On the approximation of tensor product operators

David Krieg

University of Jena

Bonn
February 9, 2017

Approximation with finite linear information

Setting: X and Y normed spaces, $F \subset X$, $T : F \to Y$.

Problem: We cannot evaluate Tf for some $f \in F$.

Capabilities: We can evaluate Af for a certain class of admissible algorithms $A \in \mathcal{A} \subset Y^F$.

Errors: The worst-case error of the algorithm $A \in \mathcal{A}$ is

$$e(A, T) = \sup_{f \in F} \|Tf - Af\|_Y .$$

The minimal worst-case error in the class $\mathcal{A}$ is

$$e(\mathcal{A}, T) = \inf_{A \in \mathcal{A}} e(A, T).$$

Approximation with finite linear information

Examples of admissible algorithms:

- The class $\mathcal{A}_n$ of algorithms that evaluate less than n linear functionals.
- The class $\mathcal{A}_n^* = \{A \in \mathcal{L}(X, Y) \mid \text{rank}(A) < n\}$.
- The class $\mathcal{A}_n^{\text{eval}}$ of algorithms that evaluate less than n function values.

Our Mission: We want to study the case

- $X = H^s_{\text{mix}}\left(G^d\right)$ and F its unit ball, $Y = L_2\left(G^d\right)$
- $T : F \to Y$, $Tf = f$
- $\mathcal{A} = \mathcal{A}_n$

Approximation with finite linear information

Theorem (Bakhvalov, Gal & Michelli, Creutzing & Wojtaszczyk, Hinrichs & Novak & Woźniakowski)

If F is the unit ball of a pre-Hilbert space and T is linear and bounded,

$$e(\mathcal{A}_n, T) = e(\mathcal{A}_n^*, T) = a_n(T)\,.$$

Contents:

- Unbounded functionals are not necessary.
- Adaption does not help.
- Linear algorithms are optimal.

If X and Y are Hilbert spaces:

T approximable with finite linear information $\Longleftrightarrow$ T compact.

Approximation with finite linear information

Fact

There is a countable orthonormal basis $\mathcal{B}$ of $N(T)^\perp$ such that $T\mathcal{B}$ is an orthogonal basis of $\overline{R(T)}$.

$\mathcal{B}$ is called singular value decomposition (SVD) of T, the values $\|Tb\|_Y$ for $b \in \mathcal{B}$ are called singular values (SVs). Clearly,

$$Tf = \sum_{b \in \mathcal{B}} \langle f, b \rangle \, Tb.$$

Fact

Moreover, $a_n(T)$ is the nth largest SV of T. The algorithm

$$A_n : F \to Y, \quad A_n(f) = \sum_{b \in \mathcal{B}_n} \langle f, b \rangle \, Tb,$$

is optimal in $\mathcal{A}_n$, if $\mathcal{B}_n \subset \mathcal{B}$ corresponds to the $n-1$ largest SVs.

Tensor product operators

For $i = 1 \dots d$, let G_i be a nonempty set and let $G = \prod_{i=1}^{d} G_i$.
The tensor product of functions $f_i : G_i \to \mathbb{C}$ is

$$f_1 \otimes \dots \otimes f_d : G \to \mathbb{C}, \quad x \mapsto f_1(x_1) \cdot \ldots \cdot f_d(x_d).$$

The tensor product of Hilbert spaces X_i of functions $G_i \to \mathbb{C}$

$$X = X_1 \otimes \dots \otimes X_d$$

is the smallest Hilbert space of functions $G \to \mathbb{C}$ that contains all tensor product functions and satisfies

$$\langle f_1 \otimes \dots \otimes f_d, g_1 \otimes \dots \otimes g_d \rangle = \langle f_1, g_1 \rangle \cdot \ldots \cdot \langle f_d, g_d \rangle .$$

Analogously let $Y = Y_1 \otimes \dots \otimes Y_d$.

Tensor product operators

For bounded linear operators $T_i : X_i \to Y_i$, the tensor product

$$T = T_1 \otimes \ldots \otimes T_d : X \to Y$$

is the unique bounded and linear operator with

$$T\left(f_1 \otimes \ldots \otimes f_d\right) = T_1 f_1 \otimes \ldots \otimes T_d f_d.$$

Fact

If T_i is compact for $i = 1 \ldots d$, then so is T.

Tensor product operators

Example

Let T_i be the embedding of $H^{s_i}(G_i)$ into $L_2(G_i)$. Then

$$L_2(G_1) \otimes \ldots \otimes L_2(G_d) = L_2(G),$$
$$H^{s_1}(G_1) \otimes \ldots \otimes H^{s_d}(G_d) = H^{\mathbf{s}}_{\mathrm{mix}}(G)$$

and T is the embedding of $H^{\mathbf{s}}_{\mathrm{mix}}(G)$ into $L_2(G)$.

Fact

If $\mathcal{B}_i$ is a SVD of T_i for each index i, then $\mathcal{B} = \mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_d$ is a SVD of T. In particular, $e(\mathcal{A}_n, T)$ is the nth largest product $\sigma_1 \cdots \sigma_d$ of singular values σ_i of T_i.

L_2-Approximation in univariate Sobolev spaces

The periodic case. Consider $P : H^s(\mathbb{T}) \hookrightarrow L_2(\mathbb{T})$.

The Fourier basis is an orthogonal basis in both spaces.

Hence,

$$a_n(P) = \left(\sum_{j=0}^{s} (2\pi \lfloor n/2 \rfloor)^{2j} \right)^{-1/2}$$

and in particular,

$$a_n(P) \overset{n\to\infty}{\sim} (\pi n)^{-s}.$$

L_2-Approximation in univariate Sobolev spaces

The nonperiodic case. Consider $S : H^s([0,1]) \hookrightarrow L_2([0,1])$.

