Numerical Simulation for Asset-Liability Management in Life Insurance

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Summary. New regulations and stronger competitions have increased the demand for stochastic asset-liability management (ALM) models for insurance companies in recent years. In this article, we propose a discrete time ALM model for the simulation of simplified balance sheets of life insurance products. The model incorporates the most important life insurance product characteristics, the surrender of contracts, a reserve-dependent bonus declaration, a dynamic asset allocation and a two-factor stochastic capital market. All terms arising in the model can be calculated recursively which allows an easy implementation and efficient evaluation of the model equations. The modular design of the model permits straightforward modifications and extensions to handle specific requirements. In practise, the simulation of stochastic ALM models is usually performed by Monte Carlo methods which suffer from relatively low convergence rates and often very long run times, though. As alternatives to Monte Carlo simulation, we here propose deterministic integration schemes, such as quasi-Monte Carlo and sparse grid methods for the numerical simulation of such models. Their efficiency is demonstrated by numerical examples which show that the deterministic methods often perform much better than Monte Carlo simulation as well as by theoretical considerations which show that ALM problems are often of low effective dimension.

1 Introduction

The scope of asset-liability management is the responsible administration of the assets and liabilities of insurance contracts. Here, the insurance company has to attain two goals simultaneously. On the one hand, the available capital has to be invested as profitably as possible (asset management), on the other hand, the obligations against policyholders have to be met (liability management). Depending on the specific insurance policies these obligations are often quite complex and can include a wide range of guarantees and option-like features, like interest rate guarantees, surrender options (with or without surrender fees) and variable reversionary bonus payments. Such bonus payments are typically linked to the investment returns of the company. Thereby,

the insurance company has to declare in each year which part of the investment returns is given to the policyholders as reversionary bonus, which part is saved in a reserve account for future bonus payments and which part is kept by the shareholders of the company. These management decisions depend on the financial situation of the company as well as on strategic considerations and legal requirements. A maximisation of the shareholders' benefits has to be balanced with a competitive bonus declaration for the policyholders. Moreover, the exposure of the company to financial, mortality and surrender risks has to be taken into account. These complex problems are investigated with the help of ALM analyses. In this context, it is necessary to estimate the medium- and long-term development of all assets and liabilities as well as the interactions between them and to determine their sensitivity to the different types of risks. This can either be achieved by the computation of particular scenarios (stress tests) which are based on historical data, subjective expectations, and guidelines of regulatory authorities or by a stochastic modelling and simulation. In the latter case, numerical methods are used to simulate a large number of scenarios according to given distribution assumptions which describe the possible future developments of all important variables, e.g. of the interest rates. The results are then analysed using statistical figures which illustrate the expected performance or the risk profile of the company.

In recent years, such stochastic ALM models for life insurance policies are becoming more and more important as they take financial uncertainties more realistically into account than an analysis of a small number of deterministically given scenarios. Additional importance arises due to new regulatory requirements as Solvency II and the International Financial Reporting Standard (IFRS). Consequently, much effort has been spent on the development of these models for life insurance policies in the last years, see, e.g., [2, 4, 7, 13, 19, 24, 33] and the references therein. However, most of the ALM models described in the existing literature are based on very simplifying assumptions in order to focus on special components and effects or to obtain analytical solutions. In this article, we develop a general model framework for the ALM of life insurance products. The complexity of the model is chosen such that most of the models previously proposed in the literature and the most important features of life insurance product management are included. All terms arising in the model can be calculated recursively which allows an straightforward implementation and efficient evaluation of the model equations. Furthermore, the model is designed to have a modular organisation which permits straightforward modifications and extensions to handle specific requirements.

In practise, usually Monte Carlo methods are used for the stochastic simulation of ALM models. These methods are robust and easy to implement but suffer from their relatively low convergence rates. To obtain one more digit accuracy, Monte Carlo methods need the simulation of a hundred times as many scenarios. As the simulation of each scenario requires a run over all time points and all policies in the portfolio of the company, often very long run times are

needed to obtain approximations of satisfactory accuracy. As a consequence, a more frequent and more comprehensive risk management, extensive sensitivity investigations or the optimisation of product parameters and management rules are often not possible. In this article we propose deterministic numerical integration schemes, such as quasi-Monte Carlo methods (see e.g. [37, 22, 40]) and sparse grid methods (see, e.g., [9, 15, 16, 23, 38, 42]) for the numerical simulation of ALM models. These methods are alternatives to to Monte Carlo simulation, which have a faster rate of convergence, exploit the smoothness and the anisotropy of the integrand and have deterministic upper bounds on the error. In this way, they often can significantly reduce the number of required scenarios and computing times as we show by numerical experiments. The performance of these numerical methods is closely related to the effective dimension and the smoothness of the problem under consideration. Here, we show that ALM problems are often of very low effective dimension (in the sense that the problem can well be approximated by sums of very low-dimensional functions) which can, to some extent, explain the efficiency of the deterministic methods. Numerical results based on a general ALM model framework for participating life insurance products demonstrate that these deterministic methods in fact often perform much better than Monte Carlo simulation even for complex ALM models with many time steps. Quasi-Monte Carlo methods based on Sobol sequences and dimension-adaptive sparse grids based on one-dimensional Gauss-Hermite quadrature formulae turn out to be the most efficient representatives of several quasi-Monte Carlo and sparse grid variants, respectively. For further details, see [17, 18, 19].

The remainder of this article is as follows: In Section 2, we describe the model framework. In Section 3, we then discuss how this model can be efficiently simulated by numerical methods for multivariate integration. In Section 4, we present numerical results which illustrate possible application of the ALM model and analyse the efficiency of different numerical approaches. The article finally closes in Section 5 with concluding remarks.

2 The ALM Model

In this section, we closely follow [19] and describe an ALM model framework for the simulation of the future development of a life insurance company. We first indicate the overall structure of the model and introduce a simplified balance sheet which represents the assets and liabilities of the company. The different modules (capital market model, liability model, management model) and the evolution of the balance sheet items are then specified in the following sections.

2.1 Overall Model Structure

The main focus of our model is to simulate the future development of all assets and liabilities of a life insurance company. To this end, the future develop-

ment of the capital markets, the policyholder behaviour and the company's management has to be modelled. We use a stochastic capital market model, a deterministic liability model which describes the policyholder behaviour and a deterministic management model which is specified by a set of management rules which may depend on the stochastic capital markets. The results of the simulation are measured by statistical performance and risk figures which are based on the company's most important balance sheet items. They are used by the company to optimise management rules, like the capital allocation, or product parameters, like the surrender fee. The overall structure of the model is illustrated in Fig. 1.

