On the approximation of tensor product operators

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Approximation with finite linear information

Setting: $X$ and $Y$ normed spaces, $F \subset X$, $T : F \to Y$.

Problem: We cannot evaluate $Tf$ for some $f \in F$.

Capabilities: We can evaluate $Af$ for a certain class of admissible algorithms $A \in \mathcal{A} \subset Y^F$.

Errors: The worst-case error of the algorithm $A \in \mathcal{A}$ is

$$e(A, T) = \sup_{f \in F} \|Tf - Af\|_Y.$$ 

The minimal worst-case error in the class $\mathcal{A}$ is

$$e(\mathcal{A}, T) = \inf_{A \in \mathcal{A}} e(A, T).$$
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Examples of admissible algorithms:

- The class $\mathcal{A}_n$ of algorithms that evaluate less than $n$ linear functionals.
- The class $\mathcal{A}_n^* = \{ A \in \mathcal{L}(X, Y) \mid \text{rank}(A) < n \}$.
- The class $\mathcal{A}_n^{\text{eval}}$ of algorithms that evaluate less than $n$ function values.

Our Mission: We want to study the case

- $X = H^s_{\text{mix}}(G^d)$ and $F$ its unit ball, $Y = L^2(G^d)$
- $T : F \rightarrow Y$, $Tf = f$
- $\mathcal{A} = \mathcal{A}_n$
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Theorem (Bakhvalov, Gal & Michelli, Creutzig & Wojtaszczyk, Hinrichs & Novak & Woźniakowski)

If $F$ is the unit ball of a pre-Hilbert space and $T$ is linear and bounded,

$$e(A_n, T) = e(A_n^*, T) = a_n(T).$$

Contents:

- Unbounded functionals are not necessary.
- Adaptation does not help.
- Linear algorithms are optimal.

If $X$ and $Y$ are Hilbert spaces:

$T$ approximable with finite linear information $\iff T$ compact.
**Fact**

There is a countable orthonormal basis $\mathcal{B}$ of $N(T)^\perp$ such that $T\mathcal{B}$ is an orthogonal basis of $R(T)$.

$\mathcal{B}$ is called singular value decomposition (SVD) of $T$, the values $\|Tb\|_Y$ for $b \in \mathcal{B}$ are called singular values (SVs). Clearly,

$$Tf = \sum_{b \in \mathcal{B}} \langle f, b \rangle Tb.$$ 

**Fact**

Moreover, $a_n(T)$ is the $n$th largest SV of $T$. The algorithm

$$A_n : F \to Y, \quad A_n(f) = \sum_{b \in \mathcal{B}_n} \langle f, b \rangle Tb,$$

is optimal in $A_n$, if $\mathcal{B}_n \subset \mathcal{B}$ corresponds to the $n - 1$ largest SVs.
For $i = 1 \ldots d$, let $G_i$ be a nonempty set and let $G = \prod_{i=1}^{d} G_i$. The tensor product of functions $f_i : G_i \to \mathbb{C}$ is

$$f_1 \otimes \ldots \otimes f_d : G \to \mathbb{C}, \quad x \mapsto f_1(x_1) \cdot \ldots \cdot f_d(x_d).$$

The tensor product of Hilbert spaces $X_i$ of functions $G_i \to \mathbb{C}$

$$X = X_1 \otimes \ldots \otimes X_d$$

is the smallest Hilbert space of functions $G \to \mathbb{C}$ that contains all tensor product functions and satisfies

$$\langle f_1 \otimes \ldots \otimes f_d, g_1 \otimes \ldots \otimes g_d \rangle = \langle f_1, g_1 \rangle \cdot \ldots \cdot \langle f_d, g_d \rangle.$$ 

Analogously let $Y = Y_1 \otimes \ldots \otimes Y_d$. 
Tensor product operators

For bounded linear operators $T_i : X_i \to Y_i$, the tensor product

$$T = T_1 \otimes \ldots \otimes T_d : X \to Y$$

is the unique bounded and linear operator with

$$T (f_1 \otimes \ldots \otimes f_d) = T_1 f_1 \otimes \ldots \otimes T_d f_d.$$

Fact

If $T_i$ is compact for $i = 1 \ldots d$, then so is $T$. 
Example

Let $T_i$ be the embedding of $H^{s_i}(G_i)$ into $L_2(G_i)$. Then

$$L_2(G_1) \otimes \ldots \otimes L_2(G_d) = L_2(G),$$

$$H^{s_1}(G_1) \otimes \ldots \otimes H^{s_d}(G_d) = H^{s\text{mix}}(G)$$

and $T$ is the embedding of $H^{s\text{mix}}(G)$ into $L_2(G)$.

Fact

If $\mathcal{B}_i$ is a SVD of $T_i$ for each index $i$, then $\mathcal{B} = \mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_d$ is a SVD of $T$. In particular, $e(A_n, T)$ is the $n$th largest product $\sigma_1 \cdots \sigma_d$ of singular values $\sigma_i$ of $T_i$. 
The periodic case. Consider $P : H^s(\mathbb{T}) \hookrightarrow L_2(\mathbb{T})$.

The Fourier basis is an orthogonal basis in both spaces. Hence,

$$a_n(P) = \left( \sum_{j=0}^{s} (2\pi \lfloor n/2 \rfloor)^{2j} \right)^{-1/2}$$

and in particular,

$$a_n(P) \stackrel{n \to \infty}{\sim} (\pi n)^{-s}.$$
The nonperiodic case. Consider $S : H^s([0, 1]) \hookrightarrow L_2([0, 1])$.

