numerical experiments will be presented.

REFERENCES

[1] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, and P. Wojtaszczyk. Convergence rates for greedy algorithms in reduced basis methods. *SIAM J. Math. Anal.*, 43(3):1457–1472, 2011.

[2] A. Buffa, Y. Maday, A. T. Patera, C. Prud'homme, and G. Turinici. *A priori* convergence of the greedy algorithm for the parametrized reduced basis method. *ESAIM Math. Model. Numer. Anal.*, 46(3):595–603, 2012.

[3] W. Dahmen, C. Huang, C. Schwab, and G. Welper. Adaptive Petrov-Galerkin methods for first order transport equations. *SIAM J. Numer. Anal.*, 50(5), 2012.

[4] C. Greif and K. Urban. Decay of the kolmogorov n-width for wave problems. *Applied Mathematics Letters*, 96, 2019.

[5] R. Löscher, O. Steinbach, and M. Zank. Numerical results for an unconditionally stable space-time finite element method for the wave equation. In *Domain Decomposition Methods in Science and Engineering XXVI*. Springer International Publishing, 2022.

[6] M. Ohlberger and S. Rave. Reduced Basis Methods: Success, Limitations and Future Challenges. *Proceedings of the Conference Algoritmy*, 2016.

SOLVING MINIMAL RESIDUAL METHODS IN $W^{-1,P'}$ WITH LARGE EXPONENTS P

Johannes Storn[1]

[1] Mathematisches Institut, Universität Leipzig, Germany, johannes.storn@uni-leipzig.de

In this talk, which bases on the paper [1], we adjust recent developments in the numerical computation of the p-Laplace problem via dual regularized Kačanov iterations in [2] to compute the minimizer of a residual in the $W^{-1,p'}(\Omega)$-norm. This minimizer has superior properties compared to minimizers of classical minimal residual methods in Hilbert spaces for challenging problems like singular perturbed problems or convection-dominated diffusion.

REFERENCES

[1] Johannes Storn. *Solving Minimal Residual Methods in $W^{-1,p'}$ with large Exponents p*, accepted in J. Sci. Comput., 2024. https://doi.org/10.48550/arXiv.2307.05178

[2] Anna Kh. Balci, Lars Diening, and Johannes Storn. *Relaxed Kačanov Scheme for the p-Laplacian with Large Exponent*. SIAM J. Numer. Anal., 2775–2794, 2023. https://doi.org/10.1137/22M1528550

SCALING-ROBUST BUILT-IN A POSTERIORI ERROR ESTIMATION FOR DISCONTINUOUS LEAST-SQUARES FINITE ELEMENT METHODS

Philipp Bringmann[1]

[1] TU Wien, Institute of Analysis and Scientific Computing, Austria,
philipp.bringmann@asc.tuwien.ac.at

A striking advantage of least-squares finite element methods (LSFEMs) is the built-in a posteriori error-estimation property that holds for every conforming ansatz space. However, the generalization to discontinuous finite element functions requires a modification of the established analysis. To this end, the talk introduces a least-squares principle on piecewise Sobolev functions by the example of the Poisson model problem in 2D with mixed boundary conditions. First, this novel approach guarantees the built-in error estimation for all discrete subspaces of the piecewise Sobolev spaces employing an integral-mean side condition on the normal jumps of the flux variable. Second, the discontinuous least-squares principle allows to measure the normal jump in its natural norm and, in this way, avoids the over-penalization in the usual discontinuous LSFEMs from the literature. The resulting new scheme allows for various discretizations including standard piecewise polynomial ansatz spaces on triangular and polygonal meshes. Standard penalty terms of discontinuous-Galerkin type weakly enforce the interelement continuity of the ansatz functions without the necessity of a sufficiently large penalty parameter. Crucially, all a priori and a posteriori error estimates are robust with respect to the size of the domain due to a suitable weighting of the least-squares residuals.

REFERENCES

[1] P. Bringmann, *Scaling-robust built-in a posteriori error estimation for discontinuous least-squares finite element methods*, submitted, 2023. Preprint available under arXiv:2310.19930.

LOCALLY CONSERVATIVE STAGGERED LEAST SQUARES METHOD ON GENERAL MESHES

Lina Zhao[1] **and** **Eun-Jae Park**[2]

[1] Department of Mathematics, City University of Hong Kong, Hong Kong
[2] Department of Computational Science and Engineering, School of Mathematics and Computing, Yonsei University, Seoul 03722, Korea, ejpark@yonsei.ac.kr

In this talk we present a novel staggered least squares method for the Poisson model problem on general meshes. Our new method can be flexibly applied to rough grids and allows hanging nodes, which is of particular interest in practical applications. Moreover, it offers the advantage of not having to deal with inf-sup conditions and yielding positive definite discrete problems. Optimal a priori error estimates in energy norm are derived. In addition, a superconvergent estimates in energy norm are also developed by employing variational error expansion. The main difficulty involved here is to show the L^2 norm error estimates for the potential variable, where duality argument and the superconvergent estimates are the key ingredients. The single valued flux over the outer boundary of the dual partition enables us to construct a locally conservative flux. Numerical experiments confirm the theoretical findings and the performance of the adaptive mesh refinement guided by the least squares functional estimator are also displayed.

REFERENCES

[1] L. Zhao and E.-J. Park *A locally conservative staggered least squares method on polygonal meshes*, Special Issue: Advancements in Polytopal Element Methods, Mathematics in Engineering, 2024, 6(2): 339-362. doi: 10.3934/mine.2024014

COMPARISON OF VARIATIONAL DISCRETIZATIONS FOR A CONVECTION-DIFFUSION PROBLEM

Cristina Bacuta[1], **Constantin Bacuta**[1] **and Daniel Hayes**[1]

[1] University of Delaware, USA, bacuta@udel.edu

In this talk, for a model convection-diffusion problem, we present new error estimates for a general upwinding finite element discretization based on bubble modification of the test space. The key analysis tool is finding representations of the optimal norms on the trial spaces at the continuous and discrete levels. We analyze and compare three methods: the standard linear discretization, the saddle point least square and the upwinding Petrov-Galerkin methods. We conclude that the bubble upwinding Petrov-Galerkin method is the most performant discretization for the one dimensional model. Our results for the model convection-diffusion problem can be extended for creating new and efficient discretizations for the multidimensional cases.

REFERENCES

[1] C. Bacuta, D. Hayes, and O'Grady, *Saddle point least squares discretization for convection-diffusion*, Applicable Analysis, 2023.

[2] Cr. Bacuta and C. Bacuta, *Connections between finite difference and finite element approximations for a convection-diffusion problem*, arXiv:2402.03574v1, February 5, 2024.

[3] Cr. Bacuta, C. Bacuta, and D. Hayes, *Comparison of variational discretizations for a convection-diffusion problem*, arXiv:2402.10281, February 15, 2024.

LEAST-SQUARES FINITE ELEMENT FORMULATIONS OF STEKLOV EIGENVALUE PROBLEMS

Fleurianne Bertrand[1] **and Önder Türk**[2]

[1] Faculty of Mathematics, TU Chemnitz, Chemnitz, Germany
[2] Institute of Applied Mathematics, Middle East Technical University, Ankara, Turkey, oturk@metu.edu.tr

In this study we present approximations of several Steklov eigenvalue problems using first order least-squares finite elements. The eigenproblems are defined by the Laplace operator in the partial differential equation, and are characterised by the existence of the spectral parameter in one of the boundary conditions. We devise a novel formulation that is based on minimising an appropriate functional in the least-squares sense which accommodates both the simplified and generalised Steklov eigenvalue problems. The convergence of the discrete solutions towards the corresponding continuous ones is analysed with the use of appropriate error estimates. We describe a number of properties of the discrete spectral operator and provide numerical results for both the classical and generalised problems defined on different geometries to demonstrate the optimality of the approach. Moreover, we aim at assessing the sensitivity of the method and quantifying its advantages and disadvantages with respect to various standard approaches.

A POSTERIORI ERROR CONTROL FOR NONLINEAR LEAST-SQUARES FINITE ELEMENT METHOD

Fleurianne Bertrand[1] **and Henrik Schneider**[2]

[1] Fakultät für Mathematik, Technische Universität Chemnitz, Germany
[2] Fakultät für Mathematik, Universität Duisburg-Essen, Germany, henrik.schneider@uni-due.de

The Least-Squares Finite Element has been successfully applied to several nonlinear problems [2, 3]. In this talk, we present a generalized framework for a posteriori error control for nonlinear problems. We describe a general approach to extend the analysis of a linear problem to a nonlinear one, where nonlinearity will be controlled by a Lipschitz condition. We show that if this condition holds the Least-Squares Functional is an efficient and reliable error estimator. We then apply this framework to a heat equation with temperature-dependent thermal conductivity [1] and present numerical experiments.

REFERENCES

[1] G. Aguirre-Ramirez and J. T. Oden, *Finite element technique applied to heat conduction in solids with temperature dependent thermal conductivity* International Journal for Numerical Methods in Engineering, 7(3):345–355, 1973.

[2] B. Müller, A. Schwarz, J. Schröder and G. Starke *A First-Order System Least Squares Method for Hyperelasticity*, SIAM Journal on Scientific Computing 36 (2014), B795-B816.

[3] F. Bertrand and H. Schneider, *Least-Squares finite element method for the simulation of sea-ice motion* arXiv:2305.11635.

SOLVING THE UNIQUE CONTINUATION PROBLEM USING AN IMPROVED CONDITIONAL STABILITY ESTIMATE.

Harald Monsuur[1], **Rob Stevenson**[1]

[1] Korteweg-de Vries (KdV) Institute for Mathematics, University of Amsterdam, The Netherlands, h.monsuur@uva.nl

On a bounded domain $\Omega \subset \mathbb{R}^n$ and a subdomain $\omega \Subset \Omega$, we consider the problem of reconstructing $u\colon \Omega \to \mathbb{R}$ from $-\Delta u$ and $u|_\omega$. Generally, only approximations $f \approx -\Delta u$ and $q \approx u|_\omega$ are available, and the task is to find an approximation to u from the (perturbed) data f and q. This problem is also known as a *data assimilation problem.* Thanks to our new improved conditional stability estimate, we can employ the least squares methodology from [1] to obtain a new method for solving this problem. For a practical method, we show how to minimize dual norms using non-conforming spaces. The possibilities for solving the Cauchy problem are also discussed.

REFERENCES

[1] Dahmen, W., Monsuur, H. & Stevenson, R. Least squares solvers for ill-posed PDEs that are conditionally stable. *ESAIM Math. Model. Numer. Anal.*. **57**, 2227-2255 (2023), https://doi.org/10.1051/m2an/2023050

SHAPE OPTIMIZATION BY CONSTRAINED FIRST-ORDER SYSTEM LEAST MEAN APPROXIMATION

Gerhard Starke

Fakultät für Mathematik, Universität Duisburg-Essen, Germany, gerhard.starke@uni-due.de

We reformulate the problem of shape optimization, subject to PDE constraints, as an L^p best approximation problem under divergence constraints to the shape tensor introduced by Laurain and Sturm in [3]. More precisely, we prove that the L^p distance of the above approximation problem is equal to the dual norm of the shape derivative considered as a functional on W^{1,p^*} (where $1/p + 1/p^* = 1$). This implies that for any given shape, one can evaluate its distance from being a stationary one with respect to the shape derivative by simply solving the associated L^p-type least mean approximation problem, which can be viewed as a generalization of constrained first-order system least squares [1]. Interestingly, the Lagrange multiplier for the divergence constraint turns out to be the shape deformation of steepest descent. This provides a way, as an alternative to the approach by Deckelnick, Herbert and Hinze [2], for computing shape gradients in W^{1,p^*} for $p^* \in (2, \infty)$. The discretization of the least mean approximation problem is done with (lowest-order) matrix-valued Raviart-Thomas finite element spaces leading to piecewise constant approximations of the shape deformation acting as Lagrange multiplier. Admissible deformations in W^{1,p^*} to be used in a shape gradient iteration are reconstructed locally. Our computational results confirm that the L^p distance of the best approximation does indeed measure the distance of the considered shape to optimality. Also confirmed by our computational tests are the observations from [2] that choosing p^* (much) larger than 2 (which means that p must be close to 1 in our best approximation problem) decreases the chance of encountering mesh degeneracy during the shape gradient iteration.

REFERENCES

[1] J. H. Adler and P. S. Vassilevski, *Error analysis for constrained first-order system least-squares finite-element methods*. SIAM J. Sci. Comput., 36, A1071–A1088, 2014.

[2] K. Deckelnick, P. J. Herbert and M. Hinze, *A novel $W^{1,\infty}$ approach to shape optimisation with Lipschitz domains*. ESAIM Control Optim. Calc. Var., 28, Paper No. 2, 29, 2022.

[3] A. Laurain and K. Sturm, *Distributed shape derivative via averaged adjoint method and applications*, ESAIM Math. Model. Numer. Anal., 50, 1241–1267, 2016.

[4] G. Starke, *Shape optimization by constrained first-order system mean approximation*, arXiv:2309.13595, 2023.

CONSTRAINED L^P APPROXIMATION OF SHAPE TENSORS AND ITS ROLE FOR THE DETERMINATION OF SHAPE GRADIENTS

Laura Hetzel[1] **and Gerhard Starke**[2]

[1] Fakultät für Mathematik, Universität Duisburg-Essen, Germany, laura.hetzel@uni-due.de
[2] Fakultät für Mathematik, Universität Duisburg-Essen, Germany

A crucial issue in numerically solving PDE-constrained shape optimization problems is avoiding mesh degeneracy. Recently, there were two suggested approaches to tackle this problem: (i) departing from the Hilbert space towards the Lipschitz topology approximated by W^{1,p^*} with $p > 2$ and (ii) using the symmetric rather than the full gradient to define a norm.
In this talk we will discuss an approach that allows to combine both. It is based on our earlier work [2] on the L^p approximation of the shape tensor of Laurain & Sturm [1]. We extend this by adding a symmetry constraint to the derived L^p least mean approximation problem and show that the distance measured in a suitably weighted L^p-norm is equal to the dual norm of the shape derivative with respect to the L^{p^*}-norm associated with the linear elastic strain of the deformation. The resulting L^p least mean problem can be viewed as a generalization of a constrained first-order least squares formulation. Moreover, it turns out that the Lagrange multiplicator associated with the divergence constraint is the direction of the steepest descent with respect to the utilized norm. This provides a way to compute shape gradients in W^{1,p^*} with respect to the norm defined by the symmetric gradient.
The discretization of the resulting least mean problem can be done by the PEERS element and its three-dimensional counterpart. We will illustrate the advantages of this approach by computational results of some common shape optimization problems.

REFERENCES

[1] A. Laurain and K. Sturm. Distributed shape derivative *via* averaged adjoint method and applications. *ESAIM Math. Model. Numer. Anal.*, 50(4):1241–1267, 2016.

[2] G. Starke. Shape optimization by constrained first-order system mean approximation, 2023. arXiv:2309.13595.

STABLE ADAPTIVE LEAST-SQUARES SPACE-TIME BEM FOR THE WAVE EQUATION

Daniel Hoonhout[1], **Richard Löscher**[2], **Olaf Steinbach**[2] **and Carolina Urzúa-Torres**[1]

[1] Delft Institute of Applied Mathematics, Technische Universiteit Delft, Netherlands, c.a.urzuatorres@tudelft.nl
[2] Institut für Angewandte Mathematik, Technische Universität Graz, Austria

We consider space-time boundary element methods for the weakly singular operator V corresponding to transient wave problems. In particular, we restrict ourselves to the one-dimensional case and work with prescribed Dirichlet data and zero initial conditions. We begin by revisiting two approaches: energetic BEM [1] and the more recent formulation proposed in [3], for which the weakly singular operator is continuous and satisfies inf-sup conditions in the related spaces. However, numerical evidence suggests that it is unstable when using low-order Galerkin-Bubnov discretisations. As an alternative, it was shown in [4] that one obtains ellipticity -and thus stability- by composing V with the modified Hilbert transform [5].

In this talk, we reformulate these variational formulations as minimisation problems in L^2. For discretisation, the minimisation problem is restated as a mixed saddle point formulation. Unique solvability can be established by combining conforming nested boundary element spaces for the mixed formulation such that the first-kind variational formulation is discrete inf-sup stable. We will analyse under which conditions the discrete inf-sup stability is satisfied, and, moreover, we will show that the mixed formulation provides a simple error estimator, which can be used for adaptivity. The theory is complemented by several numerical examples.

REFERENCES

[1] A. Aimi, M. Diligenti, C. Guardasoni and S. Panizzi, *A space-time energetic formulation for wave propagation analysis by BEMs*, Riv. Mat. Univ. Parma(7),8:171–207, 2008.

[2] D. Hoonhout, R. Löscher, O. Steinbach and C. Urzúa–Torres, *Stable least-squares space-time boundary element methods for the wave equation*, arXiv:2312.12547, 2023.

[3] O. Steinbach and C. Urzúa–Torres, *A new approach to space-time boundary integral equations for the wave equation*, SIAM J. Math. Analysis, 54(2):1370–1392, 2022.

[4] O. Steinbach, C. Urzúa–Torres and M. Zank, *Towards coercive boundary element methods for the wave equation.* J. Integral Equations Appl., 34 501–515, 2021.

[5] O. Steinbach and M. Zank, *Coercive space-time finite element methods for initial boundary value problems*, Electron.Trans. Numer. Anal., 52:154–194, 2020.

GOAL-ORIENTED ADAPTIVE SPACE-TIME FINITE ELEMENT METHODS FOR REGULARIZED PARABOLIC P-LAPLACE PROBLEMS

Bernhard Endtmayer[1], **Ulrich Langer**[2] **and Andreas Schafelner**[2]

[1] Institut für Angewandte Mathematik, Leibniz Universität, Hannover, Germany,
endtmayer@ifam.uni-hannover.de
[2] Institute of Numerical Mathematics, Johannes Kepler University Linz, Austria

In this talk, we consider goal-oriented adaptive space-time finite element discretizations of the regularized parabolic p-Laplace initial-boundary value problem on completely unstructured simplicial space-time meshes. The adaptivity is driven by the dual-weighted residual (DWR) method since we are interested in an accurate computation of some possibly nonlinear functionals at the solution. Such functionals represent goals in which engineers are often more interested than the solution itself. The DWR method requires the numerical solution of a linear adjoint problem that provides the sensitivities for the mesh refinement. This can be done by means of the same full space-time finite element discretization as used for the primal non-linear problems. The numerical experiments presented demonstrate that this goal-oriented, full space-time finite element solver efficiently provides accurate numerical results for different functionals.

ON A MODIFIED HILBERT TRANSFORMATION, THE DISCRETE INF-SUP CONDITION, AND ERROR ESTIMATES

Richard Löscher, Olaf Steinbach, Marco Zank

Institut für Angewandte Mathematik, TU Graz, Austria

In this lecture, we analyze the discrete inf-sup condition and related error estimates for a modified Hilbert transformation as used in the space-time discretization of time-dependent partial differential equations. It turns out that the stability constant depends linearly on the finite element mesh parameter, but in most cases, we can show optimal convergence. We present a series of numerical experiments which illustrate the theoretical findings.