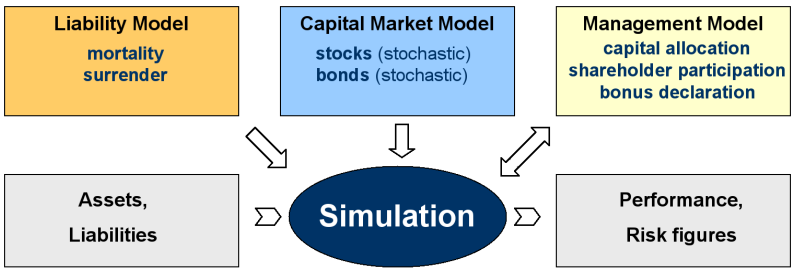

Fig. 1. Overall structure of the ALM model.

We model all terms in discrete time. Here, we denote the start of the simulation by time $t = 0$ and the end of the simulation by $t = T$ (in years). The interval $[0, T]$ is decomposed into K periods $[t_{k-1}, t_k]$ with $t_k = k\,\Delta t$, $k = 1, \ldots, K$ and a period length $\Delta t = T/K$ of one month.

The asset side consists of the market value C_k of the company's assets at time t_k. On the liability side, the first item is the book value of the actuarial reserve D_k, i.e., the guaranteed savings part of the policyholders after deduction of risk premiums and administrative costs. The second item is the book value of the allocated bonuses B_k which constitute the part of the surpluses that have been credited to the policyholders via the profit participation. The free reserve F_k is a buffer account for future bonus payments. It consists of surpluses which have not yet been credited to the individual policyholder accounts, and is used to smooth capital market oscillations and to achieve a stable and low-volatile return participation of the policyholders. The last item, the equity or company account Q_k, consists of the part of the surpluses which is kept by the shareholders of the company and is defined by

$$Q_k = C_k - D_k - B_k - F_k$$

such that the sum of the assets equals the sum of the liabilities. Similar to the bonus reserve in [24], Q_k is a hybrid determined as the difference between a market value C_k and the three book values D_k, B_k and F_k. It may be interpreted as hidden reserve of the company as discussed in [29]. The balance

sheet items at time t_k, $k = 0, \ldots, K$, used in our model are shown in Table 1. In a sensitivity analysis for sample parameters and portfolios it is shown in

Assets		Liabilities	
Capital	C_k	Actuarial reserve	D_k
		Allocated bonus	B_k
		Free reserve	F_k
		Equity	Q_k

Table 1. Simplified balance sheet of the life insurance company.

[19] that this model captures the most important behaviour patterns of the balance sheet development of life insurance products. Similar balance sheet models have already been considered in, e.g., [2, 3, 24, 33, 29].

2.2 Capital Market Model

We assume that the insurance company invests its capital either in fixed interest assets, i.e., bonds, or in a variable return asset, i.e., a stock or a basket of stocks. For the modelling of the interest rate environment we use the Cox-Ingersoll-Ross (CIR) model [11]. The CIR model is a one-factor mean-reversion model which specifies the dynamics of the short interest rate $r(t)$ at time t by the stochastic differential equation

$$dr(t) = \kappa(\theta - r(t))dt + \sqrt{r(t)}\sigma_r dW_r(t), \tag{1}$$

where $W_r(t)$ is a standard Brownian motion, $\theta > 0$ denotes the mean reversion level, $\kappa > 0$ denotes the reversion rate and $\sigma_r \geq 0$ denotes the volatility of the short rate dynamic. In the CIR model, the price $b(t, \tau)$ at time t of a zero coupon bond with a duration of τ periods and with maturity at time $T = t + \tau\Delta t$ can be derived in closed form by

$$b(t, \tau) = A(\tau)\, e^{-B(\tau)\, r(t)} \tag{2}$$

as an exponential affine function of the prevailing short interest rate $r(t)$ with

$$A(\tau) = \left(\frac{2he^{(\hat{\kappa}+h)\tau\Delta t/2}}{2h + (\hat{\kappa}+h)(e^{h\tau\Delta t} - 1)} \right)^{2\kappa\theta/\sigma_r^2}, \quad B(\tau) = \frac{2(e^{h\tau\Delta t} - 1)}{2h + (\hat{\kappa}+h)(e^{h\tau\Delta t} - 1)},$$

and $h = \sqrt{\hat{\kappa}^2 + 2\sigma_r^2}$. To model the stock price uncertainty, we assume that the stock price $s(t)$ at time t evolves according to a geometric Brownian motion

$$ds(t) = \mu s(t)dt + \sigma_s s(t)dW_s(t), \tag{3}$$

where $\mu \in \mathbb{R}$ denotes the drift rate and $\sigma_s \geq 0$ denotes the volatility of the stock return. By Itô's lemma, the explicit solution of this stochastic differential equation is given by

$$s(t) = s(0)\, e^{(\mu - \sigma_s^2/2)t + \sigma_s W_s(t)}. \tag{4}$$

Usually, stock and bond returns are correlated. We thus assume that the two Brownian motions satisfy $dW_s(t)dW_r(t) = \rho dt$ with a constant correlation coefficient $\rho \in [-1, 1]$. These and other models which can be used to simulate the bond and stock prices are discussed in detail, e.g., in [6, 25, 28].

In the discrete time case, the short interest rate, the stock prices and the bond prices are defined by $r_k = r(t_k)$, $s_k = s(t_k)$ and $b_k(\tau) = b(t_k, \tau)$. For the solution of equation (1), we use an Euler-Maruyama discretization[3] with step size Δt, which yields

$$r_k = r_{k-1} + \kappa(\theta - r_{k-1})\Delta t + \sigma_r \sqrt{|r_{k-1}|}\sqrt{\Delta t}\, \xi_{r,k}, \tag{5}$$

where $\xi_{r,k}$ is a $N(0,1)$-distributed random variable. For the stock prices one obtains

$$s_k = s_{k-1} e^{(\mu - \sigma_s^2/2)\Delta t + \sigma_s \sqrt{\Delta t}(\rho \xi_{r,k} + \sqrt{1-\rho^2}\xi_{s,k})}, \tag{6}$$

where $\xi_{s,k}$ is a $N(0,1)$-distributed random variable independent of $\xi_{r,k}$. Since

$$\mathrm{Cov}(\rho\xi_{r,k} + \sqrt{1-\rho^2}\xi_{s,k}, \xi_{r,k}) = \rho,$$

the correlation between the two Wiener processes $W_s(t)$ and $W_r(t)$ is respected. More information on the numerical solution of stochastic differential equations can be found, e.g., in [22, 30].

