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Abstract

The appearance of singularities at isolated point loads of plane load-bearing structures, like ceilings, poses severe difficulties for numerical simulations in structural engineering based on elastic plate theory, e.g., Kirchhoff-Love or Mindlin-Reissner models. In order to overcome this obstacle, we propose a general approach based on Green's functions and methods from singular analysis to explicitly determine the asymptotic behaviour of elastic plate models in the vicinity of point loads. In this context, we have studied interrelations between Green's functions and parametrices in a conical pseudo-differential algebra for the Laplace and bi-Laplace operator. Eventually our method provides a general approach for the construction of Green's functions of elastic shell models, where analytic Green's functions are presently not available. Besides a global model, we consider a local defect correction (LDC) in a neigbourhood of a point load. These models are coupled with each other and can be solved in an iterative manner. Whereas the global model can be treated numerically by a coarse discretization scheme, which is appropriate away from a point load, the local model has to take care of the singular structure by a priori subtracting the singular asymptotic behaviour of the solution which is provided by the above mentioned methods from singular analysis.

Contents

1 Introduction

The presence of singularities in physical models is a generic difficulty for their numerical simulation. Even a rather mild singular behaviour of the quantity of interest, e.g., divergent higher derivatives near the singularity, might severely restrict convergence rates of numerical discretization schemes. Various approaches exist to overcome this obstacle, among these are adaptive refinement schemes [16] and singular basis functions [2], which restore the original performance of discretization schemes in absence of singularities. In this context it is helpful to have a priori knowledge of the expected asymptotic singular behaviour of the quantity of interest. The desired asymptotic information can be obtained by using techniques from singular analysis. Within the present work, we provide a case study for the interrelation of numerical and singular analysis in order to solve singular boundary value problems related to elliptic partial differential operators.

The paper is organized as follows: In Section 2, we outline two general types of singular elliptic boundary value problems and discuss some basic techniques from singular analysis which provide a priori asymptotic information in the vicinity of a singularity. In order to make the paper reasonably self contained, this part contains a short outline of terms and definitions we have employed from singular analysis. Section 2.1 provides a brief outline of the LDC method and how we want to make use of a priori asymptotic information. This is followed in Section 2.2 by a preliminary error analysis of the LDC in this context. In Section 3, we ouline the Kirchhoff-Love and Reissner-Mindlin plate models and our Green's function based approach to transform the original singular problems into smooth problems that can be efficiently solved by numerical methods. Having specified the Green's functions, including boundary conditions, which are required by our approach, we turn to a discussion of these Green's functions from the singular analysis point of view. In particular, we highlight some specific features of these two dimensional problems which are absent in higher dimensions. For the sake of honesty, one should mention, that analytic expressions for the required Green's functions are well known in the literature, see e.g. [22, 20, 21]. Therefore readers only interested in the numerical aspects of this work can immediately jump to Section 5. The motivation behind our treatment of Green's functions in the framework of singular analysis, is the lack of a general approach that provides analytic expressions for a wide class of elliptic partial differential operators. Actually, we hope that our work paves the way for applications where no analytic expressions for the Green's function are presently available. As a first step in this direction, the present work should be considered as a feasibility study, which reveals the prospects but also potential problems of our approach. In Section 4.1, we discuss a recursive scheme for the construction of the parametrix of an elliptic operator and illustrate it for the Laplace operator. We discuss the properties of its parametrix and point out how to get the desired Green's function from it. Finally, we consider in Section 4.2 a similar construction for the parametrix of the bi-Laplace operator and derive the corresponding Green's function, including appropriate boundary conditions. The paper closes in Section 6 with concluding remarks and an outlook on our future work.

2 Local defect correction incorporating asymptotic information

Before we enter into our discussion of the LDC method, cf. [13, 14, 17] for a detailed exposition, let us depict the basic idea of our approach in a rather informal manner. The boundary value problems, we want to solve numerically are of generic form

$$\text{Type a):} \quad \mathcal{A}u^* = f \quad \text{or} \quad \text{Type b):} \quad \mathcal{A}u^* = \delta(\cdot - \tilde{x}) \text{ with } \tilde{x} \in \Omega, \tag{2.1}$$

in an open domain Ω, where $\mathcal{A}$ represents a possibly singular elliptic partial differential operator. For type a) problems, the right-hand side f is commonly supposed to be singular. We want to consider second and fourth order differential operators and boundary conditions involving $u^*|_{\partial\Omega}$, $\partial_\nu u^*|_{\partial\Omega}$ and $\partial_\nu^2 u^*|_{\partial\Omega}$, where ∂_ν denotes the normal derivative at the boundary.

It is a peculiar feature of our LDC method that it requires an a priori knowledge of the asymptotic behaviour of u^* in the vicinity of singularities. The pseudo-differential calculus of singular analysis provides a systematic approach to obtain the desired asymptotic information, for detailed expositions we refer to the monographs [6, 19, 24]. In the following, we want to sketch some basic ideas of the corresponding operator algebra and introduce appropriate function spaces which take care of the asymptotic behaviour near a singularity. The function spaces we have to consider are weighted Sobolev spaces with asymptotics, so called Kegel spaces, which replace the ordinary Sobolev spaces commonly used in numerical analysis. Within the present work, it is sufficient to restrict our discussion to point-like conical singularities, it should be mentioned, however, that the techniques discussed below can be extended to higher order edge and corner singularities as well.

Let us consider a conical singularity that can be locally modelled by an open stretched cone $\mathcal{C}^2 := \mathbb{R}_+ \times S^1$, with base S^1. For applications in plate theory it is sufficient to consider cones with base S^1 but the definitions given below work for arbitrary cones with a smooth base. For the definition of weighted Sobolev spaces $\mathcal{K}^{s,\gamma}(\mathcal{C}^2)$, we make use of the identification $\mathbb{R}^2 \setminus \{0\}$ and $\mathcal{C}^2$ via polar coordinates $\theta: \ x \to (r, \phi)$, i.e.,[1]

$$\mathcal{K}^{s,\gamma}(\mathcal{C}^2) := \omega \mathcal{H}^{s,\gamma}(\mathcal{C}^2) + (1-\omega) H^s(\mathbb{R}^2),$$

for a cut-off function ω, i.e., $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega(r) = 1$ near $r = 0$. Here $\mathcal{H}^{s,\gamma}(\mathcal{C}^2) = r^\gamma \mathcal{H}^{s,0}(\mathcal{C}^2)$, and $\mathcal{H}^{s,0}(\mathcal{C}^2)$ for $s \in \mathbb{N}_0$ is defined to be the set of all $u(r,\phi) \in r^{-1}L^2(\mathbb{R}_+ \times S^1)$ such that $(r\partial_r)^j Du \in r^{-1}L^2(\mathbb{R}_+ \times S^1)$ for all $D \in \mathrm{Diff}^{s-j}(S^1)$, $0 \le j \le s$. The definition for $s \in \mathbb{R}$ in general follows by duality and complex interpolation. Beyond a certain distance from the singularity, $\mathcal{K}^{s,\gamma}(\mathcal{C}^2)$ spaces become ordinary Sobolev spaces which means that for $u \in \mathcal{K}^{s,\gamma}(\mathcal{C}^2)$, the part $(1-\omega)u$ belongs, after back-transformation from polar to Cartesian coordinates, to the ordinary Sobolev space $H^s(\mathbb{R}^2)$. Weighted Sobolev spaces with asymptotics are subspaces of $\mathcal{K}^{s,\gamma}$ spaces which are defined as direct sums

$$\mathcal{K}_Q^{s,\gamma}(\mathcal{C}^2) := \mathcal{E}_Q^\gamma(\mathcal{C}^2) + \mathcal{K}_\Theta^{s,\gamma}(\mathcal{C}^2) \tag{2.2}$$

of flattened weighted cone Sobolev spaces

$$\mathcal{K}_\Theta^{s,\gamma}(\mathcal{C}^2) := \bigcap_{\epsilon > 0} \mathcal{K}^{s,\gamma-\vartheta-\epsilon}(\mathcal{C}^2)$$

with $\Theta = (\vartheta, 0]$, $-\infty \le \vartheta < 0$, and asymptotic spaces

$$\mathcal{E}_Q^\gamma(\mathcal{C}^2) := \left\{ \omega(r) \sum_j \sum_{k=0}^{m_j} c_{jk}(x) r^{-q_j} \ln^k r \right\}.$$

[1]The definition means that a function $u: \ \mathcal{C}^2 \to \mathbb{R}$ in $\mathcal{K}^{s,\gamma}(\mathcal{C}^2)$ can be represented in the form $\omega u + (1-\omega)u$ such that $\theta^*(1-\omega)u \in H^s(\mathbb{R}^2)$, where θ^* denotes the pullback on functions of the diffeomorphism θ.

The asymptotic space $\mathcal{E}_Q^\gamma(\mathcal{C}^2)$ is characterized by a sequence $q_j \in \mathbb{C}$ which is taken from a strip of the complex plane, i.e.,

$$q_j \in \left\{ z : \frac{3}{2} - \gamma + \vartheta < \Re z < \frac{3}{2} - \gamma \right\},$$

where the width and location of this strip are determined by its *weight data* (γ, Θ) with $\Theta = (\vartheta, 0]$ and $-\infty \leq \vartheta < 0$. Each substrip of finite width contains only a finite number of q_j. Furthermore, the coefficients c_{jk} belong to finite dimensional subspaces $L_j \subset C^\infty(S^1)$. The asymptotics of $\mathcal{E}_Q^\gamma(\mathcal{C}^2)$ is therefore completely characterized by the *asymptotic type* $Q := \{(q_j, m_j, L_j)\}_{j\in\mathbb{Z}_+}$. In the following, we employ the asymptotic subspaces

$$\mathcal{S}_Q^\gamma(\mathcal{C}^2) := \left\{ u \in \mathcal{K}_Q^{\infty,\gamma}(\mathcal{C}^2) : (1-\omega)u \in \mathcal{S}(\mathbb{R}, C^\infty(S^1))|_{\mathbb{R}_+} \right\}$$

with Schwartz type behaviour for exit $r \to \infty$. The spaces $\mathcal{K}_Q^{s,\gamma}(\mathcal{C}^2)$ and $\mathcal{S}_Q^\gamma(\mathcal{C}^2)$ are Fréchet spaces equipped with natural semi-norms according to the decomposition (2.2); we refer to [6, 19, 24] for further details.