REFERENCES

[1] R. Löscher, O. Steinbach, M. Zank: On a modified Hilbert transformation, the discrete inf-sup condition, and error estimates. arXiv:2402.08291, 2024.

SPACE–TIME VIRTUAL ELEMENTS: A PRIORI ERROR ANALYSIS, RESIDUAL ERROR ESTIMATORS, AND ADAPTIVITY

Sergio Gómez[1], Lorenzo Mascotto[1,*], Andrea Moiola[2], and Ilaria Perugia[3]

[1] Department of Mathematics and Applications, University of Milano-Bicocca, Italy
[2] Department of Mathematics, University of Pavia, Italy
[3] Faculty for Mathematics, University of Vienna, Austria
[*] lorenzo.mascotto@unimib.it

We present a space–time virtual element method for parabolic problems based on a standard Petrov-Galerkin formulation [1]. Trial and test spaces are nonforming in space, so as to allow for a unified analysis in any spatial dimension. The information between time slabs is transmitted by means of upwind terms involving polynomial projections of the discrete functions. After discussing a priori error estimates, we validate them on some numerical examples and compare the results with those of conforming space–time finite elements.

Moreover, we introduce and assess numerically several properties of a residual-type error estimator [2]: we verify its reliability and efficiency for h-adaptive refinements; compare the performance of the space–time nonconforming virtual and conforming finite element methods; investigate the quasi-efficiency of the error estimator for p- and hp-refinements.

REFERENCES

[1] S. Gómez, L. Mascotto, A. Moiola and I. Perugia, *Space-time virtual elements for the heat equation*, SIAM J. Numer. Anal., 62(1):199–228, 2024.

[2] S. Gómez, L. Mascotto and I. Perugia, *Design and performance of a space-time virtual element method for the heat equation on prismatic meshes*, Comput. Methods Appl. Mech. Engrg., 418(A):116491, 2024.

AN A POSTERIORI ESTIMATE AND SPACE-TIME ADAPTIVE BOUNDARY ELEMENTS FOR THE WAVE EQUATION

Alessandra Aimi[1], Giulia Di Credico[2], Heiko Gimperlein[3] and Chiara Guardasoni[1]

[1] Department of Mathematical, Physical and Computer Sciences, University of Parma, Italy
[2] Department of Engineering and Architecture, University of Parma, Italy
[3] Engineering Mathematics, University of Innsbruck, Austria

In this talk we discuss residual a posteriori error estimates and space-time adaptive mesh refinements for time-dependent boundary element methods for the wave equation. We obtain reliable estimates for the weakly singular and the hypersingular boundary integral equations, corresponding to Dirichlet, respectively Neumann, boundary conditions. Numerical examples confirm the theoretical results for space-, time- and space-time adaptive mesh refinements procedures. The current work extends the space-adaptive mesh refinement procedures, e.g. for geometric singularities, in [1].

REFERENCES

[1] Heiko Gimperlein, Ceyhun Ozdemir, David Stark and Ernst P. Stephan, *A residual a posteriori estimate for the time-domain boundary element method,* Numerische Mathematik 146 (2020), 239-280.

A QUASI-OPTIMAL SPACE-TIME FINITE ELEMENT METHOD FOR PARABOLIC PROBLEMS

Lars Diening[1], Rob Stevenson[2] and Johannes Storn[3]

[1] Department of Mathematics, University of Bielefeld, Germany, lars.diening@uni-bielefeld.de
[2] Department of Mathematics, University of Amsterdam, The Netherlands, rob.p.stevenson@gmail.com
[3] Department of Mathematics, Leipzig University, Germany, jstorn@math.uni-bielefeld.de

The operator resulting from a space-time variational formulation of a parabolic evolution equation is known to be well-posed w.r.t. several pairs (U, V) of domains and co-domains, each of them with their pros and cons. With the first order system formulation studied in [1], the space V is of L^2-type, which permits an easy least squares discretization. A price to be paid is that the norm on U is so strong that for rough solutions the convergence rates are sometimes disappointing. With all other pairs, V is a dual space, which has the consequence that for a least squares discretization the dual space has to be replaced by a finite dimensional test space that, for a given trial space, gives uniform inf-sup stability.

We consider (U, V) to be the pair of Bochner spaces with temporal smoothness indices being $\pm\frac{1}{2}$. We extend the usual formulation that requires homogeneous initial data to a formulation that permits any initial condition in $L^2(\Omega)$. For trial finite element spaces w.r.t. possibly locally refined partitions into parabolically scaled prisms of the space-time cylinder, we construct test finite element spaces, with dimensions proportional to that of the trial spaces, that give uniform inf-sup stability. We circumvent the evaluation of fractional norms by the construction of uniform multi-level preconditioners of linear computational complexity.

REFERENCES

[1] Thomas Führer and Michael Karkulik. Space–time least-squares finite elements for parabolic equations. *Comput. Math. Appl.*, 92:27–36, 2021.

INF-SUP THEORY FOR THE BIOT EQUATIONS: ANALYSIS AND DISCRETISATION

Christian Kreuzer[1] **and Pietro Zanotti**[2]

[1] Fakultät für Mathematik, Technische Universität Dortmund, Germany,
christian.kreuzer@tu-dortmund.de
[2] Dipartimento di Matematica, Università di Pavia, Italy

The quasi-static Biot equations in poroelasticity describe the flow of a Newtonian fluid inside an elastic porous medium. The main unknowns are the displacement of the elastic medium and the pressure of the fluid. In addition the two auxiliary variables of the total pressure and the fluid content play a central role in our new approach.

In the first part of the talk, we present a new analysis establishing existence of a unique solution based on the Banach-Necas inf-sup theory. Compared to results in the Literature, this establishes an isomorphism between data and solution and thus requires minimal regularity of the data.

Guided by the inf-sup theory, we propose in the second part of the talk a finite element discretisation for the Biot problem resorting to the the backward Euler scheme in time and discretising all variables in space by conforming Lagrange finite elements on simplicial meshes. We establish the well-posedness, the stability and the quasi-optimality of the discretisation. Moreover, we decouple the best-error into separate approximation problems for the respective problem variables by providing sophisticated interpolation operators.

A HYBRID MIXED VARIATIONAL FORMULATION AND DISCRETIZATION FOR THE LINEAR TRANSPORT EQUATION

Karsten Urban[1], **Nina Beranek**[1]

[1] Institute of Numerical Mathematics, Ulm University, Germany, nina.beranek@uni-ulm.de

A mixed finite element discretization comes with the drawback of a large system due to the additionally introduced unknown(s). One way out is the use of Lagrange multipliers within a hybridization process. This allows us to eliminate some internal degrees of freedom, leading to a smaller, symmetric positive definite linear system, and to use finite element spaces that are discontinuous from element to another. Moreover, the exploitation of the Lagrange multipliers in a postprocess may yield higher order approximations of the original variables, see e.g. [1, 2].

This work is concerned with the theoretical study of a hybrid mixed variational formulation for the linear transport equation. A stable finite element discretization for the hybridized mixed problem is developed.

Firstly, we derive the mixed variational formulation based on the ideas of [3]. Proving the inf-sup stability of the involved bilinear forms, we show the formulation to be well-posed.

Secondly, we analyse the problem and the involved function spaces in case of a domain decomposition. It turns out that the interelement jumps of the normal components of the solution need to be controlled in order to guarantee the required regularity of the solution. Following hybridization techniques, see e.g. [1, 2], we weaken the interelement continuity constraints by introducing a Lagrange multiplier.

Thirdly, we come up with a suitable finite element discretization for the hybridized mixed problem. Following the approach of [3], a slightly modified version of Raviart-Thomas elements of zeroth order are used for one of the unknowns. The problem is shown to be well-posed and stable in the chosen discrete spaces.

REFERENCES

[1] D. N. Arnold, F. Brezzi, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, ESAIM: Mathematical Modelling and Numerical Analysis, 19(1):7–32, 1985.

[2] F. Brezzi, M. Fortin, *Mixed and hybrid finite element methods*, Springer New York, 1991.

[3] J. Cartier, M. Peybernes, *Mixed variational formulation and mixed-hybrid discretization of the transport equation*, Transport Theory and Statistical Physics, 39(1):1–46, 2010.

A SPACE-TIME MULTIGRID METHOD FOR SPACE-TIME FINITE ELEMENT DISCRETIZATIONS OF PARABOLIC AND HYPERBOLIC PDES

Nils Margenberg[1], **Peter Munch**[2]

[1] Helmut Schmidt University Hamburg, Faculty of Mechanical and Civil Engineering, margenbn@hsu-hh.de
[2] Uppsala University, Division of Scientific Computing, peter.munch@it.uu.se

We develop a space-time multigrid method within the framework of space-time finite element methods. The approach includes both continuous and discontinuous variational time discretizations of high order, combined with continuous spatial finite element discretizations [4]. Multigrid methods have been proven to be efficient for large-scale problems. However, extending these advantages to the space-time domain presents challenges [3]. A critical part to achieve good performance is an effective smoother. We employ a space-time cell-wise Vanka smoother. We demonstrate its effectiveness for the heat, acoustic wave and Stokes equations. We present methods for reducing the cost of Vanka-Smoothers, which can become expensive for higher order methods.

Further, we discuss the efficient implementation of space-time multigrid methods using the matrix-free framework provided by the dealii finite element library [1, 2]. Our implementation supports h, p, and hp-Multigrid strategies across both space and time dimensions. We provide a comprehensive comparison of these approaches. We perform scaling and convergence tests on state-of-the-art high-performance computing platforms. The method is tested on unstructured meshes and problems with heterogeneous coefficients. Thereby, we demonstrate their potential to address complex, coupled problems.

REFERENCES

[1] Arndt, D., Bangerth, W., Bergbauer, M., Feder, M., Fehling, M., Heinz, J., Heister, T., Heltai, L., Kronbichler, M., Maier, M., Munch, P., Pelteret, J.-P., Turcksin, B., Wells, D., Zampini, S., 2023. The deal.II Library, Version 9.5. Journal of Numerical Mathematics 31, 231–246. https://doi.org/10.1515/jnma-2023-0089

[2] Kronbichler, M., Kormann, K., 2012. A generic interface for parallel cell-based finite element operator application. Computers & Fluids 63, 135–147. https://doi.org/10.1016/j.compfluid.2012.04.012

[3] Gander, M.J., Neumüller, M., 2016. Analysis of a New Space-Time Parallel Multigrid Algorithm for Parabolic Problems. SIAM J. Sci. Comput. 38, A2173–A2208. https://doi.org/10.1137/15M1046605

[4] Köcher, U., Bause, M., 2014. Variational Space–Time Methods for the Wave Equation. J Sci Comput 61, 424–453. https://doi.org/10.1007/s10915-014-9831-3

COST-OPTIMAL GOAL-ORIENTED ADAPTIVE FEM WITH NESTED ITERATIVE SOLVERS

Philipp Bringmann[1], Maximilian Brunner, Dirk Praetorius[1], Julian Streitberger[1]

[1] TU Wien, Institute of Analysis and Scientific Computing, Austria,
julian.streitberger@asc.tuwien.ac.at

Based on [3], this talk presents a cost-optimal goal-oriented adaptive FEM (GOAFEM) algorithm for the efficient computation of a goal value $G(u^\star)$ for the solution $u^\star$ to a nonsymmetric linear elliptic partial differential equation (PDE). The recent work [2] showed that the key to cost-optimality is full R-linear convergence of an appropriate quasi-error quantity together with optimal convergence rates with respect to the number of degrees of freedom. Therein, contraction of an iterative solver in the PDE-related norm is a crucial ingredient in the analysis. While a natural candidate for nonsymmetric PDEs is a preconditioned generalized minimal residual (GMRES) method, it only leads to contraction of the residual in a discrete vector norm and the connection to the PDE-related norm is not clear. Therefore, we follow the approach of [2] and consider a nested iterative solver, where the outer solver is a symmetrization method (the so-called Zarantonello iteration) and the inner solver is an optimal geometric multigrid method [1] for the symmetrized problem. Following this approach, we show that an embedding of nested iterative solvers into the standard GOAFEM loop `SOLVE&ESTIMATE` – `MARK` – `REFINE` guarantees full R-linear convergence of an appropriate quasi-error product so that convergence rates with respect to the number of degrees of freedom and with respect to the total runtime coincide. Finally, we prove optimal complexity of the proposed algorithm for sufficiently small adaptivity parameters. Numerical experiments investigate the performance of the algorithm and indicate that larger stopping parameters are feasible and even favorable in practice.

REFERENCES

[1] M. Innerberger, A. Miraçi, D. Praetorius, and J. Streitberger, *hp-Robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs*, ESAIM: M2AN, 58(1), 247-272, 2024.

[2] P. Bringmann, M. Brunner, M. Feischl, D. Praetorius, and J. Streitberger, *On full linear convergence and optimal complexity of adaptive FEM with inexact solver*, arXiv:2311.15738, 2023.

[3] P. Bringmann, M. Brunner, D. Praetorius, and J. Streitberger, *Optimal complexity of goal-oriented adaptive FEM for nonsymmetric linear elliptic PDEs*, arXiv:2312.00489, 2023.

PARALLEL MULTIPLE GOAL-ORIENTED ADAPTIVE SPACE-TIME FINITE ELEMENT METHODS FOR QUASI-LINEAR PARABOLIC EVOLUTION EQUATIONS

Bernhard Endtmayer[1], **Ulrich Langer**[2] and **Andreas Schafelner**[2]

[1] Institut für Angewandte Mathematik, Leibniz Universität Hannover, Hannover, Germany
[2] Institute of Numerical Mathematics, Johannes Kepler University Linz, Austria,
andreas.schafelner@jku.at

We present two extensions of our recently proposed goal-oriented adaptive space-time finite element method for the regularized parabolic p-Laplace problem on possibly unstructured space-time meshes. We first consider the case of multiple goal functionals, where we discuss the combination of multiple goal functionals into a single functional. Second, we evaluate the parallel performance of the adaptive method. Since we use an all-at-once discretization approach, the parallelization of the solver for the non-linear systems of equations is straightforward. We discuss the localization via the partition-of-unity method for distributed memory parallelization. We present numerical experiments that demonstrate the performance of the parallel goal-oriented space-time finite element solver for different kinds of functionals.

GOAL-ORIENTED ADAPTIVITY TECHNIQUES FOR CONVECTION-DOMINATED PROBLEMS

Marius P. Bruchhäuser[1] **and Markus Bause**[1]

[1] Faculty of Mechanical and Civil Engineering, Helmut Schmidt University Hamburg, Germany, {bruchhaeuser, bause}@hsu-hamburg.de

The Dual Weighted Residual (DWR) Method has attracted researchers' interest in many fields of application problems since it was introduced by Becker and Rannacher at the turn of the last millennium. With regard to an efficient numerical approximation of the underlying model problem, the DWR approach yields an a posteriori error estimator measured in goal quantities of physical interest, that can be used for adaptive mesh refinement in space and time. Here, we apply this goal-oriented error control combined with residual-based stabilization techniques to convection-dominated (transport) problems. We consider challenges regarding this type of model problems as well as the practical realization of the underlying approach. The performance properties of the space-time adaptive algorithm are studied by means of well-known benchmarks for convection-dominated problems and examples of physical relevance. We show robustness and computational efficiency results and demonstrate the importance of stabilization in a strongly convection-dominated case. Furthermore, we give insight into the application of the DWR approach to nonstationary incompressible flow problems in combination with efficient iterative solver technologies using a flexible geometric multigrid preconditioner.

REFERENCES

[1] M. P. Bruchhäuser and M. Bause, *A cost-efficient space-time adaptive algorithm for coupled flow and transport*, Comput. Methods Appl. Math., 23(4):849–875, 2023. doi:10.1515/cmam-2022-0245

[2] M. Anselmann and M. Bause *A geometric multigrid method for space-time finite element discretizations of the NavierStokes equations and its application to 3d flow simulation*, ACM Trans. Math. Softw., 49(1):1–25, 2023. doi:10.1145/3582492

[3] M. P. Bruchhäuser, U. Köcher and M. Bause, *On the implementation of an adaptive multirate framework for coupled transport and flow.* J. Sci. Comput., 93(59):1–29, 2022. doi:10.1007/s10915-022-02026-z

[4] M. P. Bruchhäuser, *Goal-oriented space-time adaptivity for a multirate approach to coupled flow and transport.* Ph.D. thesis, Helmut-Schmidt-University/University of the Federal Armed Forces Hamburg, 2022. doi: 10.24405/14380

[5] U. Köcher, M. P. Bruchhäuser and M. Bause, *Efficient and scalable data structures and algorithms for goal-oriented adaptivity of space–time FEM codes*, Software X, 10:100239, 2019. doi:10.1016/j.softx.2019.100239

STRESS-BASED FINITE ELEMENT METHODS FOR EIGENVALUE PROBLEMS

Fleurianne Bertrand

Fakultät für Mathematik, Teschnische Universität Chemnitz, Germany

Accurate flux approximations are of interest in many applications and this is particularly true for fluid-structure interaction problems. Considering the corresponding spectral problem, Stress-based methods involve the flux and the stress as independent variables approximated in a suitable H (div)-conforming finite element spaces. This talk will discuss the applicability of those methods for the determination of the corresponding elastoacoustic vibrations, and show that the resulting schemes provide a correct spectral approximation. Quasi-optimal error estimates and numerical experiments to confirm those will be provided.

REFERENCES

[1] L. Alzaben, FB, and D. Boffi. Computation of eigenvalues in linear elasticity with least-squares finite elements: Dealing with the mixed system, 14th WCCM-ECCOMAS Congress, 2021.

[2] FB. and D. Boffi. Least-squares formulations for eigenvalue problems associated with linear elasticity. *Computers and Mathematics with Applications*, 2021.

[3] FB. and Daniele Boffi. First order least-squares formulations for eigenvalue problems. *IMA Journal of Numerical Analysis*, 42 (2), 2022.