2.3 Management Model

In this section, we discuss the capital allocation, the bonus declaration mechanism and the shareholder participation.

Capital allocation

We assume that the company rebalances its assets at the beginning of each period. Thereby, the company aims to have a fixed portion $\beta \in [0, 1]$ of its assets invested in stocks, while the remaining capital is invested in zero coupon bonds with a fixed duration of τ periods. We assume that no bonds are sold before their maturity. Let P_k be the premium income at the beginning of period k and let C_{k-1} be the total capital at the end of the previous period. The part N_k of $C_{k-1} + P_k$ which is available for a new investment at the beginning of period k is then given by

[3] An alternative to the Euler-Maruyama scheme, which is more time consuming but avoids time discretization errors, is to sample from a noncentral chi-squared distribution, see [22]. In addition, several newer approaches exist to improve the balancing of time and space discretization errors, see, e.g., [20]. This and the time discretization error are not the focus of this article, though.

$$N_k = C_{k-1} + P_k - \sum_{i=1}^{\tau-1} n_{k-i}\, b_{k-1}(\tau - i),$$

where n_j denotes the number of zero coupon bonds which were bought at the beginning of period j. The capital A_k which is invested in stocks at the beginning of period k is then determined by

$$A_k = \max\{\min\{N_k,\, \beta(C_{k-1} + P_k)\},\, 0\} \tag{7}$$

so that the side conditions $0 \le A_k \le \beta(C_{k-1}+P_k)$ are satisfied. The remaining money $N_k - A_k$ is used to buy $n_k = (N_k - A_k)/b_{k-1}(\tau)$ zero coupon bonds with duration $\tau \Delta t$.[4] The portfolio return rate p_k in period k resulting from the above allocation procedure is then determined by

$$p_k = \left(\Delta A_k + \sum_{i=0}^{\tau-1} n_{k-i}\, \Delta b_{k,i} \right) / (C_{k-1} + P_k), \tag{8}$$

where $\Delta A_k = A_k(s_k/s_{k-1} - 1)$ and $\Delta b_{k,i} = b(t_k, \tau - i - 1) - b(t_{k-1}, \tau - i)$ denote the changes of the market values of the stock and of the bond investments from the beginning to the end of period k, respectively.

Bonus declaration

In addition to the fixed guaranteed interest, a variable reversionary bonus is annually added to the policyholder's account, which allows the policyholder to participate in the investment returns of the company (contribution principle). The bonus is declared by the company at the beginning of each year (principle of advance declaration) with the goal to provide a low-volatile, stable and competitive return participation (average interest principle). Various mathematical models for the declaration mechanism are discussed in the literature. In this article, we follow the approach of [24] where the declaration is based on the current reserve rate γ_{k-1} of the company, which is defined in our framework by the ratio of the free reserve to the allocated liabilities, i.e.,

$$\gamma_{k-1} = \frac{F_{k-1}}{D_{k-1} + B_{k-1}}.$$

The annual interest rate is then defined by

$$\hat{z}_k = \max\{\hat{z}, \omega(\gamma_{k-1} - \gamma)\}.$$

Here, $\hat{z}$ denotes the annual guaranteed interest rate, $\gamma \ge 0$ the target reserve rate of the company and $\omega \in [0, 1]$ the distribution ratio or participation

[4] Note that due to long-term investments in bonds it may happen that $N_k < 0$. This case of insufficient liquidity leads to $n_k < 0$ and thus to a short selling of bonds.

coefficient which determines how fast excessive reserves are reduced. This way, a fixed fraction of the excessive reserve is distributed to the policyholders if the reserve rate γ_{k-1} is above the target reserve rate γ while only the guaranteed interest is paid in the other case. In our model this annual bonus has to be converted into a monthly interest

$$z_k = \begin{cases} (1+\hat{z}_k)^{1/12} - 1 & \text{if } k \bmod 12 = 1 \\ z_{k-1} & \text{otherwise} \end{cases}$$

which is given to the policyholders in each period k of this year.

Shareholder participation

Excess returns $p_k - z_k$, conservative biometry and cost assumptions as well as surrender fees lead to a surplus G_k in each period k which has to be divided among the free reserve F_k and the equity Q_k. In case of a positive surplus, we assume that a fixed percentage $\alpha \in [0,1]$ is saved in the free reserve while the remaining part is added to the equity account. Here, a typical assumption is a distribution according to the 90/10-rule which corresponds to the case $\alpha = 0.9$. If the surplus is negative, we assume that the required capital is taken from the free reserve. If the free reserves do not suffice, the company account has to cover the remaining deficit. The free reserve is then defined by

$$F_k = \max\{F_{k-1} + \min\{G_k, \alpha\, G_k\},\, 0\}. \tag{9}$$

The exact specification of the surplus G_k and the development of the equity Q_k is derived in Section 2.5.

2.4 Liability Model

In this section, we discuss the modelling of the decrement of policies due to mortality and surrender and the development of the policyholder's accounts.

Decrement model

For efficiency, the portfolio of all insurance contracts is often represented by a reduced number m of model points. Each model point then represents a group of policyholders which are similar with respect to cash flows and technical reserves, see, e.g., [27]. By pooling, all contracts of a model point expire at the same time which is obtained as the average of the individual maturity times.

We assume that the development of mortality and surrender is given deterministically and modelled using experience-based decrement tables. Let q_k^i and u_k^i denote the probabilities that a policyholder of model point i dies or surrenders in the k-th period, respectively. The probabilities q_k^i typically depend on the age, the year of birth and the gender of the policyholder while u_k^i

often depends on the elapsed contract time. Let δ_k^i denote the expected number of contracts in model point i at the end of period k. Then, this number evolves over time according to

$$\delta_k^i = \left(1 - q_k^i - u_k^i\right) \delta_{k-1}^i. \tag{10}$$

We assume that no new contracts evolve during the simulation.