Lemma

$$a_n(P) \le a_n(S) \le a_{n-s}(P)$$

Hence,

$$a_n(S) \overset{n\to\infty}{\sim} a_n(P) \overset{n\to\infty}{\sim} (\pi n)^{-s}\,.$$

Problems:

- What about $a_n(S)$ for small values of n?
- Optimal algorithms?

L_2-Approximation in univariate Sobolev spaces

We need the eigenfunctions of $W = S^*S$.

Lemma (K.)

Let $\lambda > 0$ and $f \in H^s([0,1])$. Then,

$$Wf = \lambda f \iff \begin{cases} \sum_{k=1}^{s} (-1)^k f^{(2k)} = \left(\frac{1}{\lambda} - 1\right) f, \\ f^{(s)}(x) = 0 \quad \text{and} \quad f^{(s+k)}(x) = f^{(s-k)}(x) \\ \text{for } k = 1, \ldots, s-1 \text{ and } x \in \{0,1\}. \end{cases}$$

$\hookrightarrow$ Recipe for optimal algorithms and explicit singular values.

Tensor product sequences

Setting: $(\sigma_n)_{n\in\mathbb{N}}$ a nonincreasing zero sequence,
$(\sigma_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}^d}$ its dth tensor power,
$\left(a_{n,d}\right)_{n\in\mathbb{N}}$ its nonincreasing rearrangement.

Example: $T_1 = \ldots = T_d$ compact operator between Hilbert spaces and $\sigma_n = a_n(T_1)$. Then, $a_{n,d} = a_n(T)$.

Notation: $P^d : H^s_{\mathrm{mix}}\left(\mathbb{T}^d\right) \hookrightarrow L_2\left(\mathbb{T}^d\right)$,
$S^d : H^s_{\mathrm{mix}}\left([0,1]^d\right) \hookrightarrow L_2\left([0,1]^d\right)$.

Tensor product sequences

Theorem (K.)

If $\sigma_n \leq C\, n^{-s}$ for all $n \geq N_1$, then there is some $N_d \in \mathbb{N}$ such that

$$a_{n,d} \leq C^d\, n^{-s} \left(\frac{(\log n)^{d-1}}{(d-1)!} \right)^s \quad \text{for all } n \geq N_d. \tag{1}$$

If $\sigma_n \geq c\, n^{-s}$ for all $n \geq n_1$, then there is some $n_d \in \mathbb{N}$ such that

$$a_{n,d} \geq c^d\, n^{-s} \left(\frac{(\log n)^{d-1}}{(d-1)!} \right)^s \quad \text{for all } n \geq n_d. \tag{2}$$

Tensor product sequences

Corollary (K.)

$$e\left(\mathcal{A}_n, S^d\right) \overset{n\to\infty}{\sim} e\left(\mathcal{A}_n, P^d\right) \overset{n\to\infty}{\sim} \left(\frac{(\log n)^{d-1}}{\pi^d\,(d-1)!\,n}\right)^s .$$

Remark: Compare Kühn/Sickel/Ullrich (2015) for the periodic case.

Meaning: If n is large enough, periodicity does not affect the approximation error.

Problem: Estimate is useless for $n \leq e^{d-1}$.

Tensor product sequences

Theorem (K.)

Let $\sigma_1 = 1 > \sigma_2 > 0$ and assume that $\sigma_n \leq C\, n^{-s}$ for all $n \geq 2$. For any $n \in \{2, \ldots, 2^d\}$,

$$\left(\frac{1}{n}\right)^{\frac{\log \sigma_2^{-1}}{\log\left(1+\frac{d}{\log_2 n}\right)}} \leq a_{n,d} \leq \left(\frac{\exp\left(C^{2/s}\right)}{n}\right)^{\frac{\log \sigma_2^{-1}}{\log\left(\sigma_2^{-2/s} d\right)}} .$$

Roughly: For small n, we obtain that $a_{n,d} \approx n^{-\frac{\log \sigma_2^{-1}}{\log d}}$.

Example: Let $s \geq 2$. Then

$$a_n\left(P^d\right) \approx n^{-\frac{s \log(2\pi)}{\log d}} \quad \text{and} \quad a_n\left(S^d\right) \approx n^{-\frac{1.28}{\log d}} .$$

Tensor product sequences

Theorem (Novak & Woźniakowski)

The problem $\left\{H^s_{\mathrm{mix}}\left(G^d\right) \hookrightarrow L_2\left(G^d\right)\right\}$ *is not polynomially tractable for both* $G = \mathbb{T}$ *and* $G = [0,1]$.

Does an increasing smoothness yield tractability?

Theorem (K.)

The problem $\left\{H^{s_d}_{\mathrm{mix}}\left([0,1]^d\right) \hookrightarrow L_2\left([0,1]^d\right)\right\}$ *is not polynomially tractable for any choice of natural numbers* s_d*. The problem* $\left\{H^{s_d}_{\mathrm{mix}}\left(\mathbb{T}^d\right) \hookrightarrow L_2\left(\mathbb{T}^d\right)\right\}$ *is strongly polynomially tractable, iff it is polynomially tractable, iff* s_d *grows at least logarithmically in* d*.*

Compare Papageorgiou/Woźniakowski 2010.

Tensor product sequences

What about tensor products of different sequences?

- Results on the order of convergence by Mityagin and Nikol'skaya.
- The full asymptotic behavior is **not** determined by the asymptotic behavior of the factors.
- Preasymtotics?