ADAPTIVE FINITE ELEMENT METHODS FOR THE LINEAR ELASTICITY EIGENVALUE PROBLEM

Joscha Gedicke[1] **and Karen Petersen**[2]

[1] Institut für Numerische Simulation, Universität Bonn, Germany
[2] Institut für Numerische Simulation, Universität Bonn, Germany, petersen@ins.uni-bonn.de

The linear elasticity eigenvalue problem is considered in structural analysis for modeling and decomposing the deformation of solid objects under stress. Adaptive finite element methods are often used to approximate it, as they provide meshes that lead to optimal convergence even on domains with corner singularities. To model the problem numerically, we consider it on a Lipschitz domain $\Omega \in \mathbb{R}^2$ with a polygonal boundary. Our goal is to find an eigenpair $(\kappa, \mathbf{u}) \in \mathbb{R}_{\geq 0} \times H^1(\Omega, \mathbb{R}^2)$ such that, in the weak sense,

$$-\mathbf{div}\, \mathbb{C}\varepsilon(\mathbf{u}) = \kappa\, \mathbf{u} \quad \text{ in } \Omega$$

holds, where $\mathbb{C}$ is the fourth order strain tensor and $\varepsilon(\cdot)$ is the symmetric gradient. The boundary can be modeled using Dirichlet, Neumann or gliding boundary conditions.
In this talk we consider the two error estimators defined by Carstensen and Thiele in [**?**] and investigate their reliability and efficiency for the given linear elasticity eigenvalue problem. The estimators compute solutions to local problems using a partition of unity and locally defined residuals. The first estimator can be used for $\mathcal{P}_1$ and the second for $\mathcal{P}_{\geq 2}$ finite elements. In numerical experiments we test their performance on different domains and compare them to a standard residual-based error estimator.

REFERENCES

[1] C. Carstensen and J. Thiele, *Partition of unity for localization in implicit a posteriori finite element error control for linear elasticity*, Internat. J. Numer. Methods Engrg., 73(1):71-95, 2008.

DERIVATION AND SIMULATION OF THERMOELASTIC KIRCHHOFF PLATES

Johanna Alms[1] **and Sven Beuchler**[1]

[1] Institut für Angewandte Mathematik, Leibniz Universität Hannover, Germany,
beier@ifam.uni-hannover.de

Within the research of the Cluster of Excellence PhoenixD it is of interest to simulate thermoelastic materials on thin optical components which have the structure of Kirchhoff-Plates. This leads to a bothsided nonlinear coupled 2nd order variational system of the heat equation and the elasticity equations. The standard finite element method (FEM) is a powerful tool for the numerical solution of boundary value problems of elliptic PDEs. Because of the 2nd order of the system standard FEM cannot be applied directly. However for the biharmonic equation a mixed formulation was developed in [1] such that it is reduced to a 1st order variational problem. In this talk we will present a regularity result for the thermoelastic system and we derive a 1st order thermoelastic system on Kirchhoff-Plates by extending the mixed method for the biharmonic equation. This enables the Usage of standard FEM. We finish the talk with some FEM simulation results of our implementation in deal.ii.

REFERENCES

[1] K.Rafetseder and W. Zulehner, *A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach*, SIAM Journal on Numerical Analysis, 56(3):A1961–A1986, 2018.

GUARANTEED LOWER EIGENVALUE BOUNDS WITH HYBRID HIGH-ORDER METHODS

Ngoc Tien Tran

Institut für Mathematik, Universität Augsburg, Germany

Post-processed lower eigenvalue bounds can be computed with conforming, Crouzeix-Raviart, or mixed FEM, but they are limited to lowest-order convergence and not accessible to adaptive computations. Recently, hybrid methods have been proposed to (theoretically) overcome these limitations for the Laplace problem. Under a mild explicit condition on the maximal mesh-size, the computed eigenvalue is a lower bound of the exact eigenvalue. However, in all contributions so far, the constants in this condition are only guaranteed for the lowest-order case. This talk proposes hybrid high-order eigensolvers so that the involved constants are guaranteed for all order of discretization. In fact, they are independent of the polynomial degree and shape regularity as long as the mesh only consists of convex elements. The design is applicable to a wide range of problems such as linear elasticity or Steklov eigenvalue problems.

REFERENCES

[1] N. T. Tran, *Lower eigenvalue bounds with hybrid high-order methods*, soon available in arXiv.

A LEAST-SQUARES GRADIENT RECOVERY METHOD FOR HAMILTON-JACOBI-BELLMAN EQUATION

Amireh Mousavi[1] **and Omar Lakkis**[2]

[1] Department of Mathematics and Computer Science, Friedrich-Schiller Universität Jena, Germany, amireh.mousavi@uni-jena.de

[2] School of Mathematical and Physical Sciences, University of Sussex, England UK

This presentation introduces a conforming finite element method designed to approximate the strong solution of the second-order Hamilton-Jacobi-Bellman equation under Dirichlet boundary conditions, with coefficients satisfying the Cordes condition. The focus of the talk is on discussing the convergence of the continuum semismooth Newton method for the fully nonlinear Hamilton-Jacobi-Bellman equation. Employing such linearization approach leads to a recursive sequence of linear elliptic equations in nondivergence form. To solve these linear equations numerically, we adopt the least-squares gradient recovery method proposed in [2]. The presentation includes a detailed exploration of the optimal-rate a priori and a posteriori error bounds for the approximation. A particular emphasis is placed on utilizing a posteriori error estimators to guide an adaptive refinement procedure. We will show the effectiveness of our approach through numerical experiments on both uniform and adaptive meshes. These experiments aim to validate and reinforce the theoretical findings discussed earlier in the presentation.

REFERENCES

[1] A. Mousavi and O. Lakkis, *A least-squares Galerkin approach to gradient recovery for Hamilton-Jacobi-Bellman equation with Cordes coefficients*, arXiv:2205.07583v1, 2022.

[2] O. Lakkis and A. Mousavi, *A least-squares Galerkin approach to gradient and Hessian recovery for nondivergence-form elliptic equations*, IMA Journal of Numerical Analysis, 42(3):2151–2189, 2022.

A POSTERIORI ERROR ANALYSIS OF THE VIRTUAL ELEMENT METHOD FOR QUASILINEAR ELLIPTIC PDES

Scott Congreve[1] **and Alice Hodson**[1]

[1]Department of Numerical Mathematics, Charles University, Czech Republic,
hodson@karlin.mff.cuni.cz

In this talk we present the *a posteriori* error analysis of the virtual element method for the second order quasilinear equation: $-\nabla \cdot (\mu(\boldsymbol{x}, |\nabla u|)\nabla u) = f$ on general polygonal meshes. We treat the nonlinear coefficient μ by evaluating it using the *gradient projection*, an L^2 projection operator key to the virtual element discretisation. This is straightforward using the VEM construction from [1] where a hierarchy of projection operators for the necessary derivatives is defined, with the starting point being a constraint least squares problem. This approach is highly advantageous and can easily be included into existing software frameworks. We present computable upper and lower bounds of the error estimator and detail how we use the error estimator as part of an adaptive mesh refinement algorithm. The performance of the method is studied through numerical experiments.

REFERENCES

[1] A. Dedner, and A. Hodson, *A framework for implementing general virtual element spaces*, SIAM J. Sci. Comput., (accepted).

SPACE-TIME GOAL-ORIENTED ERROR CONTROL WITH MODEL ORDER REDUCTION DUAL-WEIGHTED RESIDUALS FOR INCREMENTAL POD-BASED ROM FOR TIME-AVERAGED GOAL FUNCTIONALS

Hendrik Fischer[1,2], Julian Roth[1,2], Thomas Wick[1,2], Ludovic Chamoin[2], Amélie Fau[2]

[1] Leibniz Universität Hannover, Institut für angewandte Mathematik, AG Wissenschaftliches Rechnen, Welfengarten 1, 30167 Hannover, Germany
[2] Université Paris-Saclay, CentraleSupélec, ENS Paris-Saclay, CNRS, LMPS - Laboratoire de Mécanique Paris-Saclay, 91190 Gif-sur-Yvette, France

In this presentation, the dual-weighted residual (DWR) method is applied to obtain an error-controlled incremental proper orthogonal decomposition (POD) based reduced order model [1]. A novel approach called MORe DWR (Model Order Reduction with Dual-Weighted Residual error estimates) is being introduced. It marries tensor-product space-time reduced-order modeling with time slabbing and an incremental POD basis generation with goal-oriented error control based on dual-weighted residual estimates. The error in the goal functional is being estimated during the simulation and the POD basis is being updated if the estimate exceeds a given threshold. This allows an adaptive enrichment of the POD basis in case of unforeseen changes in the solution behavior. Consequently, the offline phase can be skipped, the reduced-order model is being solved directly with the POD basis extracted from the solution on the first time slab or time interval and -if necessary- the POD basis is being enriched on-the-fly during the simulation with high-fidelity finite element solutions. Therefore, the full-order model solves can be reduced to a minimum, which is demonstrated on numerical tests for the heat equation, elastodynamics, and porous media [2] using time-averaged quantities of interest. One example of future interest is the extension of two-sided error estimates [3] to the space-time MORe DWR method.

REFERENCES

[1] H. Fischer, J. Roth, T. Wick, L. Chamoin, A. Fau; *MORe DWR: Space-time goal-oriented error control for incremental POD-based ROM for time-averaged goal functionals*, Journal of Computational Physics (JCP), Vol. 504, 2024, No. 112863

[2] H. Fischer, J. Roth, L. Chamoin, A. Fau, M. F. Wheeler, T. Wick; *Adaptive space-time model order reduction with dual-weighted residual (MORe DWR) error control for poroelasticity*, arXiv:2311.08907, Oct 2023

[3] B. Endtmayer, U. Langer, T. Wick; *Two-side a posteriori error estimates for the dual-weighted residual method*, SIAM Journal on Scientific Computing (SISC), Vol. 42(1), A371-A394, 2020

CONVERGENCE OF ADAPTIVE MULTILEVEL STOCHASTIC GALERKIN FEM FOR PARAMETRIC PDES

Alexander Freiszlinger[1], **Dirk Praetorius**[1]

[1] TU Wien, Institute of Analysis and Scientific Computing, Austria
alexander.freiszlinger@asc.tuwien.ac.at

In this talk, we propose and analyze an adaptive multilevel stochastic Galerkin finite element method for a second-order elliptic diffusion problem with random coefficients. The problem is discretized by means of finite generalized polynomial chaos (gpc) expansions in the parameter domain, and standard FEM-discretizations in the spatial domain. Following [1], the adaptive algorithm is driven by a residual-based error estimator, which incorporates both the error due to FEM-discretization and the error due to truncated gpc expansions. Under a local compatibility condition on the mesh sizes of the triangulations associated to an active parameter in the full parameter set, we prove that the proposed algorithm guarantees R-linear convergence of the estimator. To this end, we adopt the approach of [2], and show contraction of a suitable quasi-error quantity. We propose a novel multilevel-refinement algorithm, which simultaneously refines every grid while additionally preserving a local compatibility condition between the meshes in the hierarchy and assigns suitable triangulations to newly activated parameters. Numerical experiments illustrate the theoretical results.

REFERENCES

[1] M. Eigel, C.J. Gittelson, C. Schwab and E. Zander, *Adaptive stochastic Galerkin FEM*, Comput. Methods Appl. Mech. Eng., 270:247-269, 2014.

[2] M. Eigel, C.J. Gittelson, C. Schwab and E. Zander, *A convergent adaptive stochastic Galerkin Finite Element Method with quasi-optimal spatial meshes*, ESAIM Math. Model. Numer. Anal., 49(5):1367-1398, 2015.

AN *hp*-ADAPTIVE SAMPLING ALGORITHM ON DISPERSION RELATION RECONSTRUCTION FOR 2D PHOTONIC CRYSTALS

Yueqi Wang[1] **and Guanglian Li**[1]

[1] Department of Mathematics, The University of Hong Kong, Hong Kong,
u3007895@connect.hku.hk

Computing the dispersion relation for two-dimensional photonic crystals is a notoriously challenging task: It involves solving parameterized Helmholtz eigenvalue problems with high-contrast coefficients. To resolve the challenge, we propose a novel *hp*-adaptive sampling scheme that can detect singular points via adaptive mesh refinement in the parameter domain, and meanwhile, allow for adaptively enriching the local polynomial spaces on the elements that do not contain singular points. In this way, we obtain an element-wise interpolation on an adaptive mesh. We derive an exponential convergence rate when the number of singular points is finite, and a first-order convergence rate otherwise. Numerical tests are provided to illustrate its performance.

ADAPTIVE APPROXIMATION OF NONLINEAR STOCHASTIC PROCESSES

Michael Feischl, Máté Gerencsér

Institute for Analysis and Scientific Computing, TU Wien, michael.feischl@tuwien.ac.at

There are many stochastic processes which are known to be arbitrarily hard to approximate when using uniform timesteps. This can even happen if the coefficients (drift and diffusion) of the governing stochastic differential equation are smooth. Practically more relevant is for example the Cox-Ingersoll-Ross process, which has a square root singularity in the diffusion term. It is known that uniform approximations of the process converge with arbitrarily small algebraic rate. We demonstrate numerically, that adaptive mesh refinement in time can overcome this barrier and deliver the expected convergence rates of $1/2$. This is particularily interesting since multi-level approximations require exactly this rate to offer a significant performance gain.

FINITE ELEMENT APPROXIMATION OF THE FRACTIONAL POROUS MEDIUM EQUATION

José A. Carrillo[1], **Stefano Fronzoni**[1] **and Endre Süli**[1]

[1] Mathematical Institute, University of Oxford, OX2 6GG Oxford, United Kingdom, fronzoni@maths.ox.ac.uk

We construct a finite element method for the numerical solution of a fractional porous medium equation on a bounded open Lipschitz polytopal domain $\Omega \subset \mathbb{R}^d$, where $d = 2$ or 3. The pressure in the model is defined as the solution of a fractional Poisson equation, involving the fractional Neumann Laplacian in terms of its spectral definition. We perform a rigorous passage to the limit as the spatial and temporal discretization parameters tend to zero and show that a subsequence of the sequence of finite element approximations defined by the proposed numerical method converges to a weak solution of the initial-boundary-value problem under consideration. The convergence proof relies on results concerning the finite element approximation of the spectral fractional Laplacian and compactness techniques for nonlinear partial differential equations, together with properties of our equation, which are shown to be inherited by the numerical method.

REFERENCES

[1] Barrett, John W. and Süli, Endre, *Finite element approximation of finitely extensible non-linear elastic dumbbell models for dilute polymers*, ESAIM Math. Model. Numer. Anal., 45(1):39–89, 20.

[2] Barrett, John W. and Süli, Endre, *Existence and equilibration of global weak solutions to kinetic models for dilute polymers I: Finitely extensible nonlinear bead-spring chains* Math. Models Methods Appl. Sci., 21(6):1211– 1289, 201.

[3] Bonito, Andrea and Pasciak, Joseph E., *Numerical approximation of fractional powers of elliptic operators* Math. Comp., 84(295):2083–2110, 2.

A POSTERIORI ERROR CONTROL IN THE MAX MORM FOR THE MONGE-AMPÈRE EQUATION

Dietmar Gallistl[1] **and Ngoc Tien Tran**[2]

[1] Institut für Mathematik, Friedrich-Schiller-Universität Jena, Germany,
dietmar.gallistl@uni-jena.de
[2] Institut für Mathematik, Universität Augsburg, Germany

This talk discusses a stability result for the Monge-Ampère operator in a (potentially regularized) Hamilton-Jacobi-Bellman format as a consequence of Alexandrov's classical maximum principle. The main application is guaranteed a posteriori error control in the L^∞ norm for the difference of the Monge-Ampère solution and the convex hull of a fairly arbitrary C^1-conforming finite element approximation.

REFERENCES

[1] D. Gallistl and N. T. Tran, *Stability and guaranteed error control of approximations to the Monge–Ampére equation*, Numer. Math., 156(1):107–131,2024.

ADAPTIVE ENERGY MINIMIZATION FOR NONLINEAR VARIATIONAL PDE

Pascal Heid[1] **and Thomas P. Wihler**[2]

[1] Fachhochschule Nordwestschweiz, Switzerland
[2] Mathematisches Institut, Universität Bern, Switzerland

We present a general finite element framework for the numerical approximation of nonlinear elliptic variational-type boundary value problems, i.e., whose solutions appear as (local or global) minimizers of an underlying energy functional. Our approach relies on two components: (1) a linearized iterative energy reduction procedure, which allows to minimize the energy on arbitrary discrete subspaces, and (2) a novel adaptive methodology that exploits the local energy structure of the PDE (instead of a posteriori error indicators) in order to improve the approximate solution on a sequence of hierarchically refined finite element meshes. In addition to the theoretical foundations, a series of numerical experiments for quasi- and semi-linear PDE will illustrate our approach.

REFERENCES

[1] M. Amrein, P. Heid and T. P. Wihler, *A Numerical Energy Reduction Approach for Semilinear Diffusion-Reaction Boundary Value Problems Based on Steady-State Iterations*, SIAM J. Numer. Anal., 61(2), 2023.

[2] P. Heid, B. Stamm and T. P. Wihler, *Gradient flow finite element discretizations with energy-based adaptivity for the Gross-Pitaevskii equation*, J. Comput. Phys., 436, 2021.

[3] P. Heid and T. P. Wihler, *Variational adaptivity*, in preparation.

A POSTERIORI ERROR ESTIMATES ROBUST WITH RESPECT TO NONLINEARITIES AND ORTHOGONAL DECOMPOSITION BASED ON ITERATIVE LINEARIZATION

André Harnist[1], Koondanibha Mitra[2], Ari Rappaport[3,4] and Martin Vohralík[3,4]

[1] Laboratory of Applied Mathematics of Compiègne, CS 60319, Université de technologie de Compiègne, 60203 Compiègne, France
[2] Eindhoven University of Technology, De Rondom 70, 5612 AP Eindhoven, The Netherlands
[3] Inria, 2 rue Simone Iff, 75589 Paris, France, {ari.rappaport, martin.vohralik}@inria.fr
[4] CERMICS, Ecole des Ponts, 77455 Marne-la-Vallée, France

We discuss a posteriori error estimates for strongly monotone and Lipschitz-continuous nonlinear elliptic problems, where standard approaches do not give estimates robust with respect to the strength of the nonlinearities in the sense that the overestimation factor increases when the problem is more and more nonlinear. We derive estimates that include, and build on, common iterative linearization schemes such as Zarantonello, Picard, Newton, or M- and L-ones. We derive two approaches that give robustness: we either estimate the energy difference that we augment by the discretization error of the current linearization step, or we design iteration-dependent norms that feature weights given by the current linearization iterate. The second setting allows for error localization and an orthogonal decomposition into discretization and linearization components. Numerical experiments illustrate the theoretical findings, with the overestimation factors close to the optimal value of one for any strength of the nonlinearities. Details are given in [1, 2].

REFERENCES

[1] A. Harnist, K. Mitra, A. Rappaport, M. Vohralík, *Robust energy a posteriori estimates for nonlinear elliptic problems*, HAL Preprint 04033438, 2023.

[2] K. Mitra, M. Vohralík, *Guaranteed, locally efficient, and robust a posteriori estimates for nonlinear elliptic problems in iteration-dependent norms. An orthogonal decomposition result based on iterative linearization*, HAL Preprint 04156711, 2023.

ADAPTIVE REGULARIZATION, DISCRETIZATION, AND LINEARIZATION FOR NONSMOOTH ELLIPTIC PDE

Martin Vohralík[1,2], **Ari Rappaport**[1,2], **François Févotte**[3]

[1] Inria, 2 rue Simone Iff, 75589 Paris, France.
[2] Université Paris-Est, CERMICS (ENPC), 77455, 136 Marne-la-Vallée, France.
[3] Triscale Innov, 7 rue de la Croix Martre, 91120, Palaiseau, France.