Insurance products

In the following, we assume that premiums are paid at the beginning of a period while benefits are paid at the end of the period. Furthermore, we assume that all administrative costs are already included in the premium. For each model point $i = 1, \ldots, m$, the guaranteed part of the insurance product is defined by the specification of the following four characteristics:

- premium characteristic: $(P_1^i, \ldots, P_K^i)$ where P_k^i denotes the premium of an insurance holder in model point i at the beginning of period k if he is still alive at that time.
- survival benefit characteristic: $(E_1^{i,G}, \ldots, E_K^{i,G})$ where $E_k^{i,G}$ denotes the guaranteed payments to an insurance holder in model point i at the end of period k if he survives period k.
- death benefit characteristic: $(T_1^{i,G}, \ldots, T_K^{i,G})$ where $T_k^{i,G}$ denotes the guaranteed payment to an insurance holder in model point i at the end of period k if he dies in period k.
- surrender characteristic: $(S_1^{i,G}, \ldots, S_K^{i,G})$ where $S_k^{i,G}$ denotes the guaranteed payment to an insurance holder in model point i at the end of period k if he surrenders in period k.

The bonus payments of the insurance product to an insurance holder in model point i at the end of period k in case of survival, death and surrender, are denoted by $E_k^{i,B}$, $T_k^{i,B}$ and $S_k^{i,B}$, respectively. The total payments E_k^i, T_k^i and S_k^i to a policyholder of model point i at the end of period k in case of survival, death and surrender are then given by

$$E_k^i = E_k^{i,G} + E_k^{i,B}, \quad T_k^i = T_k^{i,G} + T_k^{i,B} \quad \text{and} \quad S_k^i = S_k^{i,G} + S_k^{i,B}. \tag{11}$$

The capital of a policyholder of model point i at the end of period k is collected in two accounts: the actuarial reserve D_k^i for the guaranteed part and the bonus account B_k^i for the bonus part. Both accounts can efficiently be computed in our framework using the recursions

$$D_k^i = \frac{1+z}{1-q_k^i}(D_{k-1}^i + P_k^i) - E_k^{i,G} - \frac{q_k^i}{1-q_k^i} T_k^{i,G} \tag{12}$$

and

$$B_k^i = \frac{1+z_k}{1-q_k^i} B_{k-1}^i + \frac{z_k - z}{1-q_k^i}(D_{k-1}^i + P_k^i) - E_k^{i,B} - \frac{q_k^i}{1-q_k^i} T_k^{i,B} \tag{13}$$

which results from the deterministic mortality assumptions, see, e.g., [2, 46].

Example 1. As a sample insurance product, an endowment insurance with death benefit, constant premium payments and surrender option is considered. Let P^i denote the constant premium which is paid by each of the policyholders in model point i in every period. If they are still alive, the policyholders receive a guaranteed benefit $E^{i,G}$ and the value of the bonus account at maturity d^i. In case of death prior to maturity, the sum of all premium payments and the value of the bonus account is returned. In case of surrender, the policyholder capital and the bonus is reduced by a surrender factor $\vartheta = 0.9$. The guaranteed components of the four characteristics are then defined by

$$P_k^i = P^i, \quad E_k^{i,G} = \chi_k(d^i)\, E^{i,G}, \quad T_k^{i,G} = k\, P^i \;\text{ and }\; S_k^{i,G} = \vartheta D_k^i,$$

where $\chi_k(d^i)$ denotes the indicator function which is one if $k = d^i$ and zero otherwise. The bonus payments at the end of period k are given by

$$E_k^{i,B} = \chi_k(d^i) B_k^i, \quad T_k^{i,B} = B_k^i \;\text{ and }\; S_k^{i,B} = \vartheta\, B_k^i.$$

We will return to this example in Section 3.

2.5 Balance Sheet Model

In this section, we derive the recursive development of all items in the simplified balance sheet introduced in Section 2.1.

Projection of the assets

In order to define the capital C_k at the end of period k, we first determine the cash flows which are occurring to and from the policyholders in our model framework. The premium P_k, which is obtained by the company at the beginning of period k, and the survival payments E_k, the death payments T_k, and the surrender payments S_k to policyholders, which take place at the end of period k, are obtained by summation of the individual cash flows (11), i.e.,

$$P_k = \sum_{i=1}^{m} \delta_{k-1}^i\, P_k^i,\;\; E_k = \sum_{i=1}^{m} \delta_k^i\, E_k^i,\;\; T_k = \sum_{i=1}^{m} q_k^i \delta_{k-1}^i\, T_k^i,\;\; S_k = \sum_{i=1}^{m} u_k^i \delta_{k-1}^i\, S_k^i, \tag{14}$$

where the numbers δ_k^i are given by (10). The capital C_k is then recursively given by

$$C_k = (C_{k-1} + P_k)\,(1 + p_k) - E_k - T_k - S_k \tag{15}$$

where p_k is the portfolio return rate defined in equation (8).

Projection of the liabilities

The actuarial reserve D_k and the allocated bonus B_k are derived by summation of the individual policyholder accounts (12) and (13), i.e.,

$$D_k = \sum_{i=1}^{m} \delta_k^i \, D_k^i \quad \text{and} \quad B_k = \sum_{i=1}^{m} \delta_k^i \, B_k^i.$$

In order to define the free reserve F_k, we next determine the gross surplus G_k in period k which consists in our model of interest surplus and surrender surplus. The interest surplus is given by the difference between the total capital market return $p_k \, (F_{k-1} + D_{k-1} + B_{k-1} + P_k)$ on policyholder capital and the interest payments $z_k \, (D_{k-1} + B_{k-1} + P_k)$ to policyholders. The surrender surplus is given by $S_k/\vartheta - S_k$. The gross surplus in period k is thus given by

$$G_k = p_k \, F_{k-1} + (p_k - z_k) \, (D_{k-1} + B_{k-1} + P_k) + (1/\vartheta - 1) S_k.$$

The free reserve F_k is then derived using equation (9). Altogether, the company account Q_k is determined by

$$Q_k = C_k - D_k - B_k - F_k.$$

Note that the cash flows and all balance sheet items are expected values with respect to our deterministic mortality and surrender assumptions from Section 2.4, but random numbers with respect to our stochastic capital market model from Section 2.2.

Performance figures

To analyse the results of a stochastic simulation, statistical measures are considered which result from an averaging over all scenarios. Here, we consider the path-dependent cumulative probability of default

$$\mathrm{PD}_k = \mathbb{P}\left(\min_{j=1,\ldots,k} Q_j < 0 \right)$$

as a measure for the risk while we use the expected future value $\mathbb{E}[Q_k]$ of the equity as a measure for the investment returns of the shareholders in the time interval $[0, t_k]$. Due to the wide range of path-dependencies, guarantees and option-like features of the insurance products and management rules, closed-form representations for these statistical measures are in general not available so that one has to resort to numerical methods. It is straightforward to include the computation of further performance and risk measures like the variance, the value-at-risk, the expected shortfall or the return on risk capital. To determine the sensitivity $f'(v) = \partial f(v)/\partial v$ of a given performance figure f to one of the model parameters v, finite difference approximations or more recent approaches, like, e.g., smoking adjoints [21], can be employed.