Type a) problems are of standard form and the singular behaviour, to be specified below, should be restricted to a finite number of points in Ω. Furthermore, let us assume that we have a left parametrix $\mathcal{P}$ of $\mathcal{A}$, which means that $\mathcal{P}$ acts as a pseudo-inverse of $\mathcal{A}$, satisfying the operator equation

$$\mathcal{P}\,\mathcal{A} = I + \mathcal{G}, \tag{2.3}$$

where $\mathcal{G}$ denotes a so-called Green operator. The operators in (2.3) belong to a pseudo-differential algebra of operators that map between weighted Sobolev spaces with and without asymptotics. More precisely, we consider a partial differential operator which represents continuous operators

$$\mathcal{A}:\ \mathcal{K}_P^{s,\gamma}(\mathcal{C}^2) \to \mathcal{K}_Q^{s-\mu,\gamma-\mu}(\mathcal{C}^2), \quad \mathcal{A}:\ \mathcal{K}^{s,\gamma}(\mathcal{C}^2) \to \mathcal{K}^{s-\mu,\gamma-\mu}(\mathcal{C}^2),$$

where $\mathcal{K}_{P,(Q)}^{s,\gamma}$ and $\mathcal{K}^{s,\gamma}$ denote weighted Sobolev spaces with and without specified asymptotic behaviour, respectively. The specified, possibly disctinct asymptotic behaviour is indicated by the subscripts P, Q. Similarly, a parametrix acts as continuous operator

$$\mathcal{P}:\ \mathcal{K}_Q^{s-\mu,\gamma-\mu}(\mathcal{C}^2) \to \mathcal{K}_P^{s,\gamma}(\mathcal{C}^2), \quad \mathcal{P}:\ \mathcal{K}^{s-\mu,\gamma-\mu}(\mathcal{C}^2) \to \mathcal{K}^{s,\gamma}(\mathcal{C}^2), \tag{2.4}$$

and the Green operator

$$\mathcal{G}:\ \mathcal{K}^{s,\gamma}(\mathcal{C}^2) \to \mathcal{S}_O^{s,\gamma}(\mathcal{C}^2), \tag{2.5}$$

maps weighted a weighted Sobolev space without specified asymptotic behaviour into a Schwartz space with specified asymptotic behaviour. Application of a parameterix from the left to a type a) equation yields

$$u^* = \mathcal{P}\,f - \mathcal{G}\,u^*. \tag{2.6}$$

Let us take a closer look at the asymptotic behaviour of the two terms on the right-hand side of (2.6). If we assume that f has a specific asymptotic behaviour, i.e., it belongs to $\mathcal{K}_Q^{s-\mu,\gamma-\mu}$ for a certain asymptotic type Q, then, according to (2.4), also $\mathcal{P}\,f$ belongs to such a space with a certain asymptotic type P. The second term depends on the unknown solution $u^* \in \mathcal{K}^{s,\gamma}$, however, because of (2.5), the Green operator provides a priori asymptotic information without an explicit knowledge of u^*. In summary, (2.6) provides rather detailed asymptotic information concerning the unkown solution u^* if the parametrix $\mathcal{P}$ and the Green operator $\mathcal{G}$ are explicitly known.

In the LDC method, an approximate solution $u^{(0)}$ on the global domain Ω, induces boundary conditions on the boundary of the local domain $\partial\omega$. The corresponding boundary value problem on ω is conveniently solved in two steps. In a first step appropriate boundary conditions, e.g., homogeneous Dirichlet or Neumann boundary conditions, are chosen for the parametrices and Green's functions of type a) and b) problems, respectively. The actual boundary conditions, imposed by the global solution on $\partial\omega$, are taken into account in a second step, by a numerical solution of a desingularized boundary value problem on ω, cf. point (iv) in the LDC schemes outlined below. Concerning the pseudo-differential calculus for type a) problems, it should be mentioned that the parametrix and corresponding Green operator are only required on the local domain ω, which actually contains the singularity. This offers the possibility to employ a local asymptotic expansion of the parametrix $\mathcal{P}_\omega$ and Green operator $\mathcal{G}_\omega$ in a neighbourhood of the singularity. In order to determine the leading order singular asymptotic terms of the exact solution u^*, we furthermore approximate u^* on the right-hand side by the approximate solution $u^{(0)}$, i.e.,

$$u^* \sim \mathcal{P}_\omega f - \mathcal{G}_\omega u^{(0)}. \tag{2.7}$$

Solutions of type b) problems correspond to fundamental solutions of the differential operator $\mathcal{A}$ and are generically singular at the point $\tilde{x}$. We assume, that an asymptotic fundamental solution is known in a local neighbourhood $\omega \subset \Omega$ of the singularity, which satisfies the equation

$$\mathcal{A}\, g^*(\cdot, \tilde{x}) = \delta(\cdot - \tilde{x}) + \mathcal{O}(|x - \tilde{x}|^m), \tag{2.8}$$

Application in LDC is simplified, if the asymptotic fundamental solution $g^*(\cdot, \tilde{x})$ satisfies homogeneous boundary conditions, i.e., $g^*(\cdot, \tilde{x})|_{\partial\Omega} = 0$, for second order differential operators, and additionally $\partial_\nu g^*(\cdot, \tilde{x})|_{\partial\Omega} = 0$ or $\partial_\nu^2 g^*(\cdot, \tilde{x})\big|_{\partial\Omega} = 0$ for fourth order differential operators.

2.1 Brief outline of the LDC approach

Let us outline the simplest version of a LDC method based on (2.7) or (2.8). Here and in the following section our discussion is based on a finite difference instead of a finite element discretization scheme in order to avoid particular reference to a specific type of finite element in the regularity estimates given below. Before we enter into this topic, let us define some spaces and operators which are required in the following. The LDC method is based on two different grids Ω_ℓ and ω_ℓ, where Ω_ℓ refers to a uniform global grid on the domain Ω and ω_ℓ to a local grid on the domain ω. On the grids Ω_ℓ and ω_ℓ, we consider discretized differential operators A_{Ω_ℓ} and A_{ω_ℓ}, respectively. In order to restrict f (u^*) to Ω_ℓ let us introduce an appropriate restriction operator $\tilde{R}_\Omega$ (R_Ω)[2], and a similar operators $\tilde{R}_\omega$ (R_ω) to restrict these functions from ω to ω_ℓ. Actually it makes sense to consider different kinds of restriction operators for the right-hand side f and (approximate) solutions because of their behaviour near the singularity. In some particular applications, the right-hand side f has a divergent behaviour near the singularity, whereas the solution itself remains finite and only its derivatives diverge. Vice versa let us also introduce prolongation operators T_{Ω_ℓ}, T_{ω_ℓ} which provides prolongations from functions on the grids Ω_ℓ and ω_ℓ to functions on Ω and ω, respectively. Furthermore let us introduce a transfer operator π_{Ω_ℓ} which maps functions on Ω_ℓ into functions on ω_ℓ. For the reverse mapping $\omega_\ell \to \Omega_\ell \cap \omega_\ell$, we introduce the transfer operator π_{ω_ℓ}. After these preliminary remarks let us discuss two basic LDC approaches.

Type a) boundary value problems:

(i) Consider Eq. (2.1)[Type a)] on the global grid Ω_ℓ

$$A_{\Omega_\ell} u_\ell^{(0)} = \tilde{R}_\Omega f, \tag{2.9}$$

which is appropriate to approximate the solution except near singularities of the right-hand side f.

[2] For type b) problems, discretization of the Dirac delta distribution requires special care, cf. [5, 26].

(ii) Interpolate the boundary values, i.e., determine $\pi_{\Omega_\ell} u_\ell^{(0)}|_{\partial\omega_\ell}$.

(iii) Approximate the singular asymptotic behaviour via

$$\tilde{u}^{(0)} = \mathcal{P}_\omega f - \mathcal{G}_\omega T_{\Omega_\ell} u_\ell^{(0)}, \tag{2.10}$$

on the local domain ω.

(iv) Solve on ω_ℓ the local problem

$$A_{\omega_\ell} \delta_\ell = \tilde{R}_\omega \big(f - \mathcal{A}\, \tilde{u}^{(0)} \big) \tag{2.11}$$

with Dirichlet boundary conditions $\delta_\ell|_{\partial\omega_\ell} = \pi_{\Omega_\ell} u_\ell^{(0)}|_{\partial\omega_\ell} - R_\omega \tilde{u}^{(0)}|_{\partial\omega_\ell}$.

(v) Set on ω_ℓ: $\tilde{u}_\ell^{(1)} := R_\omega \tilde{u}^{(0)} + \delta_\ell$ which satisfies the boundary condition $\tilde{u}_\ell^{(1)}|_{\partial\omega_\ell} = \pi_{\Omega_\ell} u_\ell^{(0)}|_{\partial\omega_\ell}$.

(vi) Solve on Ω_ℓ

$$A_{\Omega_\ell} u_\ell^{(1)} = \begin{cases} A_{\Omega_\ell} \pi_{\omega_\ell} \tilde{u}_\ell^{(1)} & \Omega_\ell \cap \omega_\ell \\ \tilde{R}_\Omega f & \Omega_\ell \setminus \omega_\ell \end{cases} \tag{2.12}$$

Type b) boundary value problems:

(i) Consider Eq. (2.1)[Type b)] on the global grid Ω_ℓ

$$A_\ell u_\ell^{(0)} = \tilde{R}_\Omega \delta(\cdot - \tilde{x}), \tag{2.13}$$

which is appropriate to approximate the solution except near the singularity at $\tilde{x} \in \Omega$ of the Dirac distribution. We do not assume that $\tilde{x}$ corresponds to a lattice point of Ω_ℓ or ω_ℓ, respectively.