We consider nonsmooth partial differential equations associated to a minimization of an energy functional. We adaptively regularize the nonsmooth nonlinearity so as to be able to apply the usual Newton linearization, which is not always possible otherwise. Then the finite element method is applied. We focus on the choice of the regularization parameter and adjust it on the basis of an posteriori error estimate for the difference of energies of the exact and approximate solutions. Importantly, our estimates distinguish the different error components, namely those of regularization, linearization, and discretization. This leads to an algorithm that steers the overall procedure by adaptive regularization, linearization, and discretization. We prove guaranteed upper bounds for the energy difference and discuss the robustness of the estimates with respect to the magnitude of the nonlinearity when the stopping criteria are satisfied. Numerical results illustrate the theoretical developments.

REFERENCES

[1] François Févotte, Ari Rappaport, Martin Vohralík, *Adaptive regularization, discretization, and linearization for nonsmooth problems based on primal–dual gap estimators*, Comput. Methods Appl. Mech. Eng., Volume 418, Part B, 2024.

A POSTERIORI ERROR ESTIMATES FOR VARIATIONAL INEQUALITIES DISCRETIZED BY HIGHER-ORDER FINITE ELEMENTS

Lothar Banz[1], **Andreas Schröder**[1]

[1] Department of Mathematics, Paris Lodron Universität Salzburg, Austria

The talk presents a posteriori error estimates for variational inequalities with linear constraints. The error estimates are derived using an abstract framework in which the error contributions representing non-penetration, non-conformity and complementarity conditions form a weighted functional. The use of the minimizer of this functional enables the derivation of reliable and efficient a posteriori error estimates. The abstract findings are applied, in particular, to the obstacle problem and to the simplified Signorini problem, where higher-order finite elements are used for discretization. Several numerical experiments are presented to discuss the properties of the error estimates and their applicability in adaptive schemes.

REFERENCES

[1] L. Banz and A. Schröder, *A posteriori error control for variational inequalities with linear constraints in an abstract framework*, J. Appl. Numer. Optim., 3(2):333–359, 2021.

A MODIFIED KAČANOV ITERATION SCHEME FOR THE NUMERICAL SOLUTION OF QUASILINEAR ELLIPTIC DIFFUSION EQUATIONS

Pascal Heid[1] **and Thomas P. Wihler**[2]

[1] FHNW, Switzerland, hepa@ma.tum.de
[2] Mathematisches Institut, Universität Bern, Switzerland

Kačanov's method is an efficient iterative nonlinear solver for a class of quasilinear elliptic diffusion equations. However, to guarantee the convergence to a solution of the given problem, the classical theorem requires the diffusion coefficient to be monotonically decreasing. In this talk, we introduce a modified Kačanov method, which allows for adaptive damping, and thereby to derive a new convergence analysis, which no longer requires the standard monotonicity assumption. We further present two different adaptive strategies for the practical selection of the damping parameter. Finally, the performance of the modified scheme is demonstrated with some numerical experiments in the context of finite element discretisations.

REFERENCES

[1] P. Heid and T. P. Wihler, *A modified Kačanov iteration scheme with application to quasilinear diffusion models*, ESAIM: M2AN, 56, 433-450, 2022.

10th International Conference on Computational Methods in Applied Mathematics
CMAM-10
10–14 June 2024, Bonn, Germany

ROBUST ITERATIVE LINEARIZATION METHODS AND ADAPTIVITY FOR NONLINEAR ELLIPTIC PROBLEMS

K. Mitra[1], I.S. Pop[2], M. Vohralík[3], and A. Javed[2]

[1] Mathematics & Computer Science Department, TU Eindhoven, The Netherlands
[2] Faculty of Science, Hasselt University, Belgium
[3] INRIA Paris & CERMICS ENPC, France

In this work, we consider a general formulation of iterative linearization schemes for nonlinear elliptic equations. For an elliptic operator $\mathcal{R} : H \to (H)^*$ (with H being a Hilbert space), the solution of the corresponding elliptic problem $u \in H$ satisfies $\mathcal{R}(u) = 0$. Then a general iterative linearization scheme for obtaining u constructs an approximating sequence $\{u^i\}_{i\in\mathbb{N}} \subset H$ by introducing a bilinear operator $\mathcal{B}_{\langle i\rangle} : H \times H \to \mathbb{R}$ depending upon $u^i \in H$, such that $u^{i+1} \in H$ solves the linear problem

$$\mathcal{B}_{\langle i\rangle}(u^{i+1} - u^i, v) = -\langle \mathcal{R}(u^i), v\rangle, \quad \forall\, v \in H. \tag{1}$$

We show that for a large class of problems (e.g., porous media flow, mean curvature flow, biological flows, mixed dimensional equations, optimal transport problems, and design of optical systems) and for almost all standard linearization schemes (Newton scheme, Picard scheme, L/M-schemes) this structure holds. Moreover, by considering an operator $\mathcal{B}$ which does not depend on iteration index i and is an inner product in H, we get schemes that are more robust in terms of discretization, nonlinearities, and degeneracies (the 'so called' Zarantonello/L-scheme). This however, comes at the cost of slower and possibly linear convergence compared to quadratic schemes such as the Newton method, unless optimal coefficient values are chosen. By following [1], based on a posteriori error estimates, we devise an adaptive scheme which automatically chooses the (quasi-)optimal parameters for convergence. In fact, the linearization schemes themselves can be used to find efficient a posteriori estimates for the equations [2]. Numerical results are presented for a wide variety of problems demonstrating the stability and efficiency of this class of schemes, and their usage.

REFERENCES

[1] J.S. Stokke, K. Mitra, E. Storvik, J.W. Both, & F.A. Radu. *An adaptive solution strategy for Richards' equation*, Computers & Mathematics with Applications, 152, 155-167, 2023.

[2] K. Mitra, & M. Vohralík. *Guaranteed, locally efficient, and robust a posteriori estimates for nonlinear elliptic problems in iteration-dependent norms. An orthogonal decomposition result based on iterative linearization*, HAL Preprint, hal-04156711, v.1., 2023.

A DECOUPLED, CONVERGENT AND FULLY LINEAR ALGORITHM FOR THE LANDAU–LIFSHITZ–GILBERT EQUATION WITH MAGNETOELASTIC EFFECTS

Hywel Normington[1], **Michele Ruggeri**[2]

[1] University of Strathclyde, UK
[2] University of Bologna, Italy

We consider the coupled system of the Landau-Lifshitz-Gilbert (LLG) equation and conservation of momentum to describe magnetic processes in ferromagnetic materials including magnetostrictive effects. For this nonlinear system of time-dependent partial differential equations, we present a decoupled and unconditionally convergent integrator based on linear finite elements in space and a one-step method in time. Compared to previous works on this problem, for our method, we prove a discrete energy law that mimics that of the continuous problem. Moreover, we do not employ a nodal projection to impose the unit-length constraint on the discrete magnetization, so that the stability of the method does not require weakly acute meshes. Furthermore, our integrator and its analysis hold for a more general setting, including body forces and traction, and a more general representation of the magnetostrain.

REFERENCES

[1] H. Normington and M. Ruggeri, *A decoupled, convergent and fully linear algorithm for the Landau–Lifshitz–Gilbert equation with magnetoelastic effects*, (under review), https://arxiv.org/abs/2309.00605v2

STABILITY PROPERTIES OF INTEGRAL AND DISCRETE PLANE WAVE REPRESENTATIONS OF HELMHOLTZ SOLUTIONS

Nicola Galante[1,2], Daan Huybrechs[3], Andrea Moiola[4] and Emile Parolin[1,2,*]

[1] Alpines, Inria, Paris, France
[2] Laboratoire Jacques-Louis Lions, Sorbonne University, Paris, France
[3] Department of Computer Science, KU Leuven, Leuven, Belgium
[4] Department of Mathematics, University of Pavia, Pavia, Italy
[*] emile.parolin@inria.fr

Standard low-order polynomial-based finite element methods struggle to resolve the fine oscillations of Helmholtz solutions in the high-frequency regime, an issue further worsened by the pollution effect. In that context, high-order methods among which are Trefftz methods, a class of schemes which rely on particular solutions to build approximation spaces, become competitive as they typically require less degrees of freedom for a comparable accuracy. A popular choice for Helmholtz problems is to use propagative plane waves, namely $\mathbf{x} \mapsto e^{\imath \mathbf{k} \cdot \mathbf{x}}$ with real-valued $\mathbf{k}$.

However, the computation of approximations of Helmholtz solutions in bounded domains using such waves is known to be numerically unstable. This is despite provably good error estimates. We cast a new light on this phenomenon, explaining that this stems from the impossibility to accurately compute discrete representations with exponentially large coefficients using finite precision arithmetic.

A remedy is to enrich the approximation spaces by also considering complex-valued $\mathbf{k}$, i.e. using evanescent plane waves. Analysis shows that all Helmholtz solutions in the ball can be exactly represented as stable continuous superpositions of evanescent plane waves [1, 2] a property unattainable with propagative plane waves alone. A discretization strategy is proposed that yields discrete representations with bounded coefficients, hence amenable to stable numerical computation. Numerical experiments using Trefftz discontinuous Galerkin formulations confirm empirically the stability analysis.

REFERENCES

[1] E. Parolin, D. Huybrechs, and A. Moiola. *Stable approximation of Helmholtz solutions in the disk with evanescent plane waves*, ESAIM: M2AN, 57(6):3499–3536, 2023.

[2] N. Galante, A. Moiola and E. Parolin. *Stable approximation of Helmholtz solutions in the ball with evanescent plane waves*, arXiv:2401.04016.

SEISMIC IMAGING OF A DAM-ROCK INTERFACE USING FULL-WAVEFORM INVERSION

Mohamed Aziz Boukraa[1], **Lorenzo Audibert**[2], **Marcella Bonazzoli**[1], **Houssem Haddar**[1] **and Denis Vautrin**[2]

[1] Inria, UMA, ENSTA Paris, Institut Polytechnique de Paris, 91120 Palaiseau, France
[2] EDF R&D, département PRISME, 6 quai Watier, BP 49, 78401 Chatou Cedex, France

In this talk we are interested in reconstructing the interface between the concrete structure of a hydroelectric dam and the underlying rock, using Full Waveform Inversion [1]. We minimize a regularized misfit cost functional by computing its shape derivative and iteratively updating the interface shape by the gradient descent method. At each iteration, we simulate time-harmonic elasto-acoustic wave propagation models, coupling linear elasticity in the solid medium with acoustics in the reservoir. Numerical results using realistic noisy synthetic data demonstrate the method ability to accurately reconstruct the dam-rock interface with a limited number of measurements.

REFERENCES

[1] A. Tarantola, B. Valette, *Generalized nonlinear inverse problems solved using the least squares criterion*, Reviews of Geophysics, 20:219–232, 1982.

THE GREEN'S FUNCTION FOR AN ACOUSTIC HALF-SPACE PROBLEM WITH IMPEDANCE BOUNDARY CONDITIONS

Stefan Sauter

Institute of Mathematics, University of Zurich, Zurich, Switzerland

In our talk we show that the acoustic Green's function for a half-space impedance problem in general d spatial dimensions can be written as a sum of (two) terms, each of which is the product of an exponential function with the eikonal in the argument and a slowly varying function. We introduce the notion of families of slowly varying functions and formulate this statement as a theorem along a sketch of its proof. This talk comprises joint work with Chuhe Lin, University of Zurich and Markus Melenk, TU Vienna.

LOCALIZED IMPLICIT TIME STEPPING FOR THE WAVE EQUATION

Dietmar Gallistl[1] and Roland Maier[2]

[1] Institute of Mathematics, Friedrich-Schiller-Universität Jena, Germany
[2] Institute for Applied and Numerical Mathematics, Karlsruhe Institute of Technology, Germany, roland.maier@kit.edu

This talk is about locally computing solutions to the acoustic wave equation with possibly highly oscillatory coefficients. We show that the localized (and especially parallel) computation on multiple overlapping subdomains is reasonable, making use of exponentially decaying entries of the global system matrices and an appropriate partition of unity. Moreover, a re-start is introduced after a certain amount of time steps to maintain a moderate overlap of the subdomains. Overall, the approach may be understood as a domain decomposition strategy in space on successive short time intervals that completely avoids inner iterations. Numerical examples are presented that confirm the theoretical findings.

REFERENCES

[1] D. Gallistl and R. Maier, *Localized implicit time stepping for the wave equation*, ArXiv Preprint, 2306.17056, 2023.

GUARANTEED LOWER ENERGY BOUNDS FOR THE GROSS–PITAEVSKII PROBLEM USING MIXED FINITE ELEMENTS

Dietmar Gallistl[1], Moritz Hauck[2], Yizhou Liang[3] and Daniel Peterseim[3,4]

[1] Institute of Mathematics, University of Jena, Ernst-Abbe-Platz 2, 07743 Jena, Germany
[2] Department of Mathematical Sciences, University of Gothenburg and Chalmers University of Technology, 41296 Göteborg, Sweden
[3] Institute of Mathematics, University of Augsburg, Universitätsstr. 12a, 86159 Augsburg, Germany
[4] Centre for Advanced Analytics and Predictive Sciences (CAAPS), University of Augsburg, Universitätsstr. 12a, 86159 Augsburg, Germany

We establish an a priori error analysis for the lowest order Raviart-Thomas finite element discretization of the nonlinear Gross-Pitaevskii eigenvalue problem. Optimal convergence rates are obtained for the primal and dual variables as well as for the eigenvalue and energy approximations. Most importantly, the proposed mixed discretization provides a guaranteed and assymptotically exact lower bound for the ground state energy. The theoretical results are illustrated by a series of numerical experiments.

REFERENCES

[1] D. Gallistl, M. Hauck, Y. Liang, D. Peterseim *Mixed finite elements for the Gross-Pitaevskii eigenvalue problem: a priori error analysis and guaranteed lower energy bound*, ArXiv:2402.06311

SPACE-TIME DISCONTINUOUS GALERKIN DISCRETIZATIONS OF MULTIPHYSICS WAVE PROPAGATION

Ilario Mazzieri[1]

[1] MOX-Laboratory for Modeling and Scientific Computing, Department of Mathematics, Politecnico di Milano, Italy, ilario.mazzieri@polimi.it

In this talk we present a combined space-time discontinuous Galerkin (dG) finite element method on polytopal grids (PolydG) for the numerical simulation of multiphysics wave propagation phenomena in heterogeneous media. In particular, we address wave phenomena in acoustic, elastic and poro-elastic domains. The coupling between different models is realized by means of (physically consistent) transmission conditions, weakly imposed on the interface between the domains. We will analyse different dG discretization strategies from the point of view of stability, accuracy and computational cost. To showcase the efficacy of our proposed methodologies, we provide several illustrative examples to highlight the robustness and versatility of our approach in tackling complex multiphysics wave propagation scenarios.

This is a joint work Alberto Artoni[1], Gabriele Ciaramella[1], Michele Botti[1] and Paola F. Antonietti[1].

LOCALIZED ORTHOGONAL DECOMPOSITION METHODS FOR PROPAGATING WAVES IN THE GROSS-PITAEVSKII EQUATION

Christian Döding[1]

[1] Institut für Numerische Simulation, Universität Bonn, Germany, doeding@ins.uni-bonn.de

In this talk we consider wave propagation in the Gross-Pitaevskii equation (GPE), a nonlinear Schödinger equation that has wide applications in modern physics, describing the propagation of light in complex media such as glass fibers or the time evolution of the wave function of Bose-Einstein condensates. We are interested in numerical approximations of the GPE, which can pose mathematical challenges due to nonlinearities, energy sensitivity, and low regularity of the solution in rough regimes. To address these problems, we propose a space discretization by localized orthogonal decomposition, a generalized finite element space originally developed in [2] in the context of multiscale problems. Combined with an energy-preserving time integrator, the resulting method proposed in [1] is of high order even for problems where the solution suffers from low regularity and classical methods fail. We demonstrate the performance of the derived method in numerical simulations, with application to Bose-Einstein condensates, and show that it can serve as an efficient solver for such problems.

REFERENCES

[1] C. Döding, P. Henning and J. Wärnegård, *A two level approach for simulating Bose-Einstein condensates by localized orthogonal decomposition*, Arxiv e-print 2212.07392, 2022.

[2] A. Målqvist and D. Peterseim, *Localization of elliptic multiscale problems*, Math. Comp., 83(290):2583–2603, 2014.

A FINITE ELEMENT METHOD FOR A TWO-DIMENSIONAL PUCCI EQUATION

Susanne C. Brenner[1], **Li-yeng Sung**[1] **and Zhiyu Tan**[2]

[1] Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA

[2] School of Mathematical Sciences and Fujian Provincial Key Laboratory on Mathematical Modeling and High Performance Scientific Computing, Xiamen University, Fujian, 361005, China, zhiyutan@xmu.edu.cn

A nonlinear least-squares finite element method for strong solutions of the Dirichlet boundary value problem of a two-dimensional Pucci equation on convex polygonal domains is investigated in this paper. We obtain *a priori* and *a posteriori* error estimates and present corroborating numerical results, where the discrete nonsmooth and nonlinear optimization problems are solved by an active set method and an alternating direction method with multipliers.

REFERENCES

[1] S. C. Brenner, L.-Y. Sung and Z. Tan, *A finite element method for a two-dimensional Pucci equation*, Comptes Rendus Mècanique, 351(S1):1–16, 2023.

HIERARCHICAL SUPER-LOCALIZED ORTHOGONAL DECOMPOSITION METHODS FOR THE SOLUTION OF MULTI-SCALE ELLIPTIC PROBLEMS

José Garay[1], **Hannah Mohr**[1], **and Daniel Peterseim**[1]

[1] Institut für Mathematik, Universität Augsburg, Germany, jose.garay.fernandez@uni-a.de

We present the construction of a sparse-compressed operator that approximates the solution operator of elliptic PDEs with rough coefficients. To derive the compressed operator, we construct a hierarchical basis of an approximate solution space, with super-localized basis functions that are orthogonal across hierarchy levels with respect to the inner product induced by the energy norm. The super localization is obtained through a novel variant of the Super-Localized Orthogonal Decomposition method. This basis not only induces a sparse compression of the solution space but also enables an orthogonal multi-resolution decomposition of the approximate solution operator, decoupling scales and solution contributions of each level of the hierarchy. With this decomposition, the solution of the PDE reduces to the solution of a set of independent linear systems with mesh-independent condition numbers that can be computed concurrently. We present an accuracy study of the compressed solution operator as well as numerical results illustrating our theoretical findings.