3 Numerical Simulation

In this section, we discuss the efficient numerical simulation of the ALM model described in Section 2. The number of operations for the simulation of a single scenario of the model is of order $O(m \cdot K)$ and takes about 0.04 seconds

on a dual Intel(R) Xeon(TM) CPU 3.06GH workstation for a representative portfolio with $m = 500$ model points and a time horizon of $K = 120$ periods. The number of scenarios which have to be generated depends on the accuracy requirements, on the model parameters[5] and on the employed numerical method. In the following, we first rewrite the performance figures of the model as high-dimensional integrals. Then, we survey numerical methods which can be applied to their computation, discuss their dependence on the effective dimension and review techniques which can reduce the effective dimension in certain cases.

3.1 Representation as High-Dimensional Integrals

It is helpful to represent the performance figures of the ALM simulation as high-dimensional integrals to see how more sophisticated methods than Monte Carlo simulation can be used for their numerical computation. To derive such a representation, recall that the simulation of one scenario of the ALM model is based on $2\,K$ independent normally distributed random numbers $\mathbf{y} = (y_1, \ldots, y_{2K}) = (\xi_{s,1}, \ldots, \xi_{s,K}, \xi_{r,1}, \ldots, \xi_{r,K}) \sim N(\mathbf{0}, \mathbf{1})$. These numbers specify the stock price process (6) and the short rate process (5). Then, the term structure, the asset allocation, the bonus declaration, the shareholder participation and the development of all involved accounts can be derived using the recursive equations of the previous sections. Altogether, the balance sheet items C_K, B_K, F_K and Q_K at the end of period K can be regarded as (usually very complicated) deterministic functions $C_K(\mathbf{y})$, $B_K(\mathbf{y})$, $F_K(\mathbf{y})$, $Q_K(\mathbf{y})$ depending on the normally distributed vector $\mathbf{y} \in \mathbb{R}^{2K}$. As a consequence, the expected values of the balance sheet items at the end of period K can be represented as $2\,K$-dimensional integrals, e.g.,

$$E[Q_K] = \int_{\mathbb{R}^{2K}} Q_K(\mathbf{y}) \, \frac{e^{-\mathbf{y}^T \mathbf{y}/2}}{(2\pi)^K} \, d\mathbf{y} \tag{16}$$

for the equity account. Often, monthly discretizations of the capital market processes are used. Then, typical values for the dimension $2K$ range from $60 - 600$ depending on the time horizon of the simulation.

Transformation

The integral (16) can be transformed into an integral over the $2K$-dimensional unit cube which is often necessary to apply numerical integration methods. By the substitution $y_i = \Phi^{-1}(x_i)$ for $i = 1, \ldots, 2K$, where Φ^{-1} denotes the inverse cumulative normal distribution function, we obtain

$$E[Q_K] = \int_{\mathbb{R}^{2K}} Q_K(\mathbf{y}) \, \frac{e^{-\mathbf{y}^T \mathbf{y}/2}}{(2\pi)^K} \, d\mathbf{y} = \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x} \tag{17}$$

[5] The model parameters affect important numerical properties of the model, e.g. the effective dimension (see Section 3.3) or the smoothness.

with $d = 2K$ and $f(\mathbf{x}) = Q_k(\Phi^{-1}(\mathbf{x}))$. For the fast computation of $\Phi^{-1}(x_i)$, we use Moro's method [35]. Note that the integrand (17) is unbounded on the boundary of the unit cube, which is undesirable from a numerical as well as theoretical point of view. Note further that different transformations to the unit cube exist (e.g. using the logistic distribution or polar coordinates) and that also numerical methods exist which can directly be applied to the untransformed integral (16) (e.g. Gauss-Hermite rules).

3.2 Numerical Methods for High-Dimensional Integrals

There is a wide range of methods (see, e.g., [12]) available for numerical multivariate integration. Mostly, the integral (17) is approximated by a weighted sum of n function evaluations

$$\int_{[0,1]^d} f(\mathbf{x})\ d\mathbf{x} \approx \sum_{i=1}^{n} w_i f(\mathbf{x}_i) \tag{18}$$

with weights $w_i \in \mathbb{R}$ and nodes $\mathbf{x}_i \in \mathbb{R}^d$. The number n of nodes corresponds to the number of simulation runs. Depending on the choice of the weights and nodes, different methods with varying properties are obtained. Here, the dimension as well as the smoothness class of the function f should be taken into account.

Monte Carlo

In practise, the model is usually simulated by the Monte Carlo (MC) method. Here, all weights equal $w_i = 1/n$ and uniformly distributed sequences of pseudo-random numbers $\mathbf{x}_i \in (0,1)^{2K}$ are used as nodes. This method is independent of the dimension, robust and easy to implement but suffers from a relative low probabilistic convergence rate of order $O(n^{-1/2})$. This often leads to very long simulation times in order to obtain approximations of satisfactory accuracy. Extensive sensitivity investigations or the optimisation of product or management parameters, which require a large number of simulation runs, are therefore often not possible.

Quasi-Monte Carlo

Quasi-Monte Carlo (QMC) methods are equal-weight rules like Monte Carlo. Instead of pseudo-random numbers, however, deterministic low-discrepancy sequences (see, e.g., [37, 22]) or lattices (see, e.g., [40]) are used as point sets which are chosen to yield better uniformity than random samples. Some popular choices are Halton, Faure, Sobol and Niederreiter-Xing sequences and extensible shifted rank-1 lattice rules based on Korobov or fast component-by-component constructions. From the Koksma-Hlawka inequality it follows that convergence rate of QMC methods is of order $O(n^{-1}(\log n)^d)$ for integrands of bounded variation which is asymptotically better than the $O(n^{-1/2})$ rate of

MC. For periodic integrands, lattice rules can achieve convergence of higher order depending on the decay of the Fourier coefficients of f, see [40]. Using novel digital net constructions (see [14]), QMC methods can also be obtained for non-periodic integrands which exhibit convergence rates larger than one if the integrands are sufficiently smooth.