(ii) Interpolate the boundary values, i.e., determine $\pi_{\Omega_\ell} u_\ell^{(0)}|_{\partial\omega_\ell}$.

(iii) Approximate the singular asymptotic behaviour via the given asymptotic fundamental solution

$$\tilde{u}^{(0)} = g^*(\cdot, \tilde{x}), \tag{2.14}$$

on the local domain ω.

(iv) Solve on ω_ℓ the local problem

$$A_{\omega_\ell} \delta_\ell = \tilde{R}_\omega \big(\delta(\cdot - \tilde{x}) - \mathcal{A}\, \tilde{u}^{(0)} \big) \tag{2.15}$$

with Dirichlet boundary conditions $\delta_\ell|_{\partial\omega_\ell} = \pi_{\Omega_\ell} u_\ell^{(0)}|_{\partial\omega_\ell} - R_\omega \tilde{u}^{(0)}|_{\partial\omega_\ell}$ for second order differential operators and an additional condition for fourth order operators, to be specified below. Here $\delta(\cdot - \tilde{x}) - \mathcal{A}\, \tilde{u}^{(0)}$ is supposed to represent a regular distribution which vanishes at the singularity.

(v) Set on ω_ℓ: $\tilde{u}_\ell^{(1)} := R_\omega \tilde{u}^{(0)} + \delta_\ell$ which satisfies the boundary condition $\tilde{u}_\ell^{(1)}|_{\partial\omega_\ell} = \pi_{\Omega_\ell} u_\ell^{(0)}|_{\partial\omega_\ell}$.

(vi) Solve on Ω_ℓ

$$A_{\Omega_\ell} u_\ell^{(1)} = \begin{cases} A_{\Omega_\ell} \pi_{\omega_\ell} \tilde{u}_\ell^{(1)} & \Omega_\ell \cap \omega_\ell \\ \tilde{R}_\Omega f & \Omega_\ell \setminus \omega_\ell \end{cases} \tag{2.16}$$

2.2 Preliminary error analysis of the LDC method

Let us perform a rough error analysis of our LDC approach for Type a) problems for a second order elliptic partial differential operator A. In order to simplify our presentation, we use some standard estimates for finite difference schemes on regular grids. In the following, we will use standard Sobolev spaces $H^s(\Omega)$, $H^s(\omega)$ on the global and local domain, as well as the corresponding discrete Sobolev spaces $H^s_\ell(\Omega_\ell)$, $H^s_\ell(\omega_\ell)$ on the grids. Restriction operators $\tilde{R}_\Omega, \ldots$ which simply act by pointwise restriction of functions require $s > 1$ (2D), according to the Sobolev embedding theorem, in order to represent bounded operators, i.e.,

$$\|\tilde{R}_\Omega\|_{H^s_\ell \leftarrow H^s} < \infty. \tag{2.17}$$

On a uniform grid, we assume $\|A_{\Omega_\ell}\|_{H^s_\ell \leftarrow H^{s+2}_\ell} < \infty$ and $\|A^{-1}_{\Omega_\ell}\|_{H^{s+2}_\ell \leftarrow H^s_\ell} < \infty$ for all $s \in \mathbb{R}$. For the canonical choice $s = -1$, e.g., $\tilde{R}_\Omega$ has to be some kind of average[3] in order to satisfy (2.17) for all $s \geq -1$.

The discretisation error of (2.9) is given by

$$A_{\Omega_\ell}\big(u^{(0)}_\ell - R_\Omega u^*\big) = \tilde{R}_\Omega f - A_{\Omega_\ell} R_\Omega u^* = \big(\tilde{R}_\Omega A - A_{\Omega_\ell} R_\Omega\big) u^*. \tag{2.18}$$

For a finite difference scheme it is reasonable to expect

$$\left\| \tilde{R}_\Omega \mathcal{A} - A_{\Omega_\ell} R_\Omega \right\|_{H^s_\ell(\Omega_\ell) \leftarrow H^{s+4}(\Omega)} \lesssim h^2_\ell \quad \text{for } s \geq -1, \tag{2.19}$$

where h_ℓ denotes the grid spacing on Ω_ℓ. Through interpolation between (2.19) and the estimate

$$\left\| \tilde{R}_\Omega \mathcal{A} - A_{\Omega_\ell} R_\Omega \right\|_{H^s_\ell(\Omega_\ell) \leftarrow H^{s+2}(\Omega)} \lesssim 1 \quad \text{for } s \geq -1,$$

one gets

$$\left\| \big(\tilde{R}_\Omega \mathcal{A} - A_{\Omega_\ell} R_\Omega\big) u^* \right\|_{H^s_\ell(\Omega_\ell)} \lesssim h^{\min\{2, t-s-2\}}_\ell \, \|u^*\|_{H^t(\Omega)} \quad \text{for } t \geq s+2 \geq 1. \tag{2.20}$$

Taking into account $\left\| A^{-1}_{\Omega_\ell} \right\|_{H^{s+2}_\ell(\Omega_\ell) \leftarrow H^s_\ell(\Omega_\ell)} \lesssim 1$ one gets from (2.18) and (2.20) the following error estimate on the global grid

$$\left\| u^{(0)}_\ell - R_\Omega u^* \right\|_{H^{s+2}_\ell(\Omega_\ell)} \lesssim h^{\min\{2, t-s-2\}}_\ell \, \|u^*\|_{H^t(\Omega)} \quad \text{for } s \geq -1. \tag{2.21}$$

The next step in the LDC error analysis is to study the error on the local grid ω_ℓ. For this let us introduce a convenient cut-off function χ_ω and consider the difference

$$\begin{aligned} \chi_\omega\big(\tilde{u}^{(0)} - u^*\big) &= \mathcal{P}_\omega f - \mathcal{G}_\omega T_{\Omega_\ell} u^{(0)}_\ell - \chi_\omega\big(\mathcal{P} f - \mathcal{G} u^*\big) \\ &= \big(\mathcal{P}_\omega - \chi_\omega \mathcal{P}\big) f - \big(\mathcal{G}_\omega - \chi_\omega \mathcal{G}\big) u^* + \mathcal{G}_\omega\big(u^* - T_{\Omega_\ell} u^{(0)}_\ell\big) \end{aligned} \tag{2.22}$$

In the case of (2.1) Type a), the last two terms in (2.22) belong to $C^\infty(\omega)$ whereas the regularity of the first term can be controlled by construction of $\mathcal{P}_\omega$. Therefore, let us assume in the following that $\chi_\omega\big(\tilde{u}^{(0)} - u^*\big)$ belongs to $C^n(\omega)$ for any $n \in \mathbb{N}$ which seems to be convenient. We can now

[3] A possible choice would be e.g. $\big(\tilde{R}_\Omega f\big)(p) := \int f(x) b_p(x) dx / \int b_p(x) dx$ with a finite element basis function f_p in the grid point p.

consider the discretisation error on the local grid. From (2.9) and (2.11), we get

$$\begin{aligned} A_{\omega_\ell}\big(\tilde{u}_\ell^{(1)} - R_\omega u^*\big) &= A_{\omega_\ell}\big(R_\omega \tilde{u}^{(0)} + \delta_\ell - R_\omega u^*\big) \\ &= A_{\omega_\ell}\delta_\ell + A_{\omega_\ell} R_\omega\big(\tilde{u}^{(0)} - u^*\big) \\ &= \big(\tilde{R}_\omega f - \tilde{R}_\omega\, \mathcal{A}\, \tilde{u}^{(0)}\big) + A_{\omega_\ell} R_\omega\big(\tilde{u}^{(0)} - u^*\big) \\ &= \tilde{R}_\omega\, \mathcal{A}\big(u^* - \tilde{u}^{(0)}\big) + A_{\omega_\ell} R_\omega\big(\tilde{u}^{(0)} - u^*\big) \\ &= \big(A_{\omega_\ell} R_\omega - \tilde{R}_\omega\, \mathcal{A}\big)\big(\tilde{u}^{(0)} - u^*\big), \end{aligned} \tag{2.23}$$

which satisfies the boundary condition

$$\big(\tilde{u}_\ell^{(1)} - R_\omega u^*\big)\Big|_{\partial\omega_\ell} = \big(\pi_{\Omega_\ell} u_\ell^{(0)} - R_\omega u^*\big)\Big|_{\partial\omega_\ell}.$$

The error on the boundary therefore correponds to the error of the initial global solution $u_\ell^{(0)}$. For sufficiently regular $\tilde{u}^{(0)} - u^*$, the consistency error $\big(A_{\omega_\ell} R_\omega - \tilde{R}_\omega\, \mathcal{A}\big)\big(\tilde{u}^{(0)} - u^*\big)$ is determined by the grid spacing h_ω of the local grid ω_ℓ. In the case of homogeneous Dirchlet boundary conditions one gets the error estimate

$$\Big\|\tilde{u}_\ell^{(1)} - R_\omega u^*\Big\|_{H_\ell^{s+2}(\omega_\ell)} \lesssim h_\ell^{\min\{2,t-s-2\}} \Big\|\tilde{u}^{(0)} - u^*\Big\|_{H^t(\omega)} \quad \text{for } s \geq -1. \tag{2.24}$$

Finally, let us consider the second and last step (2.12) of the iteration scheme. For notational simplicity, let us introduce the characteristic function χ_ω of the local grid ω_ℓ. Then (2.12) can be written in the form

$$A_{\Omega_\ell} u_\ell^{(1)} = \chi_\omega\big(A_{\Omega_\ell} \pi_{\omega_\ell} \tilde{u}_\ell^{(1)}\big) + \big(1 - \chi_\omega\big)\tilde{R}_\Omega f.$$