HODGE DECOMPOSITION FINITE ELEMENT METHOD FOR THE 3D QUAD-CURL PROBLEM

Susanne C. Brenner[1], **Casey Cavanaugh**[1] **and Li-yeng Sung**[1]

[1] Center for Computation and Technology and Department of Mathematics, Louisiana State University, USA, ccavanaugh@lsu.edu

In this talk, we present a finite element method for the quad-curl equation in three dimensions. Using the Hodge decomposition for divergence-free fields, the fourth-order problem is reformulated as three standard second-order saddle point systems. Furthermore, the Hodge decomposition approach allows for the finite element method to handle domains with general topology. Analysis and numerical results are presented using a variety of domains with different topological properties.

PRESSURE-ROBUSTNESS IN NAVIER–STOKES FINITE ELEMENT SIMULATIONS

Christian Merdon[1]

[1] Weierstrass-Institute for Applied Analysis and Stochastics, Berlin, Germany

This talk visits several (sub-)model problems derived from the instationary Navier–Stokes equations to study the error propagation that is caused by a lack of pressure-robustness.

Pressure-robustness characterizes schemes that allow for an a priori velocity error estimate that is independent of the pressure and is connected to the correct balancing of gradient forces and the pressure. It has qualitative implications in a number of related questions like convection robustness and efficient a posteriori error control [1].

The talk also presents some recent approach to design a divergence-free and convection-robust finite element schemes that can be applied on general shape-regular triangulations [2].

REFERENCES

[1] P.L. Lederer, Ch. Merdon, *Guaranteed upper bounds for the velocity error of pressure-robust Stokes discretisations* Journal of Numerical Mathematics, vol. 30, no. 4, 2022, pp. 267-294

[2] N. Ahmed, V. John, X. Li, Ch. Merdon, *Inf-sup stabilized Scott-Vogelius pairs on general simplicial grids for Navier-Stokes equations*, arXiv:2212.10909 [math.NA], 2022.

APPROXIMATION OF LAPLACE EIGENVALUES AND EIGENFUNCTIONS OF DOMAINS WITH CORNERS

Philipp Zilk[1], **Thomas Apel**[2]

[1] Institute for Mathematics and Computer-Based Simulation,
University of the Bundeswehr Munich, Germany,
philipp.zilk@unibw.de, https://www.unibw.de/imcs-en/team/zilk

[2] Institute for Mathematics and Computer-Based Simulation,
University of the Bundeswehr Munich, Germany,
thomas.apel@unibw.de, https://www.unibw.de/imcs-en/team/apel

Corner singularities play a significant role for the modeling of physical phenomena in non-smooth domains. Their presence renders simulations challenging since standard methods produce suboptimal results due to singular solutions. In this talk, we consider the Laplace eigenvalue problem on domains with corners, where we specifically investigate the regularity of the resulting eigenfunctions. Some of the eigenmodes may be smooth while others show singular behavior. As the different regularity properties need to be taken into account during the numerical approximation, we study the distribution of the different types of eigenfunctions in more detail. Based on this, we formulate guidelines for the approximation of Laplace eigenpairs on domains with corners. We present numerical results for typical model domains like circular sectors or L-shapes using a graded mesh refinement approach for isogeometric analysis, which has been proven to provide powerful tools for accurate spectral approximation of higher orders.

OPTIMAL PRESSURE CONVERGENCE FOR SCOTT-VOGELIUS TYPE ELEMENTS

Nis-Erik Bohne[1], **Benedikt Gräßle**[2] **and Stefan Sauter**[1]

[1] Institut für Mathematik, Universität Zürich, Switzerland
[2]Institut für Mathematik, Humboldt-Universität zu Berlin, 10117 Berlin, Germany

The Scott-Vogelius element $(\mathbf{S}_{k,0}(\mathcal{T}), \operatorname{div} \mathbf{S}_{k,0}(\mathcal{T}))$ is one of the simplest inf-sup stable finite element methods for the approximation of the Stokes equation. However, it is well known that the convergence order for the pressure approximation may become suboptimal in the presence of certain critical mesh points on the domain boundary. In this talk we present a simple post processing procedure to recover the optimal pressure approximation in a mesh-robust way. Further we explain this recovery strategy for the recently introduced pressure-wired Stokes element.

AN EQUILIBRATED A POSTERIORI ERROR ESTIMATOR FOR THE BIHARMONIC EIGENVALUE PROBLEM

Joscha Gedicke[1] **and Stephanie Zacharias**[2]

[1] Institut für Numerische Simulation, Universität Bonn, Germany
[2] Institut für Numerische Simulation, Universität Bonn, Germany, zacharia@ins.uni-bonn.de

The biharmonic eigenvalue problem plays an important role in physical and mechanical science, finding application in the Kirchhoff Love theory of plates. Thus solving these problems with traditional finite element methods has been subject of extensive research in the past, and in the last decade C^0 interior penalty methods have proven themselves to be promising [1]. Brenner, Monk and Sung introduced a residual a posteriori error estimator for C^0 interior penalty methods for the biharmonic problem in [3].

In this talk we consider the equilibrated a posteriori error estimator by Braess, Pechstein and Schöberl [2] for the biharmonic eigenvalue problem. Being based on the two-energies principle, the estimator is defined for the Hellan-Herrman-Johnson formulation of the biharmonic problem. Its main part is evaluated using a tensor σ_h^{eq} of bending moments, which satisfies the equilibration property

$$\operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}} = f_h.$$

The equilibrated tensor is chosen from the well known Hellan-Herrmann-Johnson space, and can be computed by a local postprocessing procedure. We prove reliability and efficiency of the estimator for the approximating eigenfunctions and eigenvalues. Moreover, numerical experiments are performed to investigate the estimator's robustness in the polynomial degree. These experiments suggest the estimator is not only more efficient, but also more robust in the polynomial degree than the residual a posteriori error estimator.

REFERENCES

[1] S. C. Brenner, P. Monk and J. Sung, C^0 interior penalty Galerkin method for biharmonic eigenvalue problems. In *Spectral and high order methods for partial differential equations-ICOSAHOM 2014*, volume 106 of *Lect. Notes Comput. Sci. Eng.*, pages 3-15. Springer, Cham, 2015.

[2] D. Braess, A. S. Pechstein and J. Schöberl. An equilibration-based a posteriori error bound for the biharmonic equation and two finite elements methods. *IMA J. Numer. Anal.,* 40(2):951-975, 2020.

[3] S. C. Brenner, T. Gudi and L. Sung. An a posteriori error estimator for a quadratic C^0-interior penalty method for the biharmonic problem. *IMA J. Numer. Anal.*, 30(3):777-798, 2010.

FRAMEWORK FOR NONCONFORMING APPROXIMATIONS OF SOME SEMILINEAR PROBLEMS

Carsten Carstensen[1], Benedikt Gräßle[1] and Neela Nataraj[2]

[1] Humboldt-Universität zu Berlin, Berlin, Germany
[2] Indian Institute of Technology Bombay, Mumbai, India

The a priori and a posteriori error analysis in [1, 3] establishes a unified analysis for different finite element approximations to regular roots of nonlinear partial differential equations with a quadratic nonlinearity. A smoother in the source and nonlinearity enables quasi-best approximations in [3] under a set of hypotheses that guarantees the existence and local uniqueness of a discrete solutions by the Newton-Kantorovich theorem. Related assumptions on some computed approximation close to a regular root allow the reliable and efficient a posteriori error analysis [1] for a general class of rough sources introduced in [2]. Applications include nonconforming discretisations for the von Kármán plate and the stream-vorticity formulation of the stationary Navier-Stokes equations in 2D by the Morley, two versions of discontinuous Galerkin, C^0 interior penalty, and WOPSIP methods.

REFERENCES

[1] C. Carstensen, B. Gräßle, and N. Nataraj. A posteriori error control for fourth-order semilinear problems with quadratic nonlinearity. *SIAM Journal on Numerical Analysis*, arXiv:2309.08427, 2024.

[2] C. Carstensen, B. Gräßle, and N. Nataraj. Unifying a posteriori error analysis of five piecewise quadratic discretisations for the biharmonic equation. *Journal of Numerical Mathematics*, (1):77–109, 2024.

[3] C. Carstensen, N. Nataraj, G. C. Remesan, and D. Shylaja. Unified a priori analysis of four second-order FEM for fourth-order quadratic semilinear problems. *Numerische Mathematik*, (3-4):323–368, 2023.

ADAPTIVE VIRTUAL ELEMENT METHODS FOR THE VIBRATION AND BUCKLING OF KIRCHHOFF PLATES

Joscha Gedicke[1] **and Luca Stefan Poensgen**[1]

[1] Institut für Numerische Simulation, Universität Bonn, Germany

We address the solution of the eigenvalue problem for the biharmonic equation using an adaptive method on a polygonal mesh. These eigenvalue problems, with L^2 or H^1 inner products on the right hand side, arise when considering the vibration and buckling of thin plates. We will utilize a virtual element method to approximate the solution of the equation. This approach not only allows for the use of polygonal meshes but also enables the construction of C^1 conforming elements of degree 2 or 3, with only 9 and 12 degrees of freedom on triangular elements, respectively. A primary objective, however, is the development of a residual error estimator. We will employ well-known arguments to prove reliability and efficiency; for example, the latter will be established using the bubble function technique. Concluding numerical experiments will illustrate the applicability and simplicity of the method, as well as its robust convergence properties in relation to different shapes of the polygonal elements and singularities in the solution.

ANALYSIS OF A STABILIZED FINITE ELEMENT METHOD SCHEME FOR A CHEMOTAXIS SYSTEM

Christos Pervolianakis[1]

[1] Institut für Mathematik und Informatik, Universität Jena, Germany

In this talk we consider a Chemotaxis system on a bounded domain $\Omega \subset \mathbb{R}^2$. We present a modification of the stabilized fully-discrete scheme that introduced in [1], where the spatial variable is discretized by finite element method while the temporal variable with Backward Euler. For the presented stabilized scheme, we prove results concerning the existence and uniqueness of the fully discrete solution. Moreover, we prove results concerning the positivity and the mass conservation of the resulting fully discrete scheme as well as error estimates in L_2 and H^1 norm. Several numerical experiments are performed to investigate the convergence rate in L_2 and H^1 norm of the error of the fully discrete scheme.

REFERENCES

[1] Strehl, R., Sokolov, A., Kuzmin, D., and Turek, S. *A flux-corrected finite element method for chemotaxis problems.* Comput. Methods Appl. Math 10 (2010), 219–232.

CONVERGENCE OF ADAPTIVE CROUZEIX-RAVIART AND MORLEY FEM FOR DISTRIBUTED OPTIMAL CONTROL PROBLEMS

Asha K. Dond[1], **Neela Nataraj**[1,2], **Subham Nayak**[1]

sn20@iisertvm.ac.in
[1] Indian Institute of Science Education and Research Thiruvananthapuram 695551, India
[2] Indian Institute of Technology Bombay, Powai, Mumbai 400076, India

This talk focuses the quasi-optimality of adaptive nonconforming finite element methods for the distributed optimal control problems governed by m-harmonic operators, with $m = 1, 2$. The variational discretization approach is adopted for the discretization of the control variable while the state and adjoint variables are discretized employing nonconforming finite elements. Error equivalence results are explored at both the continuous and discrete levels; leading to the derivation of a priori and a posteriori error estimates for the optimal control problem. Through the establishment of a general axiomatic framework encompassing stability, reduction, discrete reliability, and quasi-orthogonality; the quasi-optimality of the proposed methodology is rigorously demonstrated. Numerical experiments are conducted to validate the theoretically predicted orders of convergence.

REFERENCES

[1] Asha K. Dond, Neela Nataraj, and Subham Nayak. *Convergence of adaptive Crouzeix–Raviart and Morley FEM for distributed optimal control problems. Computational Methods in Applied Mathematics, 2024*

A POSTERIORI ERROR ESTIMATES FOR NONCONFORMING DISCRETIZATIONS OF SINGULARLY PERTURBED BIHARMONIC OPERATORS

Dietmar Gallistl[1], and Shudan Tian[2]

[1] Institut für Mathematik, Friedrich-Schiller-Universität Jena, Germany
[2] School of Mathematics and Computational Science, Xiangtan University, China
shudan.tian@xtu.edu.cn

This presentation will first introduce two families of MT-satisfying nonconforming elements. These elements are suitable for application in second-order non-divergence form equations, biharmonic equations, and fourth-order singularly perturbation problems, among others. Then, for the fourth-order singularly perturbation problems, we present a residual-based error estimator applicable to these two families elements. This estimator can also be extended to many existing H2 nonconforming elements. The error estimator involves the local best-approximation error of the finite element function by piecewise polynomial functions of the degree determining the expected approximation order, which need not coincide with the maximal polynomial degree of the element, for example if bubble functions are used. The error estimator is shown to be reliable and locally efficient up to this polynomial best-approximation error and oscillations of the right-hand side.

REFERENCES

[1] D. Gallistl and S. Tian, *A posteriori error estimates for nonconforming discretizations of singularly perturbed biharmonic operators* arXiv:2310.15665, 2023.

[2] D. Gallistl and S. Tian, *Continuous finite elements satisfying a strong discrete Miranda–Talenti identity* arXiv:2209.12500, 2022.

NUMERICAL ANALYSIS FOR ELECTROMAGNETIC SCATTERING WITH NONLINEAR BOUNDARY CONDITIONS

Jörg Nick

Seminar for Applied Mathematics, ETH Zürich, Switzerland

The talk covers a time-dependent electromagnetic scattering prob- lem from obstacles whose interaction with the wave is fully determined by a nonlinear boundary condition. In particular, the boundary condition studied enforces a power law type relation between the electric and magnetic field along the boundary. Based on time-dependent jump conditions of classical boundary operators, we derive a nonlinear system of time-dependent boundary integral equations that determines the tangential traces of the scattered electric and magnetic fields. Fully discrete schemes are obtained by discretising the nonlinear boundary integral equations with Runge–Kutta based convolution quadrature in time and Raviart–Thomas boundary elements in space. Error bounds with explic- itly stated convergence rates are presented. Numerical experiments illustrate the use of the proposed method and provide empirical convergence rates

REFERENCES

[1] J. Nick, Numerical analysis for electromagnetic scattering with nonlinear boundary conditions, *Math. Comp.*, Published electronically (2023)

ANALYSIS AND NUMERICAL APPROXIMATION OF STATIONARY SECOND-ORDER MEAN FIELD GAME PARTIAL DIFFERENTIAL INCLUSIONS

Yohance A. P. Osborne[1] **and Iain Smears**[1]

[1] University College London, Department of Mathematics, 25 Gordon Street, London, WC1H 0AY, UK, yohance.osborne.16@ucl.ac.uk

Mean Field Games (MFG) are models for Nash equilibria of large population stochastic differential games of optimal control. Under simplifying assumptions, these equilibria are described by non-linear systems in which a Hamilton— Jacobi—Bellman (HJB) equation and a Kolmogorov—Fokker—Planck (KFP) equation are coupled. In the classical formulation of the MFG system, the advective term of the KFP equation involves a partial derivative of the Hamiltonian that is assumed to be continuous. However, in many cases of practical interest, the underlying optimal control problem of the MFG may give rise to bang-bang controls, which typically lead to non-differentiable Hamiltonians.

In this talk we present results on the analysis and numerical approximation of stationary second-order MFG systems for the general case of convex, Lipschitz, but possibly non-differentiable Hamiltonians. In particular, we propose a generalization of the MFG system as a Partial Differential Inclusion (PDI) based on interpreting the partial derivative of the Hamiltonian in terms of subdifferentials of convex functions. We prove the existence of unique weak solutions to MFG PDIs under a monotonicity condition similar to one that has been considered previously by Lasry & Lions. Moreover, we introduce a monotone finite element discretization of the weak formulation of MFG PDIs and present theorems on the strong convergence of the approximations to the value function in the H^1-norm and the strong convergence of the approximations to the density function in L^q-norms. We conclude the talk with discussion of some numerical experiments involving non-smooth solutions.

REFERENCES

[1] Yohance A. P. Osborne and Iain Smears. “Analysis and numerical approximation of stationary second-order mean field game partial differential inclusions”. In: *SIAM J. Numer. Anal.* 62.1 (2024), pp. 138–166. ISSN: 0036-1429,1095-7170. DOI: 10.1137/22M1519274. URL: https://doi.org/10.1137/22M1519274.

10th International Conference on Computational Methods in Applied Mathematics
CMAM-10
10–14 June 2024, Bonn, Germany

EXACT ERROR ANALYSIS OF A LINEARIZED HARMONIC MAP PROBLEM

Sören Bartels[1] **and** **Vera Jackisch**[1]

[1] Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Germany
Contact: vera.jackisch@mathematik.uni-freiburg.de

In this talk we present a linearization of the harmonic map problem, which allows us to prove meaningful a-priori and sharp a-posteriori error estimates. We derive those error estimates via convex duality arguments, applied to both the continuous and the discrete model. As a discretization, we use Crouzeix-Raviart finite elements for the primal problem and Raviart-Thomas finite elements for the dual problem, since we can establish a connection between both discrete solutions via the Marini formula. We present numerical experiments using problems with smooth and singular solutions and compare our results to the discretized harmonic map flow.

REFERENCES

[1] S. Bartels and A. Kaltenbach, *Exact a posteriori error control for variational problems via convex duality and explicit flux reconstruction*, Preprint, 2024.

[2] S. Bartels, *Nonconforming discretizations of convex minimization problems and precise relations to mixed methods*, Comput. Math. Appl., 93:214–229, 2021.

ANALYSIS AND APPROXIMATION OF INCOMPRESSIBLE CHEMICALLY REACTING GENERALIZED NEWTONIAN FLUID

Seungchan Ko[1] **and Endre Süli**[2]

[1] Department of Mathematics, Inha University, Republic of Korea, scko@inha.ac.kr
[2] Mathematical Institute, University of Oxford, United Kingdom, Endre.Suli@maths.ox.ac.uk

We consider a system of nonlinear partial differential equations modeling the steady motion of an incompressible non-Newtonian fluid, which is chemically reacting. The governing system consists of a steady convection-diffusion equation for the concentration and the generalized steady Navier–Stokes equations, where the viscosity coefficient is a power-law type function of the shear-rate, and the coupling between the equations results from the concentration-dependence of the power-law index. This system of nonlinear partial differential equations arises in mathematical models of the synovial fluid found in the cavities of moving joints. We construct a finite element approximation of the model and perform the mathematical analysis of the numerical method. Key technical tools include discrete counterparts of the Bogovskiĭ operator, De Giorgi's regularity theorem in two dimensions, and the Acerbi-Fusco Lipschitz truncation of Sobolev functions, in function spaces with variable integrability exponents.

REFERENCES

[1] S. Ko, P. Pustějovská and E. Süli, *Finite element approximation of an incompressible chemically reacting non-Newtonian fluid*, ESAIM: M2AN Volume 52, Number 2, 2018

[2] S. Ko and E. Süli, *Finite element approximation of steady flows of generalized Newtonian fluids with concentration-dependent power-law index*, Math. Comp. 88, 1061-1090, 2019

[3] S. Ko, *Existence of global weak solutions for unsteady motions of incompressible chemically reacting generalized Newtonian fluids*, Journal of Mathematical Analysis and Applications, Volume 513, Issue 1, 2022.