Product methods

Product methods for the computation of (17) are easily obtained by using the tensor products of the weights and nodes of one-dimensional quadrature rules, like, e.g., Gauss rules (see, e.g., [12]). These methods can exploit the smoothness of the function f and converge with order $O(n^{-s/d})$ for $f \in C^s([0,1]^d)$. This shows, however, that product methods suffer from the curse of dimension, meaning that the computing cost grows exponentially with the dimension d of the problem, which prevents their efficient applications for high-dimensional ($d > 5$) applications like ALM simulations.

Sparse grids

Sparse grid (SG) quadrature formulas are constructed using certain combinations of tensor products of one-dimensional quadrature rules, see, e.g., [9, 15, 23, 38, 42]. In this way, sparse grids can, like product methods, exploit the smoothness of f and also obtain convergence rates larger than one. In contrast to product methods, they can, however, also overcome the curse of dimension like QMC methods to a certain extent. They converge with order $O(n^{-s}(\log n)^{(d-1)(s-1)})$ if the integrand belongs to the space of functions which have bounded mixed derivatives of order s. Sparse grid quadrature formula come in various types depending on the one-dimensional basis integration routine, like the trapezoidal, the Clenshaw-Curtis, the Patterson, the Gauss-Legendre or the Gauss-Hermite rule. In many cases, the performance of sparse grids can be enhanced by local adaptivity, see [5, 8], or by a dimension-adaptive grid refinement, see [16].

3.3 Impact of the Dimension

In this section, we discuss the dependence of MC, QMC and SG methods on the nominal and the effective dimension of the integral (17).

Tractability

In contrast to MC, the convergence rate of QMC and SG methods still exhibit a logarithmic dependence on the dimension. Furthermore, also the constants in the O-notation depend on the dimension of the integral. In many cases (particularly within the SG method) these constants increase exponentially with the dimension. Therefore, for problems with high nominal dimension d, such as the ALM of life insurance products, the classical error bounds of the

previous section are no longer of any practical use to control the numerical error of the approximation. For instance, even for a moderate dimension of $d = 20$ and for a computationally unfeasibly high number $n = 10^{90}$ of function evaluations, $n^{-1}(\log n)^d > n^{-1/2}$ still holds in the QMC and the MC error bounds. For classical Sobolov spaces with bounded derivatives up to a certain order, it can even be proved (see [39, 41]) that integration is intractable, meaning that for these function classes deterministic methods of the form (18) can never completely avoid the curse of dimension. For weighted Sobolov spaces, however, it is shown in [39, 41] that integration is tractable if the weights decay sufficiently fast. In the next paragraph and in Section 4.3 we will give some indications that ALM problems indeed belong to such weighted function spaces.

ANOVA decomposition and effective dimension

Numerical experiments show that QMC and SG methods often produce much more precise results than MC methods for certain integrands even in hundreds of dimensions. One explanation of this success is that QMC and SG methods can, in contrast to MC, take advantage of low effective dimensions. QMC methods profit from low effective dimensions by the fact that their nodes are usually more uniformly distributed in smaller dimensions than in higher ones. SG methods can exploit different weightings of different dimensions by a dimension-adaptive grid refinement, see [16]. The effective dimension of the integral (17) is defined by the ANOVA decomposition, see, e.g., [10]. Here, a function $f : \mathbb{R}^d \to \mathbb{R}$ is decomposed by

$$f(x) = \sum_{u \subseteq \{1,\ldots,d\}} f_u(x_u) \quad \text{with } f_u(x_u) = \int_{[0,1]^{d-|u|}} f(x) dx_{\{1,\ldots,d\}\setminus u} - \sum_{v \subset u} f_v(x_v)$$

into 2^d sub-terms f_u with $u \subseteq \{1, \ldots, d\}$ which only depend on variables x_j with $j \in u$. Thereby, the sub-terms f_u describe the dependence of the function f on the dimensions $j \in u$. The effective dimension in the truncation sense of a function $f : \mathbb{R}^d \to \mathbb{R}$ with variance $\sigma^2(f)$ is then defined as the smallest integer d_t, such that $\sum_{v \subseteq \{1,\ldots,d_t\}} \sigma_v^2(f) \geq 0.99\, \sigma^2(f)$ where $\sigma_u^2(f)$ denotes the variances of f_u. The effective dimension d_t roughly describes the number of important variables of the function f. The effective dimension in the superposition sense is defined as the smallest integer d_s, such that $\sum_{|v| \leq d_s} \sigma_v^2(f) \geq 0.99\, \sigma^2(f)$ where $|v|$ denotes the cardinality of the index set v. It roughly describes the highest order of important interactions between variables in the ANOVA decomposition. For the simple function $f(x_1, x_2, x_3) = x_1 e^{x_2} + x_2$ with $d = 3$, we obtain $d_t = 2$ and $d_s = 2$ for instance. For large d, it is no longer possible to compute all 2^d ANOVA sub-terms. The effective dimensions can still be computed in many cases, though. For details and an efficient algorithm for the computation of the effective dimension in the truncation sense we refer to [44]. For the more difficult problem to com-

pute the effective dimension in the superposition sense, we use the recursive method described in [45].

Dimension reduction

Typically, the underlying multivariate Gaussian process is approximated by a random walk discretization. In many cases, a substantial reduction of the effective dimension in the truncation sense and an improved performance of the deterministic integration schemes can be achieved if the Brownian bridge or the principal component (PCA) decompositions of the covariance matrix of the underlying Brownian motion is used instead as it was proposed in [1, 36] for option pricing problems. The Brownian bridge construction differs from the standard random walk construction in that rather than constructing the increments sequentially, the path of the Gaussian process is constructed in a hierarchical way which has the effect that more importance is placed on the earlier variables than on the later ones. The PCA decomposition, which is based on the eigenvalues and -vectors of the covariance matrix of the Brownian motion, maximises the concentration of the total variance of the Brownian motion in the first few dimensions.[6] Its construction requires, however, $O(d^2)$ operations instead of $O(d)$ operations which are needed for the random walk or for the Brownian bridge discretization. For large d, this often increases the run times of the simulation and limits the practical use of the PCA construction.

4 Numerical Results

We now describe the basic setting for our numerical experiments and investigate the sensitivities of the performance figures from Section 2.5 to the input parameters of the model. Then, the risks and returns of two different asset allocation strategies are compared. Finally, we compute the effective dimensions of the integral (17) in the truncation and superposition sense and compare the efficiency of different numerical approaches for its computation.