The error can be represented in the following manner

$$\begin{aligned} A_{\Omega_\ell}\big(u_\ell^{(1)} - R_\Omega u^*\big) &= \chi_\omega\big(A_{\Omega_\ell}\pi_{\omega_\ell}\tilde{u}_\ell^{(1)}\big) + \big(1 - \chi_\omega\big)\tilde{R}_\Omega f - A_{\Omega_\ell} R_\Omega u^* \\ &= \chi_\omega\big(A_{\Omega_\ell}\pi_{\omega_\ell}\tilde{u}_\ell^{(1)} - A_{\Omega_\ell} R_\Omega u^*\big) + (1 - \chi_\omega)\big(\tilde{R}_\Omega f - A_{\Omega_\ell} R_\Omega u^*\big) \\ &= \chi_\omega A_{\Omega_\ell}\big(\pi_{\omega_\ell}\tilde{u}_\ell^{(1)} - R_\Omega u^*\big) + (1 - \chi_\omega)\big(\tilde{R}_\Omega\, \mathcal{A}\, u^* - A_{\Omega_\ell} R_\Omega u^*\big) \\ &= \chi_\omega A_{\Omega_\ell}\big(\pi_{\omega_\ell}\tilde{u}_\ell^{(1)} - R_\Omega u^*\big) + (1 - \chi_\omega)\big(\tilde{R}_\Omega\, \mathcal{A} - A_{\Omega_\ell} R_\Omega\big) u^*, \end{aligned}$$

where the second term $(1 - \chi_\omega)\big(\tilde{R}_\Omega\, \mathcal{A} - A_{\Omega_\ell} R_\Omega\big) u^*$ corresponds to the consistency error away from the singularity, which is supposed to be fine. This part of the error can be estimated e.g. for $s = 0$ according to

$$\|\big(\tilde{R}_\Omega\, \mathcal{A} - A_{\Omega_\ell} R_\Omega\big) u^*\|_{H_\ell^s(\Omega_\ell)} \lesssim h_\ell^{\min\{2,t-s-2\}} \|u^*\|_{H^t(\Omega)}.$$

The case $s = -1$, however, requires a modified cut-off function χ_ω with bounded gradients. The error due to the first term is due to

$$\chi_\omega\big(\pi_{\omega_\ell}\tilde{u}_\ell^{(1)} - R_\Omega u^*\big) = \chi_\omega \pi_{\omega_\ell}\big(\tilde{u}_\ell^{(1)} - R_\omega u^*\big) + \chi_\omega\big(\pi_{\omega_\ell} R_\omega - R_\Omega\big) u^*$$

where the contribution of $\tilde{u}_\ell^{(1)} - R_\omega u^*$ has been allready estimated in (2.24) and we assume $\chi_\omega\big(\pi_{\omega_\ell} R_\omega - R_\Omega\big) = 0$ for simplicity. Putting things together, we obtain the final estimate

$$\begin{aligned} \|u_\ell^{(1)} - R_\Omega u^*\|_{H_\ell^{s+2}(\Omega_\ell)} &\lesssim \|\chi_\omega A_{\Omega_\ell}\big(\pi_{\omega_\ell}\tilde{u}_\ell^{(1)} - R_\Omega u^*\big) + (1 - \chi_\omega)\big(\tilde{R}_\Omega\, \mathcal{A} - A_{\Omega_\ell} R_\Omega\big) u^*\|_{H_\ell^s(\Omega_\ell)} \\ &\lesssim \|\tilde{u}_\ell^{(1)} - R_\omega u^*\|_{H_\ell^{s+2}(\omega_\ell)} + \|\big(\tilde{R}_\Omega\, \mathcal{A} - A_{\Omega_\ell} R_\Omega\big)(1 - \tilde{\chi}_\omega) u^*\|_{H_\ell^s(\Omega_\ell)} \\ &\lesssim h_\ell^{\min\{2,t-s-2\}} \max\Big\{\|\tilde{u}^{(0)} - u^*\|_{H^t(\omega)}, \|(1 - \tilde{\chi}_\omega) u^*\|_{H_\ell^t(\Omega_\ell)}\Big\}, \end{aligned}$$

which demonstrates, by comparison with (2.21), a balanced treatment of the errors on the local and global scale.

2.3 Global versus local defect corrections

In the preceding Sections, we made no explicit assumptions concerning a specific choice of the local subdomain $\omega \subset \Omega$. An important aspect for an optimal choice of ω is the need for further refinement of the local grid ω_ℓ with respect to the global grid Ω_ℓ. If further refinement turns out to be unnecessary, it is fine to choose $\omega = \Omega$ and to avoid a cumbersome matching of boundary conditions between local and global solutions and possible iterative steps altogether. In order to determine whether a global defect correction is appropriate or not depends on our a priori knowledge of the singular behaviour of the unkown solution u^*. In general, we will only assume asymptotic information, like in (2.7) and (2.8), but in certain cases much more could be known. Let us first consider Type a) problems where (2.7) requires a parametrix P_ω, Green operator G_ω and an approximate solution $u^{(0)}$. The construction of a parametrix and corresponding Green operator is an inherently asymptotic procedure, see [8] for a general outline and [7] for a specific example. Therefore it is advisable to consider the asymptotic expansions of P_ω, G_ω only in a sufficiently small neighbourhood ω of the singularity and allow for a further refinement of the grid. Furthermore, the term $G_\omega u^{(0)}$ might require an iterative treatment due to the presence of the approximate solution $u^{(0)}$. For a non-singular operator A, however, the term $G_\omega u^{(0)}$ represents a smooth function which does not contribute to the singular asymptotic behaviour of u^* and the parametrix dependent term $P_\omega f$ completely resolves the singular behaviour. In such a case $\omega = \Omega$ could be an appropriate choice. A similar argument applies to Type b) problems, where fundamental solutions or classical Green's function might be known analytically or can be obtained from an asymptotic parametrix construction. The latter case is explicitly dicussed in the present work for the Laplace and bi-Laplace operator in two dimensions[4].

Even if the defect correction allows us to completely remove the singularity from our problem it might still be beneficial to consider a locally refined grid ω_ℓ. Let us assume for example, that after the defect correction our modified problem, which must be solved numerically, has a solution of the form $\chi(x)u^*(x)$, where $\chi \in C^\infty(\Omega)$ vanishes in a neighbourhood of the singularity. Cut-off functions like χ are poorly approximated on regular grids[5] and a local refinement scheme turns out to be necessary for a balanced treatment of the discretization error.

Le us briefly summarize our previous discussion. In those cases where appropriate information concerning the singular behaviour of the exact solution u^* is available, it is often preferable to use a global defect correction which yields a simplified numerical treatment of the problem under consideration. Otherwise one has to work with a local and global domain and adjust the local grid to the asymptotic information at hand.

3 Green's functions and plate theory

In the context of the present work, we want to consider Green's functions for elliptic partial differential operators

$$\mathcal{A} = \sum_{|\alpha|\leq\mu} a_\alpha(x)\partial^\alpha \tag{3.1}$$

of order μ in an open domain $\Omega \subseteq \mathbb{R}^2$. Our focus is, in particular, the Laplace and bi-Laplace operator, which play a major role in the plate models considered in this paper. It should be mentioned, however, that from the very beginning, it was our intention to generalize this approach to models which involve more general types of operators (3.1). Despite the fact, that Green's functions provide a versatile tool for numerical simulations, applicability is hampered by their limited availability. Only for selected elliptic partial differential operators analytic expressions of the corresponding Green's functions are known. Amongst others, it is an intention of the present work, to

[4]Similar results can be obtained in dimensions greater than two, see [10] for specific examples.

[5]This argument does not refer to the asymptotic behaviour of the discretization error but considers the local error distribution for a fixed discretization.

develop more flexible tools in order to obtain analytic expressions of Green's functions for a broader class of differential operators. The label "Green's function" has been used in a variety of contexts[6], and we restrain from a general rigorous mathematical definition, instead we will refer in the following to the working definition given below.

Definition 1. *A Green's function $G \in \mathcal{D}'(\Omega)$ of an elliptic operator $\mathcal{A}$ is a distribution valued function $G : \Omega \rightarrow \mathcal{D}'(\Omega)$ which satisfies the distributional equation*

$$\mathcal{A}G(\cdot,\tilde{x}) = \delta_{\tilde{x}} \quad \textit{for all } \tilde{x} \in \Omega, \tag{3.2}$$

where $\delta_{\tilde{x}}$ denotes the shifted Dirac distribution, i.e., $\delta_{\tilde{x}}(f) = f(\tilde{x})$ for $f \in \mathcal{D}(\Omega)$. In the follwing, we will also use the commonly employed notation

$$\delta_{\tilde{x}}(f) \equiv \int \delta(x-\tilde{x}) f(x)\, dx,$$

which treats Dirac's distribution formally like a regular distribution.

This essentially corresponds to the classical notion of a Green's function, cf. [4], with special emphasize on its distributional character.

Within the present work, we want to consider two popular plate models of structural engineering. The first model can be obtained from Kirchhoff-Love plate theory and corresponds to a fourth order elliptic boundary value problem on a bounded open domain $\Omega \subset \mathbb{R}^2$, i.e.,

$$K\Delta\Delta u = P\delta(\cdot - \tilde{x}), \quad u|_{\partial\Omega} = g,\ \partial_n u|_{\partial\Omega} = f, \tag{3.3}$$

for a plate with bending stiffness K, where a point load P is given at $\tilde{x} \in \Omega$. Due to Dirac's δ distribution on the right-hand side, the solution of (3.3) should be considered as a fundamental solution in the distributional sense. For (3.3), we consider the Green's function

$$\Delta\Delta G_{bL}(\cdot,\tilde{x}) = \delta(\cdot - \tilde{x}), \quad G_{bL}(\cdot,\tilde{x})|_{\partial\Omega} = 0, \quad \partial_n G_{bL}(\cdot,\tilde{x})|_{\partial\Omega} = 0 \text{ for } \tilde{x} \in \Omega, \tag{3.4}$$

which is symmetric, i.e., $G_{bL}(x,\tilde{x}) = G_{bL}(\tilde{x},x)$, $x,\tilde{x} \in \Omega$, because of the essential self-adjointness of the bi-Laplace operator $\Delta\Delta$. In order to solve the boundary value problem (3.3) with the help of (3.4), we decompose its solution u into two parts, i.e.,

$$u(x) = u_0(x) + u_\infty(x),$$

with

$$u_0(x) := \frac{P}{K} G_{bL}(x,\tilde{x}), \tag{3.5}$$

and u_∞ given as solution of the boundary value problem

$$K\Delta\Delta u_\infty = 0, \quad u_\infty|_{\partial\Omega} = g,\ \partial_n u_\infty|_{\partial\Omega} = f. \tag{3.6}$$

For a sufficiently well behaved boundary $\partial\Omega$, e.g., circle or square, it has a smooth solution, which can be efficiently approximated by finite difference or finite element schemes.