CONVERGENCE RATE FOR A SPACE-TIME DISCRETIZATION FOR INCOMPRESSIBLE GENERALIZED NEWTONIAN FLUIDS: THE DIRICHLET PROBLEM FOR $P > 2$

Mirjam Hoferichter[1] **and Michael Růžička**[2]

[1] Abteilung für Angewandte Mathematik, Universität Freiburg, Germany,
mirjam.hoferichter@mathematik.uni-freiburg.de
[2] Abteilung für Angewandte Mathematik, Universität Freiburg, Germany

We present convergence rates for solutions of the equations describing the unsteady motion of incompressible shear-thickening fluids with homogeneous Dirichlet boundary conditions. A full space-time semi-implicit scheme based on a backward Euler scheme in time and a Finite Element discretization in space is considered. Berselli and Růžička were the first to obtain error estimates without the introduction of intermediate semi-discrete problems in [1] which strongly inspired the presented proof.
The main novelty is the consideration of the shear-thickening case $p > 2$ for which convergence rates have, up until now, only been proven for the generalized Stokes equation using intermediate semi-discrete problems (see [2]).

REFERENCES

[1] L. Berselli, M. Růžička (2021). *Optimal error estimate for a space-time discretization for incompressible generalized Newtonian fluids: the Dirichlet problem.* Partial Differ. Equ. Appl., 2.4, 1–23.

[2] S. Eckstein, M. Růžička (2018). *On the full space-time discretization of the generalized Stokes equations: The Dirichlet case.* SIAM J. Numer. Anal., 56.4, 2234–2261.

A NITSCHE METHOD FOR FLUID FLOW WITH SET-VALUED BOUNDARY CONDITIONS

Pablo Alexei Gazca Orozco[1], Franz Gmeineder[2], Erika Maringová Kokavcová[3], Tabea Tscherpel[4]

[1] Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Germany
[2] Fachbereich Mathematik und Statistik, Universität Konstanz, Germany
[3] Institute of Science and Technology Austria (ISTA), Austria
[4] Fachbereich Mathematik, TU Darmstadt, Germany, tscherpel@mathematik.tu-darmstadt.de

Fluids may exhibit complex behaviour at the boundary such as stick-slip behaviour, possibly including a time dependence. Typically, this cannot be described by linear boundary conditions such as no-slip or Navier slip conditions. To describe phenomena like that nonlinear boundary conditions are used, that may be set-valued or even non-monotone.

In this talk we present a mixed finite element approximation for incompressible fluids with such nonlinear boundary conditions. We employ the Nitsche method to impose the non-penetration condition of the boundary conditions by penalisation, see e.g. [2]. This is motivated by the fact that a direct imposition of boundary conditions on polygonal approximations of a curved boundary may lead to a Babuška type paradox [1]. Due to the penalisation, the convergence proof requires a novel Korn inequality involving trace terms. The possibly set-valued nature of the boundary conditions is treated by means of monotone graph approximation. We present numerical experiments for several types of boundary conditions.

REFERENCES

[1] R. Verfürth (1985). *Finite element approximation of steady Navier-Stokes equations with mixed boundary conditions.* RAIRO Modél. Math. Anal. Numér. 19, no. 3, 461–475.

[2] I. G. Gjerde, L. R. Scott (2022). *Nitsche's method for Navier-Stokes equations with slip boundary conditions.* Math. Comp. 91, no. 334, 597–622.

AN ADAPTIVE ITERATIVE LINEARISED FINITE ELEMENT METHOD FOR THE NUMERICAL SOLUTION OF STATIONARY BINGHAM FLUID FLOW PROBLEMS

Pascal Heid[1] **and Endre Süli**[2]

[1] FHNW, Switzerland, hepa@ma.tum.de
[2] Mathematical Institute, University of Oxford, UK

In this talk, we further extend the theory in [C. Kreuzer and E. Süli, Adaptive finite element approximation of steady flows of incompressible fluids with implicit power-law-like rheology, ESAIM: Math. Model. Numer. Anal.50 (5) (2016) 1333–1369] on the adaptive finite element analysis of implicitly constituted incompressible fluid flow problems by taking into account the approximation of the nonlinear finite element solutions by an iterative solver. For simplicity of the presentation, we shall solely focus on Bingham fluids, both with and without the convective term. We will present a computable algorithm with the favourable property that a subsequence of the sequence of iterates generated converges weakly to a solution of the given problem. Moreover, under a small data assumption, we will verify the uniqueness of the solution. The performance of the adaptive iterative linearised finite element algorithm will be illustrated by a numerical experiment.

REFERENCES

[1] P. Heid and E. Süli, An adaptive iterative linearised finite element method for implicitly constituted incompressible fluid flow problems and its application to Bingham fluids, Appl. Numer. Math. 181, 2022.

ERROR ESTIMATES FOR A FINITE ELEMENT DISCRETIZATION OF GENERALIZED NAVIER–STOKES EQUATIONS

Julius Jeßberger[1] **and Alex Kaltenbach**[1]

[1] Department of Applied Mathematics, University of Freiburg, Germany,
julius.jessberger@mathematik.uni-freiburg.de
[2] Institute of Mathematics, Technical University of Berlin, Germany,
kaltenbach@math.tu-berlin.de

We propose a finite element discretization for the steady, generalized Navier–Stokes equations for fluids with shear-dependent viscosity, completed with inhomogeneous Dirichlet boundary conditions and an inhomogeneous divergence constraint. We establish a priori error estimates for the velocity vector field and the scalar kinematic pressure. Numerical experiments complement the theoretical findings: while they confirm the quasi-optimality of velocity error rates, they indicate that there is still room for improvement of pressure error rates, at least for some values of the shear rate exponent.

REFERENCES

[1] J. Jeßberger, A. Kaltenbach, *Finite element discretization of the steady, generalized Navier–Stokes equations with inhomogeneous Dirichlet boundary conditions*, Arxiv, 2023.

REACHING THE EQUILIBRIUM: LONG-TERM STABLE NUMERICAL SCHEMES FOR DETERMINISTIC AND STOCHASTIC p-STOKES SYSTEMS

Jérôme Droniou[1,2], Kim-Ngan Le[1] and Jörn Wichmann[1]

[1] School of Mathematics, Monash University, Australia
[2] IMAG, Univ. Montpellier, CNRS, Montpellier, France

We propose a general class of numerical schemes for deterministic and stochastic p-Stokes systems that are stable for arbitrary long times. We show that these approximations asymptotically concentrate in an equilibrium – a steady state solution and an invariant measure for the deterministic and stochastic evolution, respectively. Moreover, we quantify the time to reach the equilibrium. The theoretical findings are consolidated by numerical experiments.

REFERENCES

[1] K-N. Le and J. Wichmann, *A class of space-time discretizations for the stochastic p-Stokes system*, arXiv, 2307.13253.

COUPLED 3D-1D SYSTEMS: DERIVATION, ERROR ANALYSIS, AND DISCONTINUOUS GALERKIN METHODS

Rami Masri[1]

[1] Department of Numerical Analysis and Scientific Computing, Simula Research Laboratory

This work is in collaboration with Miroslav Kuchta, Marius Zeinhofer and Marie E. Rognes at Simula and with Beatrice Riviere at Rice University

We consider 3D-1D coupled systems resulting from topological model order reduction techniques. Such systems model diffusion in a 3D domain containing a small inclusion reduced to its 1D centerline. We discuss the derivation and model error analysis for these systems. Further, we propose interior penalty discontinuous Galerkin (DG) methods for the 3D-1D systems. Due to the dimensionality gap, the 3D solution lacks the regularity properties that are typically needed for the error analysis of DG methods. We show convergence to weak solutions of a steady state problem via deriving a posteriori error estimates and bounds on residuals defined with suitable lift operators. For the time dependent problem, a backward Euler DG formulation is also presented and analyzed. Further, we propose a DG method for networks embedded in 3D domains, which is, up to jump terms, locally mass conservative on bifurcation points. Numerical examples in idealized geometries portray our theoretical findings, and simulations in realistic 1D networks show the robustness of our method.

A SIMPLE FINITE ELEMENT SCHEME FOR H(DIV DIV) INTERFACE PROBLEM

Shuo Zhang[1,2]

[1] Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China
[2] University of Chinese Academy of Sciences, Beijing, China
szhang@lsec.cc.ac.cn

In this talk, we study the finite element discretization of H(div div) interface problem. The talk contains two parts, namely to construct simple H(div div) finite element scheme and to construct simple finite element scheme for H(div div) interface problem. The theory of adjoint operators and the structure of finite element complexes serve as the theoretical basis. The specific structural features hint optimal solution strategy to the numerical schemes. The work is partially supported by Chinese Academy of Sciences (XDB0640000) and NSFC (12271512).

REFERENCES

[1] Jinchao Xu, Shuo Zhang, *Optimal finite element methods for interface problems*, in Proceedings of the 22nd International Conference of Domain Decomposition, 77–91, Springer International Publishing, 2016.

[2] Shuo Zhang, *Nonconforming finite element de Rham complexes*, submitted, 2024.

[3] Shuo Zhang, *Partially adjoint discretizations of adjoint operators*, *arXiv:2206.12114*, 2022.

[4] Shuo Zhang, *Low-degree* H(divdiv) *finite element schemes in* $\mathbb{R}^n$, in preparation.

BETWEEN MINIMAL REGULARITY AND STRONG SYMMETRY: FINITE ELEMENTS FOR THE REISSNER-MINDLIN PLATE

Adam Sky[1], Jack S. Hale[1], Michael Neunteufel[2] and Andreas Zilian[1]

[1] Institute of Computational Engineering, Department of Engineering, Faculty of Science, Technology and Medicine, University of Luxembourg, Esch-sur-Alzette, Luxembourg, {adam.sky@uni.lu, jack.hale@uni.lu, andreas.zilian@uni.lu}.

[2] Department of Mathematics and Statistics, Portland State University, USA, mneunteu@pdx.edu

In this talk we discuss mixed variational formulations of Hellinger-Reissner type of the Reissner-Mindlin plate problem and appropriate finite element constructions for their numerical solution. We focus on the relative merits of the respective constructions, specifically, their regularity, conformity, and computational cost.

A formulation based on weak symmetry of the bending moment tensor is presented in [1]. The bending moments are chosen $\boldsymbol{M} \in [H(\mathrm{div}, A)]^2$, the Sobolev space of square-integrable tensor-valued functions with row-wise square-integrable divergence. Because of this row-wise construction, the natural symmetry of the bending moments is not satisfied *a priori* and must be enforced weakly with an additional field.

In our recent paper [2] we present a new formulation with bending moments discretised using Hu-Zhang [3] finite elements $\mathcal{HZ}^p(A) \subset H^{\mathrm{sym}}(\mathrm{Div}, A)$ that are strongly symmetric by construction. This removes the need to enforce symmetry weakly as in [1]. Furthermore, stability and convergence follows automatically from the associated discrete de Rham complexes.

REFERENCES

[1] Beirão da Veiga, L., Mora, D. and Rodríguez, R., Numerical analysis of a locking-free mixed finite element method for a bending moment formulation of Reissner-Mindlin plate model. *Numerical Methods for Partial Differential Equations*, 29(1), 40–63, 2013.

[2] Sky, A., Neunteufel, M., Hale, J. S. and Zilian A., A Reissner-Mindlin plate formulation using symmetric Hu-Zhang elements via polytopal transformations. *Computer Methods in Applied Mechanics and Engineering*, Vol. 416, pp. 116291, 2023.

[3] Hu, J. and Zhang, S., Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^n$: The higher order case. *Journal of Computational Mathematics*, 33(3), 283-296, 2019.

LOCAL BOUNDED COMMUTING PROJECTION OPERATORS FOR DISCRETE GRADGRAD COMPLEXES

Jun Hu[1], **Yizhou Liang**[2] **and Ting Lin**[1]

[1] School of Mathematical Sciences, Peking University, China
[2] Institute of Mathematics, University of Augsburg, Germany, yizhou.liang@uni-a.de

In this talk we present the construction of local bounded commuting projections for discrete subcomplexes of the gradgrad complexes in two and three dimensions, which play an important role in the finite element theory of elasticity (2D) and general relativity (3D). The construction first extends the local bounded commuting projections to the discrete de Rham complexes to other discrete complexes. Moreover, the argument also provides a guidance in the design of new discrete gradgrad complexes.

REFERENCES

[1] J. Hu, Y. Liang, T. Lin. *Local Bounded Commuting Projection Operators for Discrete Gradgrad Complexes*, ArXiv:2304.11566

A NEW DIV-DIV-CONFORMING SYMMETRIC TENSOR FINITE ELEMENT SPACE WITH APPLICATIONS TO THE BIHARMONIC EQUATION

Long Chen[1] **and** **Xuehai Huang**[2]

[1] Department of Mathematics, University of California at Irvine, USA
[2] School of Mathematics, Shanghai University of Finance and Economics, China,
huang.xuehai@sufe.edu.cn

A new $H(\operatorname{div}\operatorname{div})$-conforming finite element is presented, which avoids the need for supersmoothness by redistributing the degrees of freedom to edges and faces. This leads to a hybridizable mixed method with superconvergence for the biharmonic equation. Moreover, new finite element divdiv complexes are established. Finally, new weak Galerkin and C^0 discontinuous Galerkin methods for the biharmonic equation are derived.

REFERENCES

[1] L. Chen and X. Huang, *A new div-div-conforming symmetric tensor finite element space with applications to the biharmonic equation*, Math. Comp., accepted, 2024.

FINITE ELEMENT DIVDIV COMPLEXES ON TETRAHEDRAL AND CUBOID MESHES

Jun Hu[1], Yizhou Liang[2], Rui Ma[3] and Min Zhang[4]

[1] Peking University, China
[2] University of Augsburg, Germany
[3] Beijing Institute of Technology, China, rui.ma@bit.edu.cn
[4] Beijing Forestry University, China

This talk will introduce some conforming finite element divdiv complexes on tetrahedral and cuboid meshes. Besides, this talk will present the applications to algebraic structure-preserving finite element discretization of both the biharmonic equation and the linearized Einstein-Bianchi system.

REFERENCES

[1] J. Hu, Y. Liang and R. Ma, *Conforming finite element divdiv complexes and the application for the linearized Einstein-Bianchi system*, SIAM J. Numer. Anal., 60(3): 1307-1330, 2022.

[2] J. Hu, Y. Liang, R. Ma and M. Zhang, *New conforming finite element divdiv complexes in three dimensions*, arXiv:2204.07895, 2022.

[3] J. Hu, R. Ma and M. Zhang. *A family of mixed finite elements for the biharmonic equations on triangular and tetrahedral grids. Sci. China Math.*, 64: 2793-2816, 2021.

DISTRIBUTIONAL COMPLEXES: HESSIAN, DIVDIV AND ELASTICITY

Ting Lin

[1] Peking University, China

Recently, there has been a growing interest in discretizing differential complexes beyond the de Rham case, including the Hessian, divdiv, and elasticity complexes. A conforming finite element discretization may involve high polynomial degrees. In contrast, using distributional spaces is much more computationally efficient. Moreover, distributional spaces can be categorized as elements from the dual mesh, which formally establishes a connection between finite element exterior calculus and discrete exterior calculus.

In this talk, I will discuss the distributional Hessian, divdiv, and elasticity complexes and their cohomologies in both 2D and 3D settings. We will prove that the cohomologies are isomorphic to their continuum counterparts. For the Hessian and divdiv complexes, we will first construct the discretization. As for the elasticity complex, we will delve into the Regge complex. Additionally, we will explore the twisted complex of the Regge complex, which can be seen as a differential complex perspective of the microstructure elasticity model.

STABILIZED SPACE-TIME FINITE ELEMENT SCHEMES ON ANISOTROPIC MESHES FOR LINEAR PARABOLIC EQUATIONS,

Ioannnis Toulopoulos[1]

[1] Department of Informatics, University of Western Macedonia, GREECE

Consider the following diffusion-advection-reaction problem: find $u(x,t) : \bar{Q}_T \to \mathbb{R}$ such that

$$u_t - \operatorname{div}(\varepsilon \nabla_x u) + \boldsymbol{\beta} \cdot \nabla_x u + ru = f \qquad \text{in } Q_T = (0,T] \times \Omega \tag{1a}$$

$$u = u_\Sigma = 0 \qquad \text{on } \Sigma := \partial\Omega \times [0,T], \tag{1b}$$

$$u(x,0) = u_0(x) \qquad \text{on } \Sigma_0 := \Omega \times \{0\}, \tag{1c}$$

where Ω is a bounded cuboid domain in $\mathbb{R}^{d_x}$, with $d_x = 1, 2, 3$, $T > 0$ a fixed time, $\nabla_x u$ is the spatial gradient of u, $\varepsilon > 0$, $r \geq 0$ are the diffusion and reaction coefficients, and $\boldsymbol{\beta} := (\beta_x, \beta_y, \beta_z)$ constant vector. In this work, we extend the methodology of space-time finite element methods (STFEMs) of pure parabolic problems to present a stable STFEM for (1) on anisotropic meshes. Again the main idea is to consider the temporal variable t as another spatial variable and to discretize (1) in a unified way by applying finite element methodologies in the whole Q_T. Here, the numerical scheme is stabilized by adding appropriate extra consistent terms. These terms are weighted by coefficients which are constructed by taking into account the anisotropic character of the mesh. The aim is to derive an anisotropic discretization error analysis and to present estimates uniform with respect to ε. We show that the produced discrete bilinear form is elliptic with respect to the discrete norm and show a-priori error estimates taking into account the anisotropic character of the meshes. In the last part of the work, a series of numerical examples are presented which support the theoretical results and illustrate the performance of the proposed STFEM. This work is based on [1].

Acknowledgment. The author is grateful to the Department of Informatics, University of Western Macedonia GREECE, for supporting his participation to the 10th International Conference on Computational Methods in Applied Mathematics (CMAM-10), Bonn, Germany, June 10-14, 2024

REFERENCES

[1] I. Toulopoulos, *Designing a stable space-time finite element scheme on anisotropic meshes for general parabolic problems*, under-review, 2024.

SUPERCLOSENESS AND ASYMPTOTIC ANALYSIS OF THE CROUZEIX-RAVIART AND ENRICHED CROUZEIX-RAVIART ELEMENTS

Wei Chen[1], Hao Han[1] and Limin Ma[2]

[1] LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China
[2] School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, P. R. China

In this talk we consider the asymptotic expansions of eigenvalues by the Crouzeix-Raviart element and enriched Crouzeix-Raviart element by establishing two pseudostress interpolations, which admit a full one-order supercloseness with respect to the numerical velocity and the pressure, respectively. The design of these interpolations overcomes the difficulty caused by the lack of supercloseness of the canonical interpolations for the two nonconforming elements, and leads to an intrinsic and concise asymptotic analysis of numerical eigenvalues, which proves an optimal superconvergence of eigenvalues by the extrapolation algorithm. Meanwhile, an optimal superconvergence of postprocessed approximations is proved by use of this supercloseness. We provide numerical experiments to verify the theoretical results.