4.1 Setting

We consider a representative model portfolio with $50,000$ contracts which have been condensed into 500 equal-sized model points. The data of each model point i is generated according to the following distribution assumptions: entry age $\underline{x}^i \sim N(36, 10)$, exit age $\overline{x}^i \sim N(62, 4)$, current age $x_0^i \sim U(\underline{x}^i, \overline{x}^i)$ and monthly premium $P^i \sim U(50, 500)$ where $N(\mu, \sigma)$ denotes the normal

[6] Note that without further assumptions on f it is not clear which construction leads to the minimal effective dimension due to possibly non-linear dependencies of f on the underlying Brownian motion. As a remedy, also more complicated covariance matrix decompositions can be employed which take into account the function f as explained in [26].

	stock price model		interest rate model					correlation
	$\mu = 8\%$	$\sigma_s = 20\%$	$\kappa = 0.1$	$\theta = 4\%$	$\sigma_r = 5\%$	$r_0 = 3\%$	$\lambda_0 = -5\%$	$\rho = -0.1$
$\mathbb{E}[Q_K]$	0.028	0.035	0.007	0.085	-0.001	0.156	-0.001	-0.0008
$\mathbb{E}[F_K]$	0.039	-0.008	0.009	0.136	-0.0014	0.212	-0.0014	-0.0002
PD_K	-0.431	0.219	-0.172	-0.884	0.729	-2.122	0.005	0.04

Table 2. Capital market parameters p used in the simulation and their partial derivatives $f'(p)/f(p)$ for $f \in \{\mathrm{PD}_K, \mathbb{E}[Q_K], \mathbb{E}[F_K]\}$.

	asset allocation		bonus declaration		shareholder	product parameters		solv. rate
	$\beta = 10\%$	$\tau = 3$	$\omega = 25\%$	$\gamma = 15\%$	$\alpha = 90\%$	$\vartheta = 90\%$	$z = 3\%$	$\gamma_0 = 10\%$
$\mathbb{E}[Q_K]$	0.083	0.004	-0.002	0.009	-0.101	-0.006	-0.086	0.011
$\mathbb{E}[F_K]$	0.002	0.002	-0.009	0.03	0.013	-0.01	-0.22	0.034
PD_K	0.265	-0.054	0	-0.002	0.001	0.08	2.706	-0.504

Table 3. Solvency rate, management and product parameters p used in the simulation and their partial derivatives $f'(p)/f(p)$ for $f \in \{\mathrm{PD}_K, \mathbb{E}[Q_K], \mathbb{E}[F_K]\}$.

distribution with mean μ and variance σ, and $U(a, b)$ denotes a uniform distribution in the interval $[a, b]$. In addition, the side conditions $15 \leq \underline{x}^i \leq 55$ and $55 \leq \overline{x}^i \leq 70$ are respected. The probability that the contracts of a model point belong to female policyholders is assumed to be 55%. From the difference of exit age and current age the maturity time $d^i = \overline{x}^i - \underline{x}^i$ of the contracts is computed. As sample insurance product, an endowment insurance with death benefit, constant premium payments and surrender option is considered as described in Example 1. For simplicity, we assume that the policies have not received any bonus payments before the start of the simulation, i.e., $B_0^i = 0$ for all $i = 1, \ldots, m$. We take the probabilities q_k^i of death from the DAV 2004R mortality table and choose exponential distributed surrender probabilities $u_k^i = 1 - e^{-0.03\Delta t}$. At time t_0, we assume a uniform bond allocation, i.e., $n_j = (1-\beta)C_0 / \sum_{i=0}^{\tau-1} b_0(i)$ for $j = 1-\tau, \ldots, 0$. We assume $Q_0 = 0$ which means that the shareholders will not make additional payments to the company to avoid a ruin. This way, $\mathbb{E}[Q_k]$ serves as a direct measure for the investment returns of the shareholders in the time interval $[0, t_k]$. The total initial reserves of the company are then given by $F_0 = \gamma_0 D_0$. In the following, we choose a simulation horizon of $T = 10$ years and a period length of $\Delta t = 1/12$ years, i.e., $K = 120$. In our numerical tests we use the capital market, product and management parameters as displayed in the second rows of Table 2 and 3 unless stated otherwise. In Table 2 and 3 also the sensitivities $f'(v)/f(v)$ (see Section 2.5) are displayed for different functions $f \in \{\mathrm{PD}_K, \mathbb{E}[Q_K], \mathbb{E}[F_K]\}$ and different model input parameter v, e.g., $\partial \mathrm{PD}_K / (\partial \mu \, \mathrm{PD}_K) = -0.431$.

4.2 Capital Allocation

To illustrate possible applications of the ALM model, we compare the constant-mix capital allocation strategy of Section 2.3 with an CPPI (constant proportion portfolio insurance) capital allocation strategy (see, e.g., [34]) with respect to the resulting default risk PD_K and returns $\mathbb{E}[Q_K]$. Within the CPPI strategy, the proportion of funds invested in (risky) stocks is linked to the current amount of reserves. The strategy is realised in our model framework by replacing $\beta(C_{k-1} + P_k)$ in equation (7) by $\beta\, F_{k-1}$ with $\beta \in \mathbb{R}^+$. The resulting risk-return profiles of the constant-mix strategy and of the CPPI strategy are displayed in Fig. 2 for different choices of β.

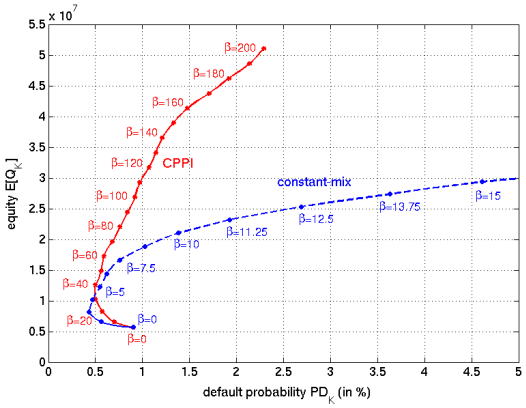

Fig. 2. Risk-return profiles of the different capital allocation strategies.

We see that the slightly negative correlation $\rho = -0.1$ results in a diversification effect such that the lowest default risk is not attained at $\beta = 0$ but at about $\beta = 2.5\%$ in the constant-mix case and at about $\beta = 40\%$ in the CPPI case. Higher values of β lead to higher returns but also to higher risks. As an interesting result we further see that the CPPI strategy almost always leads to portfolios with much higher returns at the same risk and is therefore clearly superior to the constant-mix strategy almost independently of the risk aversion of the company. The only exception is a constant-mix portfolio with a stock ratio β of $2.5 - 4\%$, which could be an interesting option for a very risk averse company.