Another model, we want to consider belongs to Reissner-Mindlin plate theory. Like the Kirchhoff-Love model discussed before, it corresponds to a fourth order elliptic boundary value problem on a bounded open domain $\Omega \subset \mathbb{R}^2$, i.e.,

$$K\Delta\Delta u = P\delta(\cdot - \tilde{x}) - \frac{(2-\nu)h^2}{10(1-\nu)} P\Delta\delta(\cdot - \tilde{x}), \quad u|_{\partial\Omega} = g,\ \partial_n u|_{\partial\Omega} = f, \tag{3.7}$$

[6]The notion of a Green's function might refer to classical Green's function in potential theory, many-particle Green's function in quantum many-particle theory or propagators in quantum field theory.

where h, ν denote the thickness and Poisson's ratio of the plate, respectively. In order to get rid of the singular part of its solution, we can make use again of the Green's function (3.4) and of another Green's function for the Laplace operator,i.e.,

$$\Delta G_L(\cdot,\tilde{x}) = \delta(\cdot - \tilde{x}), \quad G_L(\cdot,\tilde{x})|_{\partial\Omega} = 0, \text{ for } \tilde{x} \in \Omega. \tag{3.8}$$

For this purpose, let us decompose the solution into three parts, i.e.,

$$u(x) = u_0(x) + u_1(x) + u_\infty(x),$$

with

$$u_1(x) := -\frac{(2-\nu)h^2}{10(1-\nu)}\frac{P}{K}G_L(x,\tilde{x}),$$

and u_0, u_∞ given by (3.5) and (3.6), respectively.

4 Green's functions from a singular analysis point of view

In the second part of the paper, we want to discuss a general approach, based on techniques from singular analysis, in order to explicitly determine the singular behaviour of solutions of singular PDEs, fundamental solutions and Green's functions of elliptic operators. The motivation behind our work is that one can take significant advantages from the knowledge of the singular asymptotic behaviour in the design of algorithms for numerical solutions of singular problems. The LDC method, discussed in the first part of the paper, provides a convincing example for such an approach. Besides some well known singular problems, however, it is in generally difficult to get access to the required asymptotic information. Singular analysis provides a unified framework for PDEs with various types of singularities. It is in particular possible to derive rather detailed asymptotic information concerning solutions of type a) problems, see e.g. [7, 8, 9], as well as fundamental solutions and Green's functions for elliptic partial differential operators [10] in dimensions greater than two. For some technical reasons, to be discussed below, two dimensional problems are different. First we consider the well known Green's functions of the Laplace operator in $\mathbb{R}^2$, which reveals some subtleties of this function from the singular analysis point of view.

Actually we have to consider two different representations of Laplace operators, depending on whether one considers the Laplace operator with respect to Cartesian coordinates $x := (x_1, x_2)$

$$\Delta_2 := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{4.1}$$

or polar coordinates defined on the stretched cone $\mathcal{C}^2 := \mathbb{R}^+ \times S^1$ with base S^1. In polar coordinates, the Laplace operator is represented by

$$\tilde{\Delta}_2 := \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 + \Delta_{S^1}\right]. \tag{4.2}$$

where $\Delta_{S^1} = \frac{\partial^2}{\partial\phi^2}$ denotes the Laplace-Beltrami operator on the circle S^1. The Laplace-Beltrami operator Δ_{S^1} has a pure point spectrum with eigenvalues $\lambda_\ell = -\ell^2$, $\ell = 0, 1, 2, \ldots$ and each eigenvalue λ_ℓ has multiplicity two. In the following, we denote by P_ℓ, $\ell = 0, 1, 2, \ldots$, the projection operators from $L_2(S^1)$ onto the corresponding eigenspaces. Herewith, we can form the spectral resolution

$$\Delta_{S^1} = -\sum_{\ell=0}^{\infty} \ell^2 P_\ell \tag{4.3}$$

which is crucial for the following considerations.

In the singular cone algebra, the Laplace operator (4.2) is represented by a Mellin pseudo-differential operator with corresponding operator valued Mellin symbol h, i.e.,

$$\tilde{\Delta}_2 u = r^{-2}\,\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)u := r^{-2}\int_{\mathbb{R}}\int_0^\infty \left(\frac{r}{\tilde{r}}\right)^{-(1-\gamma+i\rho)} h(1-\gamma+i\rho)\,u(\tilde{r},\phi)\,\frac{d\tilde{r}}{\tilde{r}}đ\rho$$

with $đ\rho := \frac{d\rho}{2\pi}$, acting on functions $u \in \tilde{\mathcal{D}}(\mathcal{C}^2)$, here $\tilde{\mathcal{D}}(\mathcal{C}^2) := \{\varphi^* g \mid g \in \mathcal{D}(\mathbb{R}^2)\}$, where $\varphi^* g$ denotes the pullback under the diffeomorphism $\varphi:\ \mathcal{C}^2 \to \mathbb{R}^2 \setminus \{0\}$ given by charts of polar coordinates. The operator valued Mellin symbol of the pseudo-differential operator is given by

$$h(w) = w^2 + \Delta_{S^1} = w^2 - \sum_{\ell=0}^{\infty} \ell^2 P_\ell, \tag{4.4}$$

we refer, e.g., to [6, Chapter 8] for further details. Actually, the Laplace operator $\tilde{\Delta}_2$ is not elliptic on the streched cone $\mathcal{C}^2$, therefore in order to construct a parametrix, it would be necessary to consider the shifted Laplace operator $\tilde{\Delta}_2 - \kappa^2$ for which one can perform an asymptotic parametrix construction, outlined in Ref [7]. However, these additional asymptotic terms vanish in the limit $\kappa \to 0$ and eventually do not contribute to the Green's function, therefore we neglect them from the beginning.

The corresponding parametrix

$$\mathcal{P}\,u = r^2\,\mathrm{op}_M^{\gamma-\frac{5}{2}}(h^{(-1)})u := r^2\int_{\mathbb{R}}\int_0^\infty \left(\frac{r}{\tilde{r}}\right)^{-(3-\gamma+i\rho)} h^{(-1)}(3-\gamma+i\rho)\,u(\tilde{r},\phi)\,\frac{d\tilde{r}}{\tilde{r}}đ\rho,$$

with operator valued Mellin symbol $h^{(-1)}$, can be constructed in the usual manner. First, we consider the operator product

$$\begin{aligned}
\mathcal{P}\,\tilde{\Delta}_2 u &= r^2\,\mathrm{op}_M^{\gamma-\frac{5}{2}}(h^{(-1)})r^{-2}\,\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)u \\
&= \mathrm{op}_M^{\gamma-\frac{1}{2}}(T^2 h^{(-1)})\,\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)u \\
&= \mathrm{op}_M^{\gamma-\frac{1}{2}}(T^2 h^{(-1)} h)u,
\end{aligned}$$

where T^n, $n \in \mathbb{N}$, denotes shift operators acting on Mellin symbols via $T^n g(w) = g(w+n)$. The operator valued symbol of the parametrix has to satisfy the equation

$$\big(T^2 h^{(-1)}(w)\big)h(w) = 1$$

which can be solved for

$$\begin{aligned}
h^{(-1)}(w) &= \frac{1}{h(w-2)} \\
&= \frac{1}{(w-2)^2 + \Delta_{S^1}} \\
&= \sum_{\ell=0}^{\infty} \underbrace{\frac{P_\ell}{(w-2+\ell)(w-\ell-2)}}_{=:h_\ell^{(-1)}(w)}.
\end{aligned} \tag{4.5}$$

The last step is an exact inversion and therefore we have

$$\mathcal{P}\,\tilde{\Delta}_2 u = u$$

on the space of test functions $u \in \tilde{\mathcal{D}}(\mathcal{C}^2)$. Vice versa, we have

$$
\begin{aligned}
\tilde{\Delta}_2 \mathcal{P} u &= r^{-2} \operatorname{op}_M^{\gamma-\frac{1}{2}}(h) r^2 \operatorname{op}_M^{\gamma-\frac{5}{2}}(h^{(-1)}) u \\
&= \operatorname{op}_M^{\gamma-\frac{5}{2}}(T^{-2}h) \operatorname{op}_M^{\gamma-\frac{5}{2}}(h^{(-1)}) u \\
&= \operatorname{op}_M^{\gamma-\frac{5}{2}}\big((T^{-2}h) h^{(-1)}\big) u \\
&= u,
\end{aligned}
$$

on the space of test functions $\tilde{\mathcal{D}}(\mathcal{C}^2)$, where the last step follows from $(T^{-2}h)h^{(-1)} = I$. It can be easily that the terms in the sum (4.5) have only simple poles for $\ell \geq 1$ but has a pole of order 2 at $\ell = 0$. It turns out, that for this particular reason, the Laplace operator in two dimensions is special.

4.1 Construction of Green's functions from the kernel function of the parametrix

In this section we discuss a general approach of how to recover a classical Green's function of the Laplace operator from the kernel function of a parametrix. As already mentioned before, the two-dimensional case is different because of a pole of order 2 in the $\ell = 0$ term of (4.5). This requires some modifications of the general approach discussed in Ref. [10], which we first discuss for the Laplace operator and in the following section for the bi-Laplace operator.