REFERENCES

[1] Wei Chen, Hao Han and Limin Ma *Supercloseness and asymptotic analysis of the Crouzeix-Raviart and enriched Crouzeix-Raviart elements for the Stokes problem*, arXiv:2401.17702 (2024).

[2] Jun Hu and Limin Ma *Asymptotic expansions of eigenvalues by both the Crouzeix-Raviart and enriched Crouzeix-Raviart elements*, Math. Comp. **91** (2021), no. 333, 75–109

A NONCONFORMING FINITE ELEMENT METHOD FOR STOKES INTERFACE PROBLEMS ON A LOCAL ANISOTROPIC HYBRID MESH

Hua Wang[1] **and Fengren Zou**[1]

[1] School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China, wanghua@xtu.edu.cn

In this talk, we introduce a low-order nonconforming finite element method for Stokes interface problems. The method is implemented on a local anisotropic hybrid mesh, which is generated by connecting intersection points of the interface with an underlying unfitted mesh.

By utilizing this hybrid mesh, we develop a CR/rotated Q_1 type element for the velocity space, accompanied by a piecewise P_0 element for the pressure space. Moreover, we establish the inf-sup condition without the need for stabilization terms and demonstrate the linear convergence rates for both velocity and pressure spaces in the H^1 and L^2 norms.

Numerical experiments are provided to validate our theoretical results.

MODELING FLUID FILTRATION IN POROUS MEDIA BY AN OVERLAPPING APPROACH

Marco Discacciati[1], **<u>Paola Gervasio</u>**[2]

[1] Department of Mathematical Sciences, Loughborough University, United Kingdom, M.Discacciati@lboro.ac.uk
[2] DICATAM, University of Brescia, Italy, paola.gervasio@unibs.it

The talk focuses on the validation of the Interface Control Domain Decomposition (ICDD) method in the context of the Stokes-Darcy problem to model the filtration of fluids in porous media.

Differently from the commonly used approach that imposes the Beavers-Joseph-Saffman coupling conditions at a sharp interface between the fluid region and the porous medium, the ICDD method [1, 2] considers an overlapping decomposition of the computational domain and it looks for local velocities and pressures such that the velocity and pressure jumps are minimized on interfaces internal to the fluid domain and to the porous medium domain, respectively.

To validate the ICDD method, its solution is compared with the one computed by solving the Stokes equations at the microscale. The analysis allows us to identify the best width of the overlapping region and its position inside the transition zone that separates the free–fluid regime from the porous-medium regime. Finally, in the case of homogeneous porous media, we show that the ICDD solution is an approximation of order ε of the Stokes solution at the microscale, where ε is the ratio between the micro and the macroscale.

REFERENCES

[1] M. Discacciati, P. Gervasio, A. Giacomini, and A. Quarteroni. The interface control domain decomposition method for Stokes-Darcy coupling. *SIAM J. Numer. Anal.* (2014) **54**:1039–1068

[2] M. Discacciati, P. Gervasio. A coupling concept for Stokes-Darcy systems: the ICDD method *arXiv:2401.12602* (2024)

PRECONDITIONERS FOR STOKES–DARCY PROBLEMS

Iryna Rybak[1,*] **and Paula Strohbeck**[1]

[1] Institute of Applied Analysis and Numerical Simulation, University of Stuttgart, Germany
[*] iryna.rybak@ians.uni-stuttgart.de

Coupled systems of free flow and porous-medium flow arise routinely in environmental, industrial and medical settings. Such flow problems are usually described by the Stokes equations in the free-flow domain, Darcy's law in the porous medium and appropriate coupling conditions on the fluid–porous interface. Discretisations of these Stokes–Darcy problems lead to large, sparse, ill-conditioned and non-symmetric linear systems. Therefore, efficient preconditioners are needed to accelerate convergence of the applied Krylov method.

In this talk we present several preconditioners for the Stokes–Darcy problems with different sets of coupling conditions: block diagonal, block triangular and constraint preconditioners [1, 2]. We also provide spectral and field-of-values analysis, and illustrate efficiency and robustness of the proposed preconditioners in numerical experiments.

REFERENCES

[1] P. Strohbeck and I. Rybak, *Efficient preconditioners for coupled Stokes–Darcy problems*, SIAM J. Sci. Comput. (in review), 2024.

[2] P. Strohbeck, C. Riethmüller, D. Göddeke and I. Rybak, *Robust and efficient preconditioners for Stokes–Darcy problems*, in: E. Franck et al. (eds.) Finite Volumes for Complex Applications X - Volume 1, Elliptic and Parabolic Problems. pp. 375–383. Springer Nature Switzerland, 2023.

CUT-CELL DISCRETIZATIONS FOR HYPERBOLIC CONSERVATION LAWS

Gunnar Birke[1] **and Christian Engwer**[1]

[1] Angewandte Mathematik, Universität Münster, Deutschland

Cut-Cell methods offer a way to handle complex geometries, models on different subdomains or interfaces. The complex meshing proceedure is avoided by directly incorporating imaging data to define sub-cells/cut-cells, belonging to different sub-domains.

A lot of progress has been observed for conforming or discontinuous Galerkin discretizations of elliptic or parabolic PDEs on such cut-cell meshes. All boils down to questions of stability and introducing additional penalty terms to ensure stability of the arising operator. We present recent work on such stabilization techniques for hyperbolic conservations laws, e.g. transport and wave equation. We can proof L^2-stability of the semi-discrete operator and observe the expected higher-order convergence rates.

PRECONDITIONING STRATEGIES FOR PRECIPITATION AND DISSOLUTION

Cedric Riethmüller[1,*], Dominik Göddeke[1,2], Christian Rohde[1,2], Lars von Wolff[1]

[1] Institute of Applied Analysis and Numerical Simulation, University of Stuttgart, Stuttgart, Germany
[*] cedric.riethmueller@ians.uni-stuttgart.de
[2] Stuttgart Center for Simulation Science, University of Stuttgart, Stuttgart, Germany

In recent years, much effort has been put into the development of mathematical models for reactive transport in porous media while less focus has been on efficient numerical techniques to simulate them. In this talk we focus on precipitation and dissolution effects. We investigate multi-phase flow problems, comprising the Navier-Stokes equations for evaluating the flow field and the Cahn-Hilliard equation for calculating the evolving diffuse interfaces between the fluid and solid phases. We use Newton's method in order to solve the discrete, nonlinear problem arising from the spatio-temporal discretization. A key focus of the talk is the solution of the resulting large, sparse and ill-conditioned linear systems. Using problem-adapted and parameter-robust preconditioning, these systems can be solved much more efficiently than by using stock techniques. We discuss both monolithic as well as partitioned approaches.

A DECOUPLED SOLVER FOR BIOT'S MODEL

Francisco J. Gaspar[1], **Alvaro Pe de la Riva**[1], **Carmen Rodrigo**[1], **James Adler**[2], **Xiaozhe Hu**[2] **and Ludmil Zikatanov**[3]

[1] Zaragoza University, Spain
[2] Tufts University, USA
[3] Penn State, University Park, USA

In this work, we propose a new stabilization method aimed at removing the spurious oscillations in the pressure approximation of the Biot's model for low permeabilities. The new discretization allows us to iterate the fluid and mechanic problems in a fashion similar to the well-known fixed-stress split method. We also present numerical results illustrating the robust behavior of the iterative solver with respect to the physical and discretization parameters of the model.

TIME-CONTINUOUS STRONGLY CONSERVATIVE SPACE-TIME FINITE ELEMENT METHODS FOR THE DYNAMIC BIOT MODEL

Johannes Kraus[1] **and Maria Lymbery**[1] **and Kevin Osthues**[1] **and Fadi Philo**[1]

[1]Lehrstuhl für Numerische Mathematik, Universität Duisburg-Essen, Germany,
johannes.kraus@uni-due.de

We consider the dynamic Biot model describing the interaction between fluid flow and solid deformation including wave propagation phenomena in both the liquid and solid phases of a saturated porous medium. The model couples a hyperbolic equation for momentum balance to a second-order in time dynamic Darcy law and a parabolic equation for the mass balance and is considered here in four-field formulation with the displacement of the elastic matrix, its time derivative, the fluid velocity, and the fluid pressure being the physical fields of interest.

A family of variational space-time finite element methods is proposed that combines a continuous-in-time Galerkin ansatz of arbitrary polynomial degree with inf-sup stable $H(\mathrm{div})$-conforming approximations of discontinuous Galerkin (DG) type of the displacement and its time derivative, and a mixed approximation of the flux-pressure pair. We prove error estimates in a combined energy norm as well as L^2 error estimates in space for the individual fields for both maximum and L^2 norm in time which are optimal for the displacement and pressure approximations [1].

REFERENCES

[1] J. Kraus, M. Lymbery, K. Osthues, F. Philo, *Analysis of a family of time-continuous strongly conservative space-time finite element methods for the dynamic Biot model*, arXiv:2401.04609 [math.NA], 2024.

QUANTIFYING DOMAIN UNCERTAINTY IN LINEAR ELASTICITY

Helmut Harbrecht[1], Viacheslav Karnaev[1] and Marc Schmidlin[1]

[1] Departement Mathematik und Informatik, Universität Basel, Switzerland

The numerical solution of the equations of linear elasticity is well understood if the input parameters are known. This, however, is often not the case in practical applications. In this talk uncertainties in the description of the computational domain is considered. To this end, we model the random domain as the image of some given fixed, nominal domain under random domain mapping. We then prove the analytic regularity of the random solution with respect to the countable random input parameters which enter the problem through the Karhunen-Loève expansion of the random domain mappings. In particular, we provide appropriate bounds on arbitrary derivatives of the random solution with respect to those input parameters, which enable the use of state-of-the-art quadrature methods to compute quantities of interest such as the mean and variance of the random von Mises stress in a dimensionally robust way.

REFERENCES

[1] H. Harbrecht, V. Karnaev, and M. Schmidlin. *Quantifying domain uncertainty in linear elasticity*, Preprint 2023-06, Fachbereich Mathematik, Universität Basel, Switzerland, 2023 (to appear in SIAM/ASA J. Uncertain. Quantif.)

MODEL ORDER REDUCTION FOR PARAMETRIC TIME-DEPENDENT PROBLEMS USING THE LAPLACE TRANSFORM

Fernando Henríquez and Jan S. Hesthaven

Chair of Computational Mathematics and Simulation Science, EPFL, Lausanne, Switzerland

We propose a reduced basis method for solving parametric, time-dependent partial differential equations using the Laplace transform. Unlike traditional approaches, we begin by applying the said transform to the evolution problem. This yields a time-independent boundary value problem that depends on the complex Laplace variable and the problem's parametric input.

Firstly, in an offline stage, we systematically sample the Laplace variable and the parameter space, and solve the underlying collection of full-order or high-fideliy problems. Subsequently, we employ Proper Orthogonal Decomposition (POD) on this set of solutions to obtain a basis of reduced dimensiom. Next, we project the original parametric onto this basis and solve the problem using any suitable time-stepping scheme, for any given new parametric input.

Numerical experiments for parabolic and second-order hyperbolic problems validate our theoretical claims and demonstrate the advantages of the proposed method, both in terms of accuracy and speed-up, in comparison to existing approaches.

REFERENCES

[1] F. Henríquez and J. S.Hesthaven, J. S. Fast Numerical Approximation of Parabolic Problems Using Model Order Reduction and the Laplace Transform. arXiv preprint arXiv:2403.02847.

[2] J. S. Hesthaven, C. Pagliantini, C., G. Rozza. Reduced basis methods for time-dependent problems. *Acta Numerica*, 31, 265-345, 2002

[3] D.B. Phuong Huynh, D. J. Knezevic, and A. T. Patera. A Laplace transform certified reduced basis method; application to the heat equation and wave equation. *Comptes Rendus. Mathématique*, 349.7-8 (2011): 401-405.

ON UNCERTAINTY QUANTIFICATION OF EIGENPAIRS WITH HIGHER MULTIPLICITY

Jürgen Dölz[1] and David Ebert[2]

[1] Institut für Numerische Simulation, Universität Bonn, Germany, doelz@ins.uni-bonn.de
[2] Institut für Numerische Simulation, Universität Bonn, Germany, ebert@ins.uni-bonn.de

We consider generalized operator eigenvalue problems in variational form with random perturbations in the bilinear forms. This setting is motivated by variational forms of partial differential equations with random input data. The considered eigenpairs can be of higher but finite multiplicity. We investigate stochastic quantities of interest of the eigenpairs and discuss why, for eigevalues of multiplicity greater than 1 in some parts of the parameter space, only the stochastic properties of the eigenspaces are meaningful, but not the ones of individual eigenpairs. To that end, we characterize the Fréchet derivatives of the eigenpairs with respect to the perturbation and provide a new linear characterization for eigenpairs of higher multiplicity. For the uncertainty quantification of eigenspaces we consider meaningful sampling strategies as well as perturbation approaches. Numerical examples are presented to illustrate the theoretical results.

REFERENCES

[1] J. Dölz and D. Ebert, *On uncertainty quantification of eigenvalues and eigenspaces with higher multiplicity*, SIAM Journal on Numerical Analysis, 62 (2024), pp. 422–451.

[2] J. Dölz, D. Ebert, S. Schöps, and A. Ziegler, *Shape uncertainty quantification of Maxwell eigenvalues and -modes with application to TESLA cavities*, 2024. arXiv:2401.11890.

THE DIMENSION WEIGHTED FAST MULTIPOLE METHOD FOR SCATTERED DATA APPROXIMATION

Helmut Harbrecht[1], **Michael Multerer**[2] **and** **Jacopo Quizi**[2]

[1]Departement für Mathematik und Informatik, Universität Basel, Switzerland
[2] Istituto Eulero, Università della Svizzera italiana, Switzerland

This presentation is concerned with scattered data approximation for higher dimensional data sets which exhibit an anisotropic behavior in the different dimensions. Tailoring sparse polynomial interpolation to this specific situation, we demonstrates very efficient degenerate kernel approximations which we then use in a dimension weighted fast multipole method. This method enables us to deal with many more dimensions than the standard black-box fast multipole method based on interpolation. A thorough analysis of the method is provided including rigorous error estimates. The accuracy and the cost of the approach are validated by extensive numerical results. As a relevant application, we apply the approach to a shape uncertainty quantification problem.

REFERENCES

[1] H. Harbrecht, M. Multerer and J. Quizi, *The dimension weighted fast multipole method for scattered data approximation*, arXiv preprint arXiv:2402.09531, 2024.

MULTILEVEL DOMAIN UNCERTAINTY QUANTIFICATION IN COMPUTATIONAL ELECTROMAGNETICS

Ruben Aylwin-Pincheira[1], Carlos Jerez-Hanckes[2], Christoph Schwab[3] and Jakob Zech[4]

[1] Institut für Numerische Mathematik, Universität Ulm, Germany
[2] Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibañez, Chile
[3] Seminar for Applied Mathematics, ETH Zürich, Switzerland
[4] Interdisciplinary Center for Scientific Computing, Universität Heidelberg, Germany

In this talk we present the numerical approximation of the time-harmonic Maxwell equations on uncertain geometries. Previously, we considered the approximation of the Maxwell equations through the use of (single level) Monte Carlo, Quasi-Monte Carlo and Sparse grid quadratures [1]. We extend our previous results to Multi-level quadrature schemes and consider both Multi-level Monte Carlo and sparse-grid quadratures [2]. We map Maxwells equations from uncertain domains to a single *nominal domain* through a Curl-conforming pullback [3]. This allows us to prove the piecewise regularity of the pullback solutions in the nominal domain, which is then leveraged to prove dimension independent rates of convergence for the Multi-level quadrature methods under consideration. We also provide a fully discrete error analysis taking into consideration errors originating from inexact inregration of the coefficients depending on the uncertain pullbacks (geometries) [4]. Our results are then confirmed by numerical examples which verify the superiority of sparse-grid methods.

REFERENCES

[1] R. Aylwin-Pincheira, C. Jerez-Hanckes, C. Schwab and J. Zech *Domain uncertainty quantification in computational electromagnetics*, SIAM/ASA Journal on Uncertainty Quantification, 8(1): 301–341, 2020.

[2] R. Aylwin-Pincheira, C. Jerez-Hanckes, C. Schwab and J. Zech *Multilevel domain uncertainty quantification in computational electromagnetics*, Mathematical Models and Methods in Applied Sciences, 33(4): 877–921, 2023.

[3] C. Jerez-Hanckes, C. Schwab and J. Zech *Electromagnetic wave scattering by random surfaces: Shape holomorphy*, Mathematical Models and Methods in Applied Sciences, 27(12), 2229–2259, 2017

[4] R. Aylwin-Pincheira and C. Jerez-Hanckes *The effect of quadrature rules on finite element solutions of Maxwell variational problems: Consistency estimates on meshes with straight and curved elements*, Numerische Mathematik, 147(4): 903–936, 2021.

A POSTERIORI ESTIMATES FOR A COUPLED PIEZOELECTRIC MODEL WITH UNCERTAIN DATA

Tatiana Samrowski[1]

[1] Institut für Mathematik, Universität Zürich, Switzerland, tatiana@math.uzh.ch

The paper is concerned with a coupled problem describing piesoelectric effects in an elastic body with uncertain data. For this problem, we deduce majorants of the distance between the exact solution and any approximation in the respective energy class of functions satisfying the boundary conditions.

REFERENCES

[1] U. Langer, S. Repin and T. Samrowski, *Reliable a posteriori estimates for coupled systems generated by piezoelectric models*, Russian J. Numer. Anal. Math. Mod., 32(4): 259–267, 2017 SIAM J. Sci. Comput., 41(6):A3938–A3953, 2019.

HIGH ORDER IMMERSED HYBRIDIZED FINITE DIFFERENCE METHOD FOR ELLIPTIC INTERFACE PROBLEMS

Youngmok Jeon[1]

[1] Department of Mathematics, Ajou University, Suwon, Korea

Consider an elliptic interface problem:

$$\begin{aligned} -\bigtriangledown .(\kappa \bigtriangledown u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

together with the jump conditions on the interface

$$[[u]]_\Gamma = w, \quad [[\kappa \partial_\nu u]]_\Gamma = v.$$

In this talk we present high order immersed hybridized difference (IHD) methods for the above elliptic interface problem. The key ingredients of high order methods lies in a systematic and unique way of constructing the high order VR(*virtual to real*)-transformation on multi-variable polynomial spaces. Numerical experiments are performed in two and three dimensions. Numerical results achieving up to the 6th order convergence in the L_2-norm are presented for the two dimensional case, and a three dimensional example with a 4th order convergence in the L_2-norm is presented.

REFERENCES

[1] Y. Jeon, *High order immersed hybridized finite difference method for elliptic interface problems*, to appear J. Numer. Math.