4.3 Effective Dimension

For the setting of Section 4.1, we determine in this section the effective dimensions d_t and d_s of the integral (17) in the truncation and superposition sense, respectively, see Section 3.3. The effective dimensions depend on the

nominal dimension d, on the discretization of the underlying Gaussian process and on all other model parameters. In Table 4, the effective dimensions d_t are displayed which arise by the methods described in [44] for different nominal dimensions d if the random walk, the Brownian bridge and the principal component (PCA) path construction is employed, respectively. One can see that the Brownian bridge and PCA path construction lead to a large reduction of the effective dimension d_t compared to the random walk discretization. In the latter case, the effective dimension d_t is almost as large as the nominal dimension d while in the former cases the effective dimensions are almost insensitive to the nominal dimensions and are bounded by only $d_t = 16$ even for very large dimensions as $d = 512$. In case of the PCA construction, d_t is even slightly decreasing for large d which is related to the so-called concentration of measure phenomenon, see [31]. Further numerical computations using the method described in [45] show that the ALM problem is also of very low effective dimension d_s in the superposition sense. Here, we only consider moderately high nominal dimensions due to the computational costs which increase with d. For $d \leq 32$, we obtain that the integral (17) is 'nearly' additive, i.e. $d_s = 1$, independent of d and independent of the covariance matrix decomposition. Note that the effective dimensions are affected by several parameters of the ALM model. More results which illustrate how the effective dimensions in the truncation sense vary in dependence of the capital market model and of other parameters can be found in [18].

d	Random walk	Brownian bridge	Principal comp.
32	32	7	12
64	64	7	14
128	124	13	12
256	248	15	8
512	496	16	8

Table 4. Truncation dimensions d_t of the ALM integrand (17) for different nominal dimensions d and different covariance matrix decompositions.

4.4 Convergence Rates

In this section, we compare the following methods for the computation of the expected value (17) with the model parameters specified in Section 4.1:

- MC Simulation,
- QMC integration based on Sobol point sets (see [32, 43]),
- dimension-adaptive SG based on the Gauss-Hermite rule (see [16]).

In various numerical experiments, the Sobol QMC method and the dimension-adaptive Gauss-Hermite SG method turned out to be the most efficient representatives of several QMC variants (we compared Halton, Faure, Sobol low

discrepancy point sets and three different lattice rules with and without randomisation) and of several SG variants (we compared trapezoidal, Clenshaw-Curtis, Patterson, Gauss-Legendre and Gauss-Hermite rule and different grid refinement strategies), respectively. The results for $d = 32$ and $d = 512$ are summarised in Fig. 3 where the number n of function evaluations is displayed which is needed to obtain a given accuracy. In both cases we used the Brownian bridge path construction for the stock prices and short interest rates. One

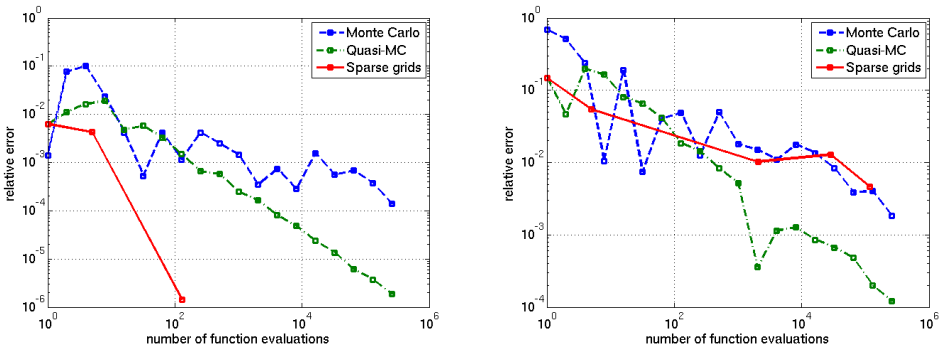

Fig. 3. Errors and required number of function evaluations of the different numerical approaches to compute the expected value (17) with $d = 32$ (left) and with $d = 512$ (right) for the model parameters specified in Section 4.1.

can see that the QMC method clearly outperforms MC simulation in both examples. The QMC convergence rate is close to one and nearly independently of the dimension. Moderate accuracy requirements of about $10^{-3}-10^{-4}$ are obtained by the QMC method about 100-times faster as by MC simulation. For higher accuracy requirements, the advantage of the QMC method is even more pronounced. Recall that these results can not be explained by the Koksma-Hlawka inequality but by the very low effective dimension of the ALM problem, see Section 4.3. The performance of the SG method deteriorates for very high dimensions. In the high dimensional case $d = 512$, the SG method is not competitive to QMC. For the moderately high dimension $d = 32$, sparse grids are the most efficient method with a very high convergence rate of almost three. With 129 function evaluation already an accuracy of 10^{-6} is achieved. Further numerical experiments indicate that the performance of the SG method is more sensitive than (Q)MC to different choices of model parameters which affect the smoothness of the integrand, like more aggressive bonus declaration schemes and more volatile financial markets.

5 Concluding remarks

In this article, we first described a discrete time model framework for the asset-liability management of life insurance products. The model incorporates fairly general product characteristics, a surrender option, a reserve-dependent bonus declaration, a dynamic capital allocation and a two-factor stochastic capital market model. The recursive formulation of the model allows for an efficient computation of the model equations. Furthermore, the model structure is modular and allows to be extended easily. Numerical experiments illustrate that the model captures the main behaviour patterns of the balance sheet development of life insurance products. In the second part of this article, we investigated the application of deterministic integration schemes, such as quasi-Monte Carlo and sparse grid methods for the numerical simulation of ALM models in life insurance. Numerical results demonstrate that quasi-Monte Carlo and sparse grid methods can often outperform Monte Carlo simulation for the ALM of participating life insurance products. Furthermore, quasi-Monte Carlo methods converge nearly independently of the dimension and produce even for high dimensions $d = 512$ more precise results than MC. Sparse grids are the most efficient method for moderately high dimensions, but their performance is more sensitive to different choices of model parameters which affect the smoothness of the integrand and deteriorates for very high dimensions. In these cases additional transformations are required to improve the smoothness of the integrand. To explain the efficiency of the deterministic methods we computed the effective dimension of the ALM problem with and without dimension reduction techniques and showed that ALM problem are often of very low effective dimension in the truncation and also in the superposition sense.

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