As already mentioned before, the Laplace operator has a pole of order 2 at $w = 2$ which belongs to the term with $\ell = 0$. The residuum at $w = 2$ is given by

$$
\begin{aligned}
\operatorname{Res}_{w=2} \left(\frac{r}{\tilde{r}}\right)^{-w} h^{(-1)}(w) &= \frac{d}{dw}\left[(w-2)^2 \left(\frac{r}{\tilde{r}}\right)^{-w} h^{(-1)}(w)\right]_{w=2} \\
&= -\ln\left(\frac{r}{\tilde{r}}\right)\left(\frac{r}{\tilde{r}}\right)^{-2} P_0.
\end{aligned}
$$

Because of the poles at $w \in \mathbb{Z}$, we have to choose $\gamma \notin \mathbb{Z}$. For reasons to be discussed below, we choose either $0 < \gamma < 1$ or $1 < \gamma < 2$ for the integration contour $\Gamma_{3-\gamma}$. Depending on our choice, we obtain

$$
r^2 \operatorname{op}_M^{\gamma-\frac{5}{2}}(h_0^{(-1)}) u = \begin{cases} -\int_r^\infty \ln\left(\frac{r}{\tilde{r}}\right) P_0 u \, \tilde{r} d\tilde{r} & \text{for } 0 < \gamma < 1 \\ \int_0^r \ln\left(\frac{r}{\tilde{r}}\right) P_0 u \, \tilde{r} d\tilde{r} & \text{for } 1 < \gamma < 2 \end{cases}
$$

and for $\ell \geq 1$ one gets

$$
r^2 \operatorname{op}_M^{\gamma-\frac{5}{2}}(h_\ell^{(-1)}) u = -\int_0^\infty \left(\frac{r_<}{r_>}\right)^\ell \frac{1}{2\ell} P_\ell u \, \tilde{r} d\tilde{r}
$$

with

$$
P_\ell u := \int_{S^1} p_\ell u \, d\mu, \quad p_0 = \frac{1}{2\pi} \quad \text{and} \quad p_\ell = \frac{1}{2\pi}\left(e^{-i\ell(\phi-\tilde{\phi})} + e^{i\ell(\phi-\tilde{\phi})}\right), \quad \ell = 1, 2, \ldots. \tag{4.6}
$$

Putting things together, one gets

$$
K_2(r, \phi | \tilde{r}, \tilde{\phi}) = -\Theta(\tilde{r} - r) \ln\left(\frac{r}{\tilde{r}}\right) p_0 - \sum_{\ell=1}^\infty \left(\frac{r_<}{r_>}\right)^\ell \frac{1}{2\ell} p_\ell(\phi, \tilde{\phi}), \quad \text{for } 0 < \gamma < 1, \tag{4.7}
$$

and

$$
K_2(r, \phi | \tilde{r}, \tilde{\phi}) = \Theta(r - \tilde{r}) \ln\left(\frac{r}{\tilde{r}}\right) p_0 - \sum_{\ell=1}^\infty \left(\frac{r_<}{r_>}\right)^\ell \frac{1}{2\ell} p_\ell(\phi, \tilde{\phi}), \quad \text{for } 1 < \gamma < 2, \tag{4.8}
$$

where Θ denotes the Heaviside step function and $r_< := \min\{r, \tilde{r}\}$, $r_> := \max\{r, \tilde{r}\}$, respectively.

In the following, we will make use of Hadamard's notion of a pseudofunction, cf. [25]. By the diffeomorphism $\varphi : \mathcal{C} \to \mathbb{R}^2 \setminus \{0\}$, a function $u(r,\phi)$ on $\mathcal{C}^2$ corresponds to a function $\tilde{u}(x)$ on $\mathbb{R}^2 \setminus \{0\}$. Therefore a function u on $\mathcal{C}^2$ can be regarded as a regular distribution on $\mathbb{R}^2$ if $\tilde{u}$ can be identified with an element in $L^1_{loc}(\mathbb{R}^2)$, the set of equivalence classes of locally integrable functions in $\mathbb{R}^2$, and therefore with a regular distribution in $\mathcal{D}'(\mathbb{R}^2)$. We call this regular distribution the pseudofunction corresponding to u and denote it by $\mathrm{Pf.}\, u$.

Let us now consider the kernel K_2 as a distributional kernel in $\mathcal{D}'(\mathbb{R}^2)$, i.e., we define $K_2(x,\tilde{x}) := \mathrm{Pf.}\, K_2(r,\phi|\tilde{r},\tilde{\phi})$, where we tentatively assume $\tilde{x} \neq 0$. A straightforward calculation, given in Appendix C, reveals

$$\Delta_2 K_2(\cdot,\tilde{x}) = \delta(\cdot - \tilde{x}) - \delta, \quad \text{for } 0 < \gamma < 1, \tag{4.9}$$

and

$$\Delta_2 K_2(\cdot,\tilde{x}) = \delta(\cdot - \tilde{x}), \quad \text{for } 1 < \gamma < 2. \tag{4.10}$$

On a first glance, (4.8) seems to be a suitable candidate, however it cannot be considered as a classical Green's function because of its singular behaviour for $\tilde{r} \to 0$ outside the diagonal D. Let us therefore look a little bit closer at the shortcoming of (4.7), which consists of the additional Dirac distribution on the right-hand side of (4.9). This shortcoming indicates the need for a more general approach which involves an additional Green operator, in addition to the parametrix that maps onto the fundamental solution of the Laplace operator, i.e., we consider the ansatz

$$\tilde{\Delta}_2\big(\mathcal{P} + \mathcal{G}\big) = I, \quad \tilde{\Delta}_2\, \mathcal{G} = 0$$

on the cone $\mathcal{C}^2$. Such an approach, however, seems a bit like a snake who bites its own tail. The original idea was to devise a general scheme to calculate Green's functions and now it seems that we require such a fundamental solution for repair. A closer look at the properties of the parametrix, however, resolves this discrepancy. According to a general result, cf. [6][Section 7.2.3, Theorem 9],

$$\omega r^{-2} \operatorname{op}_M^{\gamma-\frac{1}{2}}(h)\tilde{\omega} \;:\; \mathcal{H}^{s,\gamma}\big(\mathcal{C}^2\big) \;\to\; \mathcal{H}^{s-2,\gamma-2}\big(\mathcal{C}^2\big)$$

is continuous for all $s \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, where $\omega, \tilde{\omega}$ denote arbitrary cut-off functions. Let us suppose, there exists a function u which satisfies $\tilde{\Delta}_2 u = 0$ on $\mathcal{C}^2$, and $P_0 u = u$, such that $\mathrm{Pf.}\, u$ represents a fundamental solution of Δ_2. The existence of such a u follows from the general theory of fundamental solutions of partial differential operators [15]. Due to the local character of the Mellin operator[7], it satisfies

$$r^{-2} \operatorname{op}_M^{\gamma-\frac{1}{2}}(h)\tilde{\omega} u = g,$$

with $g \in \mathcal{H}^{s-2,\gamma-2}\big(\mathcal{C}^2\big)$ for some $s \in \mathbb{R}$ and $\gamma \in \mathbb{R}$. By choosing a cut-off function $\omega \prec \tilde{\omega}$, we get

$$\omega r^{-2} \operatorname{op}_M^{\gamma-\frac{1}{2}}(h)\tilde{\omega} u = 0 \quad \text{and} \quad r^{-2} \operatorname{op}_M^{\gamma-\frac{1}{2}}(h)\tilde{\omega} u = (1-\omega) g$$

Acting with the parametrix from the left, this yields

$$\tilde{\omega} u = \mathcal{P}(1-\omega) g$$

For r sufficiently small, i.e., $\tilde{\omega}(r) = 1$, we get from (4.7),

$$\begin{aligned} u(r) &= \mathcal{P}(1-\omega) g(r) \\ &= \int_{S^1} \int_0^\infty K_2(r,\phi|\tilde{r},\tilde{\phi})\big(1-\omega(\tilde{r})\big) g(\tilde{r}) \tilde{r}\, d\tilde{r}\, d\tilde{\phi} \\ &= -\ln(r) \underbrace{\int_0^\infty \big(1-\omega(\tilde{r})\big) g(\tilde{r}) \tilde{r}\, d\tilde{r}}_{:=C_1} + \underbrace{\int_0^\infty \ln(\tilde{r})\big(1-\omega(\tilde{r})\big) g(\tilde{r}) \tilde{r}\, d\tilde{r}}_{:=C_2}. \end{aligned} \tag{4.11}$$

[7] The Mellin pseudo-differential operator represents a local partial differential operator $\tilde{\Delta}_2$.