EDGEWISE ITERATIVE SCHEME

Mi-Young Kim[1] **and Dongwook Shin**[2]

[1] Department of Mathematics, Inha University, Korea, mikim@inha.ac.kr
[2] Department of Mathematics, Ajou University, Korea, dws@ajou.ac.kr

An edgewise iterative scheme is developed for the discrete system of the approximate solution to the Poisson's equation. Discontinuous Galerkin method with Lagrange multiplier (DGLM) is considered in the approximation of the PDE. The iteration for the interior degrees of freedom is totally local at each iteration level. The solution is computed element by element. Lagrange multiplier is edgewise updated, which is given as the average of the Robin type information on the elements sharing the edge. Analysis of the convergence of the scheme is given with the discrete maximum norm over all the edges. It is shown that the rate of convergence is independent of the mesh size h. Several numerical experiments are presented.

DISCONTINUOUS GALERKIN METHODS WITH LAGRANGE MULTIPLIERS FOR CONVECTION-DIFFUSION-REACTION PROBLEMS

Mi-Young Kim[1] **and Dongwook Shin**[2]

[1] Department of Mathematics, Inha University, Republic of Korea
[2] Department of Mathematics, Ajou University, Republic of Korea, dws@ajou.ac.kr

Discontinuous Galerkin methods with Lagrange multipliers (DGLM) have been developed for convection-diffusion-reaction problems. Lagrange multiplier is defined on the edge/face of each element by introducing a weak divergence and weak derivative in the method. The local weak formulation can be derived by weakly imposing the continuity of normal fluxes and solutions on each edges. The global weak formulation is then obtained by collecting all the local formulations. In the previous study [1], theoretical results such as stability and error estimate were analyzed. It was shown that the DGLM well captured singularities and internal/boundary layers without spurious oscillations in numerical experiments. On the other hand, for the second-order elliptic problems, the auxiliary variable was introduced to address the DGLM method in the form of mixed finite element methods [2]. In this case, the solutions are also well approximated by DGLM without showing spurious oscillations for the reaction dominated diffusion-reaction problems. In this talk, the primal and mixed forms of the DGLM are compared, and the application of a new iterative solver is introduced.

REFERENCES

[1] M.-Y. Kim and D. Shin, *A high order discontinuous Galerkin method with skeletal multipliers for convection-diffusion-reaction problems*, Computer Methods in Applied Mechanics and Engineering, 343, (2019), pp.207-233.

[2] M.-Y. Kim and D. Shin, *A high order discontinuous Galerkin method with Lagrange multipliers for second-order elliptic problems*, Applied Numerical Mathematics, 135, (2019), pp.47-68.

CONDITIONS FOR SPLINES TO ADMIT A FINITE ELEMENT CONSTRUCTION

Jun Hu[1], **Ting Lin**[2], **Qingyu Wu**[2] **and Beihui Yuan**[3]

[1] LMAM and School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China.
[2] School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China.
[3] Beijing Institute of Mathematical Sciences and Applications, Huairou, Beijing, 101400, P. R. China.

In this talk, we address a sufficient and necessary condition for the construction of C^r conforming finite element spaces on general triangulations. It has been commonly conjectured that such spaces can be generated using the piecewise polynomials with degrees $\geq 2^d r+1$ and an additional $C^{2^{d-s}r}$ smoothness on s-subsimplices. Under these conditions, Hu-Lin-Wu [2] first provided a rigorous construction for any continuity in any dimension. In this talk, we prove that this condition is also tight for finite element construction. Specifically, we introduce the concept of extendability for pre-element space – a generalization of (super)spline spaces and finite element spaces. We show that the superspline space is extendable if and only if such a condition holds, while the finite element space is always extendable under mild conditions. The theory is then established by combining both directions. This concept of extendability not only clarifies the essential connection between spline theory and finite element methods, but also provides valuable insights into the fundamental requirements for constructing conforming finite element spaces on general triangulations.

REFERENCES

[1] J. Hu, T. Lin, Q. Wu and B. Yuan, *Conditions for splines to admit a finite element construction*, to be published.

[2] J. Hu, T. Lin and Q. Wu, *A construction of C^r conforming finite element spaces in any dimension*, Foundations of Computational Mathematics: 1-37, 2023.

RIEMANNIAN SHAPE OPTIMIZATION OF THIN SHELLS USING ISOGEOMETRIC ANALYSIS

Rozan Rosandi[1,*] **and Bernd Simeon**[1]

[1] Differential-Algebraic Systems and Numerical Analysis Group,
RPTU Kaiserslautern-Landau, Germany
[*] rosandi@mathematik.uni-kl.de

Structural optimization is concerned with finding an optimal design for a structure under mechanical load. In this contribution, we consider thin elastic shell structures [1] based on a linearized Koiter model, whose shape can be described by a surface embedded in three-dimensional space. We regard the set of unparametrized embeddings of the surface as an infinite-dimensional Riemannian shape manifold [2] and perform optimization in this setting using the Riemannian shape gradient [3]. Non-uniform rational B-splines (NURBS) are employed to discretize the mid-surface and numerically solve the underlying equations that govern the mechanical behavior of the shell via isogeometric analysis [4]. By representing NURBS patches as B-spline patches in real projective space, NURBS weights can also be incorporated into the optimization routine. We discuss the practical implementation of the method and demonstrate our approach on the compliance minimization of a half-cylindrical shell under static load and fixed area constraint. For numerical experiments, we use the GeoPDEs package [5] in MATLAB, extended by the computation of shape sensitivities and Riemannian shape optimization methods.

REFERENCES

[1] M. Bischoff, K.-U. Bletzinger, W. Wall, and E. Ramm. *Models and finite elements for thin-walled structures.* In Encyclopedia of Computational Mechanics, vol. 2, ch. 3, pp. 59–137. John Wiley & Sons, 2004.

[2] M. Bauer, P. Harms, and P. W. Michor. *Sobolev metrics on shape space of surfaces.* Journal of Geometric Mechanics, 3(4):389–438, 2011.

[3] V. Schulz, M. Siebenborn, and K. Welker. *PDE constrained shape optimization as optimization on shape manifolds.* In Geometric Science of Information, pp. 499–508. Lecture Notes in Computer Science, vol. 9389. Springer, 2015.

[4] T. J. R. Hughes, J. Cottrell, and Y. Bazilevs. *Isogeometric analysis: CAD, finite elements, NURBS, exact geometry, and mesh refinement.* Computer Methods in Applied Mechanics and Engineering, 194(39–41):4135–4195, 2005.

[5] R. Vázquez. *A new design for the implementation of isogeometric analysis in Octave and MATLAB: GeoPDEs 3.0.* Computers and Mathematics with Applications, 72(3):523–554, 2016.

MULTISCALE METHODS FOR ELLIPTIC EIGENVALUE PROBLEMS WITH RANDOMLY PERTURBED COEFFICIENTS

Dilini Kolombage and Barbara Verfürth

Institut für Numerische Simulation, Universität Bonn, Germany, kolombag@ins.uni-bonn.de

Random multiscale coefficients are the key ingredients in modelling the mistakes occurring in modern materials of multiscale nature. The Localized Orthogonal Decomposition (LOD) method [2] is an efficient way of solving multiscale problems with rough coefficients. In the case of many samples, it is required to solve the multiscale problem for each perturbed multiscale coefficient. As a consequence, it demands the re-computation of the LOD space many times. Given any perturbed coefficient, the offline-online strategy [1] introduces a two-phase computational technique based on a reference element that yields the entries to the global LOD matrix. This construction no longer demands the computation of the LOD space for each sample. In this talk, we consider eigenvalue problems with periodic coefficients and boundary conditions in the presence of random defects. We introduce a modified Petrov Galerkin version of the LOD method combined with the offline-online strategy for approximating the (lowest non-trivial) eigensolutions. Numerical experiments illustrate the convergence properties and the general applicability of the scheme.

REFERENCES

[1] A. Målqvist and B. Verfürth *An offline-online strategy for multiscale problems with random defects*, ESAIM: Mathematical Modelling and Numerical Analysis, 56(1), 237-260, 2022.

[2] A. Målqvist and D. Peterseim *Numerical homogenization by localized orthogonal decomposition*, Society for Industrial and Applied Mathematics, 2020.

A QUASI-TREFFTZ DG METHOD FOR THE DIFFUSION-ADVECTION-REACTION EQUATION WITH PIECEWISE-SMOOTH COEFFICIENTS

Lise-Marie Imbert-Gérard[1], Andrea Moiola[2], Chiara Perinati[*,2] and Paul Stocker[3]

[1] Department of Mathematics, University of Arizona, USA
[2] Department of Mathematics, University of Pavia, Italy
[3] Faculty of Mathematics, University of Vienna, Austria
[*] chiara.perinati01@universitadipavia.it

Trefftz schemes are high-order Galerkin methods whose discrete functions are elementwise exact solutions of the underlying PDE. Since a family of local exact solutions is needed, Trefftz basis functions are usually restricted to PDEs that are linear, homogeneous and with piecewise-constant coefficients. If the equation has varying coefficients construction of suitable discrete Trefftz spaces is usually of reach. Quasi-Trefftz methods have been introduced to overcome this limitation, relying on discrete functions that are elementwise "approximate solutions" of the PDE, in the sense of Taylor polynomials. The main advantage of Trefftz and quasi-Trefftz schemes over more classical ones is the higher accuracy for comparable numbers of degrees of freedom.

In this talk, we present polynomial quasi-Trefftz spaces for general linear PDEs with smooth coefficients, describe their optimal approximation properties and provide a simple algorithm to compute the basis functions, based on the Taylor expansion of the PDE's coefficients. Then, we focus on a quasi-Trefftz DG method for the diffusion-advection-reaction equation with varying coefficients, showing stability and high-order convergence of the scheme. We also extend the method to non-homogeneous problems with piecewise-smooth source term, constructing a local quasi-Trefftz particular solution and then solving for the difference. We present numerical experiments in 2 and 3 space dimensions that show excellent properties in terms of approximation and convergence rate.

REFERENCES

[1] L.M. Imbert-Gérard, A. Moiola, C. Perinati and P. Stocker, *A quasi-Trefftz DG method for the diffusion-advection-reaction equation with piecewise-smooth coefficients*, in preparation.

[2] C. Perinati, *A quasi-Trefftz discontinuous Galerkin method for the homogeneous diffusion-advection-reaction equation with piecewise-smooth coefficients*, Master's thesis, University of Pavia, 2023. arXiv preprint arXiv:2312.09919.

FAST MULTIVARIATE NEWTON INTERPOLATION FOR DOWNWARD CLOSED POLYNOMIAL SPACES

Phil-Alexander Hofmann [1], Damar Wicaksono[1] and Michael Hecht[1,2]

[1] CASUS - Center for Advanced System Understanding, Helmholtz-Zentrum Dresden-Rossendorf e.V. (HZDR), Görlitz, Germany
[2] University of Wrocław , Mathematical Institute, Wrocław, Poland, m.hecht@hzdr.de

We introduce a fast Newton interpolation algorithm with a runtime complexity of $\mathcal{O}(Nn)$, where N denotes the dimension of the underlying downward closed polynomial space and n its l_p-degree, where $p > 1$. We demonstrate that the algorithm achieves the optimal geometric approximation rate for analytic *Bos-Levenberg-Trefethen functions* in the hypercube. In this case, the Euclidean degree ($p = 2$) emerges as the pivotal choice for mitigating the curse of dimensionality. The spectral differentiation matrices in the Newton basis are sparse, enabling the implementation of fast pseudo-spectral methods on flat spaces, polygonal domains, and regular manifolds. In particular, we discuss applications for high-dimensional PDEs and reaction-diffusion systems on surfaces.

REFERENCES

[1] Trefethen, L. N. *Multivariate polynomial approximation in the hypercube.* Proceedings of the American Mathematical Society, 145 (11), 4837–4844., 2017

[2] Bos, L. and N. Levenberg *Bernstein-Walsh theory associated to convex bodies and applications to multivariate approximation theory.* Computational Methods and Function Theory, 18 (2), 361–388. 2018

[3] Hecht, M., K. Gonciarz, J. Michelfeit, V. Sivkin, and I. F. Sbalzarini . *Multivariate interpolation in unisolvent nodes–lifting the curse of dimensionality. arXiv preprint arXiv:2010.10824.*, 2020

[4] Hernandez Acosta, U., D. C. Wicaksono, S. K. Thekke Veettil, J. Michelfeit, and M. Hecht (2023). *Minterpy - multivariate polynomial interpolation (version 0.2.0-alpha).* Rodare: http://doi.org/10.14278/rodare.2062, GitHub: https://github.com/casus/minterpy/, 2023

SPECTRUM ANALYSIS USING LEAST-SQUARES SPECTRAL ELEMENT APPROACH

L.K. Balyan[1], **Himanshu Garg**[2], **Felipe Lepe**[3], **Subhashree Mohapatra**[2]

[1] IIIT DM-Jabalpur, India, balyan@iiitdmj.ac.in
[2] IIIT Delhi, India, himanshug@iiitd.ac.in, **subhashree@iiitd.ac.in**
[3] Universidad del Bio-Bio , Chile, flepe@ubiobio.cl

In this work, we investigate elliptic eigen value problems related to a non-self-adjoint differential operator. A least squares spectral element formulation has been used for setup of discrete problem. Eigenvalues and eigenfunctions are proven to be of exponential accuracy. Numerical results on various domains (two and three dimensional) with different boundary conditions (Dirichlet and mixed) confirm the proposed theoretical claims. Few conjectures related to Laplace eigenvalue problems will be discussed.

REFERENCES

[1] Ashbaugh, M.S. (1999), Open problems on eigenvalues of the Laplacian, Analytic and geometric inequalities and applications, 13-28.

[2] Balyan, L.K., Dutt, P.K., Rathore, R.S. (2012), Least squares $h-p$ spectral element method for elliptic eigenvalue problems, Appl Math Comput., 218, 9596-9613.

[3] Bertrand, F., & Boffi, D. (2021). First order least-squares formulations for eigenvalue problems, IMA J. Numer. Anal., 42(2), 1339–1363.

[4] Carstensen, C., Gedicke, J., Mehrmann, V., Miedlar, A., An adaptive homotopy approach for non-selfadjoint eigenvalue problems, Numer. Math., 119, 557-583.

[5] Boffi, D., Gardini, F., & Gastaldi, L. (2022). Virtual element approximation of Eigenvalue problems, In: Antonietti, P.F., Beirão da Veiga, L., Manzini, G. (eds) The Virtual Element Method and its Applications. SEMA SIMAI Springer Series, 31, 275–320.

[6] Lepe, F., Rivera, G., VEM discretization allowing small edges for the reaction–convection–diffusion equation: source and spectral problems, ESAIM: M2AN, 57(5), 3139-3164.

[7] Trefethen, L. N., & Betcke, T. (2006). Computed eigenmodes of planar regions. Contemp. Math., 297–314.

THE PML-METHOD FOR A SCATTERING PROBLEM FOR A LOCAL PERTURBATION OF AN OPEN PERIODIC WAVEGUIDE

Andreas Kirsch[1] **and** **Ruming Zhang**[2]

[1] Institut für Angwandete und Numerische Mathematik, Karlsruher Institut für Technologie, Germany
[2] Institut für Mathematik, Technische Universität Berlin, Germany, zhang@math.tu-berlin.de

The understanding of waves scattered by locally perturbed periodic open waveguides has been a challenging topic in the scattering theory, the major difficulty lies in the existence of guided waves. Numerical simulations to this problem suffer from different types of singularities and the unbounded domain. Based on the Floquet-Bloch transform and the complex curve modification method, we rewrite the solution to the problem into the integral of a coupled family of quasi-periodic problems with respect to the quasi-periodicity. The carefully designed curve avoids all the guided waves thus each quasi-periodic problem is well posed and can be solved by standard methods. Based on this approach, we are also able to prove that the perfectly matched layers, which is an efficient tool to truncate scattering problems into finite domains, converge exponentially with respect to the parameters.

REFERENCES

[1] A. Kirsch and R. Zhang, *The PML-Method for a Scattering Problem for a Local Perturbation of an Open Periodic Waveguide*, Preprint 2024. https://arxiv.org/pdf/2401.00730.pdf

HIGH-ORDER STABLE COMPUTATIONAL ALGORITHM FOR SPACE-TIME FRACTIONAL STOCHASTIC NONLINEAR DIFFUSION WAVE MODEL

Anant Pratap Singh[1], **Dr. Vineet Kumar Singh** [1]

[1] Indian Institute of Technology (BHU), Varanasi, anantpratapiit@gmail.com

In the current work a numerical method is developed and examined for the space–time fractional stochastic nonlinear diffusion wave model. The implicit numerical scheme is designed by embedding the matrix transform approach for discretizing the Riesz-space fractional derivative, and via incorporating $(3 - \alpha)$ order approximation to the Caputo-fractional derivative in temporal direction. Further, Taylor's series method is utilized to linearize the nonlinear source term, and has been efficiently employed to compute the solution of a class of nonlinear fractional diffusion wave equation. We demonstrate that the implicit scheme converges with β-order in space and $(3 - \alpha)$ order in time. The theoretical investigation of the unconditional stability of the implicit scheme and the optimal error estimates in the temporal-spatial direction are conducted. Moreover, the consistency and high efficacy of the proposed numerical algorithms are further supported by several numerical tests, which shows that the designed numerical technique is easy to implement and reduces the computing costs.

REFERENCES

[1]] A.P. Singh, V.K. Singh, *Analysis of a robust implicit scheme for space–time fractional stochastic nonlinear diffusion wave model, IJCM 100(7), pp. 1625-1645.*

A NONCONFORMING LEAST-SQUARES SPECTRAL ELEMENT METHOD FOR STOKES INTERFACE PROBLEMS IN TWO DIMENSIONS

Naraparaju. Kishore Kumar[1], **Subhashree Mohapatra**[2] **and Shivangi Joshi** [1]

[1] BITS Pilani Hyderabad Campus, India, Email: naraparaju@hyderabad.bits-pilani.ac.in
[2] IIIT Delhi, India, Email: subhashree@iiitd.ac.in

In this talk, we present a higher-order spectral element approach for Stokes interface problems with smooth interfaces. The given domain is discretized into a finite number of subdomains, so that the division matches along the interface. The interface is resolved exactly using blending elements. The higher order spectral element functions are used, and they are nonconforming. A suitable least-squares functional is proposed. The interface conditions across the interface are enforced in appropriate Sobolev norms in the minimizing functional. The method is shown to be exponentially accurate, and various numerical examples are presented to validate the theoretical estimates.

REFERENCES

[1] S. Mohapatra, Pravir K. Dutt, B. V. Rathish Kumar and Marc I. Gerritsma, Nonconforming least squares spectral element method for Stokes equations on non-smooth domains, J. of Comp. Appl. Math., 372, 112696, 2020.

[2] N. Kishore Kumar and G. Naga Raju, Nonconforming least-squares method for elliptic partial differential equations with smooth interfaces, J. of Sci. Comp., 53(2), 295-319, 2012.

Index of Speakers