Approximation of high-dimensional multivariate periodic functions based on rank-1 lattice sampling

We consider the approximation of functions $f\colon \mathbb{T}^d \to \mathbb{C}$ belonging to subspaces of the Wiener algebra, namely to the Banach spaces $\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d) := \left\{f\colon \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta}(\mathbf{k})|\hat{f}_{\mathbf{k}}| < \infty\right\}$ and the Hilbert spaces $\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d) := \left\{f\colon \sqrt{\sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta+\lambda}(\mathbf{k})^2|\hat{f}_{\mathbf{k}}|^2} < \infty\right\}$ with $\beta \geq 0$, $\alpha > -\beta$, $\lambda > 1/2$, where $\hat{f}_{\mathbf{k}}$ are the Fourier coefficients of f and the weights $\omega^{\alpha,\beta}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^\alpha \ \prod_{s=1}^d \max(1, |k_s|)^\beta$ for $\mathbf{k} := (k_1, \ldots, k_d)^\top$. For the approximation of f, a trigonometric polynomial $p(\mathbf{x}) := \sum_{\mathbf{k}\in I} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}}$ with frequencies supported on an index set $I \subset \mathbb{Z}^d$ is used and the Fourier coefficients $\hat{p}_{\mathbf{k}} \in \mathbb{C}$ are computed by sampling the function f along a rank-1 lattice. For this computation, a fast algorithm can be used, which is based mainly on a single one-dimensional fast Fourier transform, and the arithmetic complexity of the algorithm is $\mathcal{O}(|I|^2 \log |I| + d|I|)$. We show upper bounds for the approximation error and achieve an optimal order of convergence, when using suitable frequency index sets I. Numerical results are presented up to dimension $d = 10$, which confirm the theoretical findings.

This is joint work with Lutz Kämmerer and Daniel Potts.