A simple calculation gives

$$\Delta_2\,\mathrm{Pf.}\,u = -\tfrac{C_1}{p_0}\,\delta,$$

which shows that u is just a constant multiple of a fundamental solution of Δ_2. More precisely, the previous consideration provides, up to a multiplicative constant, a fundamental solution of Δ_2, which can be used to correct (4.7) in order to get the correct behaviour in (4.9). Adding a corresponding counterterm[8] $\ln(r)p_0$ to (4.7) yields

$$\begin{aligned}
\tilde{K}_2(r,\phi|\tilde{r},\tilde{\phi}) &= K_2(r,\phi|\tilde{r},\tilde{\phi}) + \ln(r)p_0 \\
&= \big(\Theta(r-\tilde{r})\ln(r) + \Theta(\tilde{r}-r)\ln(\tilde{r})\big)\,p_0 - \sum_{\ell=1}^{\infty}\left(\frac{r_<}{r_>}\right)^{\ell}\frac{1}{2\ell}\,p_\ell(\phi,\tilde{\phi}), \\
&= \ln(r_>)p_0 - \sum_{\ell=1}^{\infty}\left(\frac{r_<}{r_>}\right)^{\ell}\frac{1}{2\ell}\,p_\ell(\phi,\tilde{\phi}).
\end{aligned} \tag{4.12}$$

Our correction term corresponds to adding an additional Green operator

$$\mathcal{G}\,u := \ln(r)\int_0^{\infty} P_0 u\,\tilde{r}d\tilde{r},$$

which obviously satisfies $\tilde{\Delta}_2\,\mathcal{G} = 0$ to the parameterix $\mathcal{P}$ and $\tilde{K}_2$ represents the corresponding kernel function of the operator $\mathcal{P}+\mathcal{G}$ in the pseudo-differential algebra. Taking the limit $\lim_{\tilde{r}\to 0}$ in (4.12), one recovers the fundamental solution $\ln(r)p_0$, which means that we eventually obtained a Green's function $G_2(\cdot,\tilde{x})$ of the Laplace operator in a separable form given by the Laplace expansion, i.e.,

$$G_2(\cdot,\tilde{x}) := \begin{cases} \mathrm{Pf.}\,\tilde{K}_2(\cdot|\tilde{r},\tilde{\phi}) & \text{for } \tilde{x} = \varphi(\tilde{r},\tilde{\phi}) \neq 0 \\ \lim_{\tilde{r}\to 0}\mathrm{Pf.}\,\tilde{K}_2(\cdot|\tilde{r},\tilde{\phi}) & \text{for } \tilde{x} = 0 \end{cases}.$$

It can be easily seen, that a similar line of arguments doesn't work for (4.8), just because (4.11) becomes zero in this case. This is due to the fact, that a fundamental solution u does not belong to $\mathcal{H}^{s,\gamma}(\mathcal{C}^2)$ for $1<\gamma<2$. For comparison, we have calculated the Laplace expansion for the Green's function G_2 of the Laplace operator Δ_2 in Appendix A by a conventional approach, cf. Eq. (A.2). It can be seen that this expansion agrees with (4.12) for all $\ell \geq 0$. Furthermore, it is worth mentioning that in contrast to dimensions ≥ 3, the Green's function G_2 cannot be obtained from Green's functions G_κ of the shifted Laplace operator $\Delta - \kappa^2$ by analytic continuation with respect to the parameter $\kappa \to 0$. Here too, it is only the $\ell = 0$ term in the spectral resolution (4.3) which causes a problem. A detailed discussion of this problem is given in Appendix B.

In order to impose boundary conditions the operator $\mathcal{P}+\mathcal{G}$ can be further modified. Let us consider a homogeneous Dirichlet boundary value problem

$$\Delta_2 u = f, \quad u|_{\partial B_R} = 0$$

on the open ball $B_R(0)$ of radius R. Green's functions which satisfy homogeneous Dirichlet boundary condition are called Green's functions of the first kind. In our particular case, the Green's function of first kind can be constructed by the method of images, cf. [11, 18], where one obtains

$$\hat{K}_2(r,\phi|\tilde{r},\tilde{\phi}) = \big(\ln(r_>) - \ln(R)\big)p_0 - \sum_{\ell=1}^{\infty}\left[\left(\frac{r_<}{r_>}\right)^{\ell} - \left(\tfrac{r_< r_>}{R^2}\right)^{\ell}\right]\frac{1}{2\ell}p_\ell(\phi,\tilde{\phi}). \tag{4.13}$$

The additional terms which impose the boundary condition correspond to the Green operator

$$\mathcal{G}_\partial\,u = -\ln(R)\int_0^{\infty} P_0 u\tilde{r}d\tilde{r} + \sum_{\ell=1}^{\infty}\frac{r^\ell}{2\ell R^{2\ell}}\int_0^{\infty}\tilde{r}^\ell P_\ell u\tilde{r}d\tilde{r}$$

[8] The second constant C_2 is irrelevant for a fundamental solution and can be discarded.

which maps u into a harmonic polynomial $\mathcal{G}_\partial u$. Our previous considerations can be summarized in the following proposition:

Proposition 1. *The classical Green's function G_2 of Δ_2 on the open ball $B_R(0)$ with homogeneous Dirichlet boundary conditions is given by to $G_2 := \mathrm{Pf.}\,\hat{K}_2$ where $\hat{K}_2$ denotes the kernel of the operator*

$$\mathcal{P} + \mathcal{G} + \mathcal{G}_\partial : \; L_2(B_R) \;\longrightarrow\; H^2(B_R), \tag{4.14}$$

and satisfies the following conditions

$$\begin{aligned} (i) &\quad \Delta_2 G_2(\cdot, \tilde{x}) = \delta(\cdot - \tilde{x}) \\ (ii) &\quad G_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \setminus D) \\ (iii) &\quad G_2(x, \tilde{x}) = G_2(\tilde{x}, x) \text{ for } (x, \tilde{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus D \\ (iv) &\quad G_2(x, \tilde{x}) = G_2(\tilde{x}, x) = 0 \quad \text{for } x \in B_R(0),\; \tilde{x} \in \partial B_R(0). \end{aligned} \tag{4.15}$$

It's Laplace expansion is given by (4.13).

In order to treat an inhomogeneous Dirichlet boundary value problem

$$\Delta_2 u = f, \quad u|_{\partial B_R} = g$$

we have to supplement the parametrix (4.14) by a Poisson operator [12]

$$\mathcal{K}\, g = \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^\ell} P_\ell g \quad \text{with } g \in L_2(\partial B_R),$$

where $\mathcal{K}\, g$ is a harmonic function on $B_R(0)$ which satisfies the boundary condition $u|_{\partial B_R} = g$. Putting things together, we obtain the operator

$$\begin{pmatrix} \mathcal{P} + \mathcal{G} + G_\partial & \mathcal{K} \end{pmatrix} \;:\; \begin{pmatrix} L_2(B_R) \\ L_2(\partial B_R) \end{pmatrix} \quad \longrightarrow \quad H^2(B_R).$$

4.2 Classical Green's functions for the bi-Laplace operator

The bi-Laplace operator $\Delta_2\Delta_2$, cf. (4.1) plays a prominent role in the two dimensional Kirchhoff-Love and Reissner-Mindlin plate theories. In polar coordinates, the biharmonic operator is given by

$$\begin{aligned} \tilde{\Delta}_2\tilde{\Delta}_2 \;\; := \;\; & \frac{1}{r^4}\Bigg[\left(-r\frac{\partial}{\partial r}\right)^4 + 4\left(-r\frac{\partial}{\partial r}\right)^3 + 4\left(-r\frac{\partial}{\partial r}\right)^2 + 2\left(-r\frac{\partial}{\partial r}\right)^2 \Delta_{S^1} \\ & + 4\left(-r\frac{\partial}{\partial r}\right)\Delta_{S^1} + 4\Delta_{S^1} + \Delta_{S^1}\Delta_{S^1}\Bigg]. \end{aligned} \tag{4.16}$$

Similar to the Laplace operator, we can represent the bi-Laplace operator in the cone algebra by a Mellin type pseudo-differential operator, i.e.,

$$\tilde{\Delta}_2\tilde{\Delta}_2 u = r^{-4}\,\mathrm{op}_M^{\gamma-\frac{1}{2}}(b)u$$

for $u \in \tilde{\mathcal{D}}(\mathcal{C}^n)$, with operator valued Mellin symbol

$$b(w) = w^4 + 4w^3 + 4w^2 + 2\big(w^2 + 2w + 2\big)\Delta_{S^1} + \Delta_{S^1}\Delta_{S^1}.$$

For the parametrix, we choose the ansatz

$$\mathcal{P}\, u = r^4 \operatorname{op}_M^{\gamma-\frac{9}{2}}(b^{(-1)})u$$

and consider the operator product

$$\begin{aligned}\mathcal{P}\big(\tilde{\Delta}_2\tilde{\Delta}_2\big) &= r^4 \operatorname{op}_M^{\gamma-\frac{9}{2}}(b^{(-1)}) r^{-4} \operatorname{op}_M^{\gamma-\frac{1}{2}}(b) \\ &= \operatorname{op}_M^{\gamma-\frac{1}{2}}(T^4 b^{(-1)}) \operatorname{op}_M^{\gamma-\frac{1}{2}}(b) \\ &= \operatorname{op}_M^{\gamma-\frac{1}{2}}(T^4 b^{(-1)} b).\end{aligned}$$

The operator valued symbol of the parametrix has to satisfy the equation

$$T^4 b^{(-1)}(w) b(w) = 1$$

which can be solved for

$$\begin{aligned}b^{(-1)}(w) &= \frac{1}{b(w-4)} \\ &= \frac{1}{(w-4)^2(w-2)^2 + 2(w^2-6w+10)\Delta_{S^1} + \Delta_{S^1}\Delta_{S^1}} \\ &= \sum_{\ell=0}^{\infty} \frac{P_\ell}{(w-4)^2(w-2)^2 - 2(w^2-6w+10)\ell^2 + \ell^4} \\ &= \sum_{\ell=0}^{\infty} \underbrace{\frac{P_\ell}{(w-\ell-4)(w+\ell-4)(w-\ell-2)(w+\ell-2)}}_{=:b_\ell^{(-1)}(w)},\end{aligned} \tag{4.17}$$

where we applied a spectral resolution with projection operators (4.6). The term $\ell = 0$ in (4.17) has two poles of order 2 at $w_1 = 2$ and $w_2 = 4$, respectively and the term $\ell = 1$ has a pole of order 2 at $w_1 = 3$. Otherwise all the poles in the sum (4.17) are simple. For $\ell = 0$ the residuum at $w_1 = 2$ is given by

$$\begin{aligned}\operatorname{Res}_{w=2}\left(\frac{r}{\tilde{r}}\right)^{-w} b_0^{(-1)}(w) &= \frac{d}{dw}\left[(w-2)^2\left(\frac{r}{\tilde{r}}\right)^{-w} b_0^{(-1)}(w)\right]_{w=2} \\ &= \frac{1}{4}\left(\frac{r}{\tilde{r}}\right)^{-2} P_0\left[1-\ln\left(\frac{r}{\tilde{r}}\right)\right]\end{aligned}$$

and at $w_2 = 4$ it is given by

$$\begin{aligned}\operatorname{Res}_{w=4}\left(\frac{r}{\tilde{r}}\right)^{-w} b_0^{(-1)}(w) &= \frac{d}{dw}\left[(w-4)^2\left(\frac{r}{\tilde{r}}\right)^{-w} b_0^{(-1)}(w)\right]_{w=4} \\ &= -\frac{1}{4}\left(\frac{r}{\tilde{r}}\right)^{-4} P_0\left[1+\ln\left(\frac{r}{\tilde{r}}\right)\right]\end{aligned}$$