Lemma

\[ a_n(P) \leq a_n(S) \leq a_{n-s}(P) \]

Hence,

\[ a_n(S) \xrightarrow{n \to \infty} a_n(P) \xrightarrow{n \to \infty} (\pi n)^{-s}. \]

Problems:
- What about $a_n(S)$ for small values of $n$?
- Optimal algorithms?
L₂-Approximation in univariate Sobolev spaces

We need the eigenfunctions of \( W = S^* S \).

**Lemma (K.)**

Let \( \lambda > 0 \) and \( f \in H^s([0, 1]) \). Then,

\[
W f = \lambda f \iff \begin{cases}
\sum_{k=1}^{s} (-1)^k f^{(2k)} = \left( \frac{1}{\lambda} - 1 \right) f, \\
f^{(s)}(x) = 0 \quad \text{and} \quad f^{(s+k)}(x) = f^{(s-k)}(x) \\
\text{for } k = 1, \ldots, s-1 \text{ and } x \in \{0, 1\}.
\end{cases}
\]

\[\rightarrow\] Recipe for optimal algorithms and explicit singular values.
**Tensor product sequences**

**Setting:** \((σ_n)_{n∈N}\) a nonincreasing zero sequence, 
\((σ_n)_{n∈N^d}\) its \(d\)th tensor power, 
\((a_{n,d})_{n∈N}\) its nonincreasing rearrangement.

**Example:** \(T_1 = \ldots = T_d\) compact operator between Hilbert spaces and \(σ_n = a_n(T_1)\). Then, \(a_{n,d} = a_n(T)\).

**Notation:** \(P^d : H^s_{mix}(\mathbb{T}^d) \leftrightarrow L_2(\mathbb{T}^d)\), 
\(S^d : H^s_{mix}([0,1]^d) \leftrightarrow L_2([0,1]^d)\).
**Theorem (K.)**

If \( \sigma_n \leq C n^{-s} \) for all \( n \geq N_1 \), then there is some \( N_d \in \mathbb{N} \) such that

\[
a_{n,d} \leq C^n n^{-s} \left( \frac{(\log n)^{d-1}}{(d-1)!} \right)^s \quad \text{for all } n \geq N_d. \tag{1}
\]

If \( \sigma_n \geq c n^{-s} \) for all \( n \geq n_1 \), then there is some \( n_d \in \mathbb{N} \) such that

\[
a_{n,d} \geq c^n n^{-s} \left( \frac{(\log n)^{d-1}}{(d-1)!} \right)^s \quad \text{for all } n \geq n_d. \tag{2}
\]
Tensor product sequences

**Corollary (K.)**

\[
\lim_{n \to \infty} e\left( \mathcal{A}_n, S^d \right) \sim e\left( \mathcal{A}_n, P^d \right) \sim \left( \frac{(\log n)^{d-1}}{\pi^d (d-1)! n} \right)^s.
\]

**Remark:** Compare Kühn/Sickel/Ullrich (2015) for the periodic case.

**Meaning:** If \( n \) is large enough, periodicity does not affect the approximation error.

**Problem:** Estimate is useless for \( n \leq e^{d-1} \).
Theorem (K.)

Let $\sigma_1 = 1 > \sigma_2 > 0$ and assume that $\sigma_n \leq C n^{-s}$ for all $n \geq 2$. For any $n \in \{2, \ldots, 2^d\}$,

$$
\left( \frac{1}{n} \right)^{\frac{\log \sigma_2^{-1}}{\log \left(1 + \frac{d}{\log_2 n}\right)}} \leq a_{n,d} \leq \left( \frac{\exp \left(\frac{C^2}{s}\right)}{n} \right)^{\frac{\log \sigma_2^{-1}}{\log \left(\sigma_2^{-2/s} d\right)}}.
$$

Roughly: For small $n$, we obtain that $a_{n,d} \approx n^{-\frac{\log \sigma_2^{-1}}{\log d}}$.

Example: Let $s \geq 2$. Then

$$
a_n \left( P^d \right) \approx n^{-\frac{s \log(2\pi)}{\log d}} \quad \text{and} \quad a_n \left( S^d \right) \approx n^{-\frac{1.28}{\log d}}.
$$
Theorem (Novak & Woźniakowski)

The problem \( \{ H^s_{\text{mix}} (G^d) \hookrightarrow L_2 (G^d) \} \) is not polynomially tractable for both \( G = \mathbb{T} \) and \( G = [0, 1] \).

Does an increasing smoothness yield tractability?

Theorem (K.)

The problem \( \{ H^s_{\text{mix}} ([0, 1]^d) \hookrightarrow L_2 ([0, 1]^d) \} \) is not polynomially tractable for any choice of natural numbers \( s_d \). The problem \( \{ H^s_{\text{mix}} (\mathbb{T}^d) \hookrightarrow L_2 (\mathbb{T}^d) \} \) is strongly polynomially tractable, iff it is polynomially tractable, iff \( s_d \) grows at least logarithmically in \( d \).

Compare Papageorgiou/Woźniakowski 2010.
What about tensor products of different sequences?

- Results on the order of convergence by Mityagin and Nikol’skaya.
- The full asymptotic behavior is *not* determined by the asymptotic behavior of the factors.
- Preasymptotics?