For $\ell = 1$ the residuum at $w_1 = 3$ is given by

$$\begin{aligned}\operatorname{Res}_{w=3}\left(\frac{r}{\tilde{r}}\right)^{-w} b_1^{(-1)}(w) &= \frac{d}{dw}\left[(w-3)^2\left(\frac{r}{\tilde{r}}\right)^{-w} b_1^{(-1)}(w)\right]_{w=3} \\ &= \frac{1}{4}\left(\frac{r}{\tilde{r}}\right)^{-3} P_1 \ln\left(\frac{r}{\tilde{r}}\right)\end{aligned}$$

Let $u \in \tilde{\mathcal{D}}(\mathcal{C}^n)$, the action of the parametrix $\mathcal{P}$ is given by the double integral

$$r^4 \operatorname{op}_M^{\gamma-\frac{9}{2}}(b^{(-1)})u = r^4 \int_{\mathbb{R}} \int_0^\infty \left(\frac{r}{\tilde{r}}\right)^{-(5-\gamma+i\rho)} b^{(-1)}(5-\gamma+i\rho)\, u(\tilde{r},\phi)\, \frac{d\tilde{r}}{\tilde{r}} đ\rho. \tag{4.18}$$

For the bi-Laplace operator, we consider the intervals $1 < \gamma < 2$ and $2 < \gamma < 3$ as natural choice. Like before, one can apply Cauchy's residue theorem to the spectral resolution (4.17) of the operator valued symbol. The calculations yield

$$\begin{aligned} r^4 \operatorname{op}_M^{\gamma-\frac{9}{2}}(b_0^{(-1)})u &= \int_0^r \frac{1}{4}\tilde{r}^2 \left[1 + \ln\left(\frac{r}{\tilde{r}}\right)\right] (P_0 u)(\tilde{r})\, \tilde{r} d\tilde{r} \\ &\quad + \int_r^\infty \frac{1}{4} r^2 \left[1 - \ln\left(\frac{r}{\tilde{r}}\right)\right] (P_0 u)(\tilde{r})\, \tilde{r} d\tilde{r} \\ &= \int_0^\infty \frac{1}{4} r_<^2 \left[1 - \ln\left(\frac{r_<}{r_>}\right)\right] (P_0 u)(\tilde{r})\, \tilde{r} d\tilde{r} \end{aligned}$$

$$r^4 \operatorname{op}_M^{\gamma-\frac{9}{2}}(b_1^{(-1)})u = \begin{cases} -\int_0^\infty \frac{1}{16}\frac{r_<^3}{r_>} (P_1 u)(\tilde{r},\tilde{\phi})\, \tilde{r} d\tilde{r} + \int_r^\infty \frac{1}{4} r\tilde{r} \ln\left(\frac{r}{\tilde{r}}\right) (P_1 u)(\tilde{r},\tilde{\phi})\, \tilde{r} d\tilde{r} & \text{for } 1 < \gamma < 2 \\ -\int_0^\infty \frac{1}{16}\frac{r_<^3}{r_>} (P_1 u)(\tilde{r},\tilde{\phi})\, \tilde{r} d\tilde{r} - \int_0^r \frac{1}{4} r\tilde{r} \ln\left(\frac{r}{\tilde{r}}\right) (P_1 u)(\tilde{r},\tilde{\phi})\, \tilde{r} d\tilde{r} & \text{for } 2 < \gamma < 3 \end{cases}$$

and for $\ell \geq 2$ one gets

$$r^4 \operatorname{op}_M^{\gamma-\frac{9}{2}}(b_\ell^{(-1)})u = \int_0^\infty \frac{1}{8}\left(\frac{r_<}{r_>}\right)^\ell \left(\frac{r_>^2}{\ell(\ell-1)} - \frac{r_<^2}{\ell(\ell+1)}\right) (P_\ell u)(\tilde{r},\tilde{\phi})\, \tilde{r} d\tilde{r}.$$

Putting things together, one gets for

$$\begin{aligned} K_4(r,\phi|\tilde{r},\tilde{\phi}) &= \frac{1}{4} r_<^2 \left[1 - \ln\left(\frac{r_<}{r_>}\right)\right] p_0 + \left[\frac{1}{4}\Theta(\tilde{r}-r) r\tilde{r} \ln\left(\frac{r}{\tilde{r}}\right) - \frac{1}{16}\frac{r_<^3}{r_>}\right] p_1 \\ &\quad + \frac{1}{8}\sum_{\ell=2}^\infty \left(\frac{r_<}{r_>}\right)^\ell \left(\frac{r_>^2}{\ell(\ell-1)} - \frac{r_<^2}{\ell(\ell+1)}\right) p_\ell \quad \text{for } 1 < \gamma < 2, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} K_4(r,\phi|\tilde{r},\tilde{\phi}) &= \frac{1}{4} r_<^2 \left[1 - \ln\left(\frac{r_<}{r_>}\right)\right] p_0 - \left[\frac{1}{4}\Theta(r-\tilde{r}) r\tilde{r} \ln\left(\frac{r}{\tilde{r}}\right) + \frac{1}{16}\frac{r_<^3}{r_>}\right] p_1 \\ &\quad + \frac{1}{8}\sum_{\ell=2}^\infty \left(\frac{r_<}{r_>}\right)^\ell \left(\frac{r_>^2}{\ell(\ell-1)} - \frac{r_<^2}{\ell(\ell+1)}\right) p_\ell \quad \text{for } 2 < \gamma < 3, \end{aligned} \tag{4.20}$$

For comparison, we have calculated the generalized Laplace expansion for the Green's function G_4 of the bi-Laplace operator in Appendix A, cf. Eq. (A.3), in a conventional manner. It can be seen that the expansions (4.19), (4.20) and (A.3) agree for $\ell \geq 2$. The first term, corresponding to $\ell = 0$ is the same for (4.19) and (4.20), whereas the second term differs, depending on the choice of γ, i.e.,

$$G_4(r,\phi|\tilde{r},\tilde{\phi}) - K_4(r,\phi|\tilde{r},\tilde{\phi}) = \tfrac{1}{4}\big(r^2\ln(r) + \tilde{r}^2\ln(\tilde{r})\big)p_0 - \big(\tfrac{1}{4}\ln(r) + \tfrac{1}{8}\big)r\tilde{r}p_1, \quad \text{for } 1 < \gamma < 2, \tag{4.21}$$

$$G_4(r,\phi|\tilde{r},\tilde{\phi}) - K_4(r,\phi|\tilde{r},\tilde{\phi}) = \tfrac{1}{4}\big(r^2\ln(r) + \tilde{r}^2\ln(\tilde{r})\big)p_0 - \big(\tfrac{1}{4}\ln(\tilde{r}) + \tfrac{1}{8}\big)r\tilde{r}p_1, \quad \text{for } 2 < \gamma < 3, \tag{4.22}$$

In Appendix C, we performed some explicit calculations in order to reveal the properties of the corresponding distributions. For $1 < \gamma < 2$, we getcontinuationGreen

$$\Delta_2\Delta_2\, \mathrm{Pf.}\, K_4(\cdot,\tilde{x}) = \delta(\cdot - \tilde{x}) - \delta + \mathcal{S}_{\tilde{x}}, \tag{4.23}$$

with non-regular distribution

$$\mathcal{S}_{\tilde{x}}(u) := -\lim_{x\to 0}\left(\frac{P_1 u(x)}{|x|}\right)\frac{1}{\sqrt{\pi}}\tilde{x}$$

For $2<\gamma<3$, we get

$$\Delta_2\Delta_2\,\mathrm{Pf.}\,K_4(\cdot,\tilde{x}) = \delta(\cdot-\tilde{x})-\delta, \tag{4.24}$$

Following the same line of arguments as for the Laplace operator discussed before, we can repair the kernel for $2<\gamma<3$ by adding a fundamental solution in order to obtain a Green's function for the bi-Laplace operator. For the bi-Laplace operator, we have

$$\omega r^{-4}\,\mathrm{op}_M^{\gamma-\frac{1}{2}}(b)\tilde{\omega}\;:\;\mathcal{H}^{s,\gamma}\big(\mathcal{C}^2\big)\;\to\;\mathcal{H}^{s-4,\gamma-4}\big(\mathcal{C}^2\big),$$

which is continuous for all $s\in\mathbb{R}$ and $\gamma\in\mathbb{R}$, where $\omega,\tilde{\omega}$ denote arbitrary cut-off functions. Let u satisfy $\tilde{\Delta}\tilde{\Delta}u=0$ on $\mathcal{C}^2$, and $P_0u=u$, such that $\mathrm{Pf.}\,u$ represents a fundamental solution of $\Delta\Delta$. It satisfies

$$\omega r^{-4}\,\mathrm{op}_M^{\gamma-\frac{1}{2}}(b)\tilde{\omega}u = 0\quad\text{and}\quad r^{-4}\,\mathrm{op}_M^{\gamma-\frac{1}{2}}(b)\tilde{\omega}u=(1-\omega)g$$

with $\omega\prec\tilde{\omega}$ and $g\in\mathcal{H}^{s-4,\gamma-4}\big(\mathcal{C}^2\big)$ for some $s\in\mathbb{R}$ and $\gamma\in\mathbb{R}$. Acting with the parametrix from the left, we get

$$\tilde{\omega}u=\mathcal{P}(1-\omega)g$$

For r sufficiently small, i.e., $\omega(r)=1$, we get from (4.20),

$$\begin{aligned} u(r) &= \mathcal{P}(1-\omega)g(r)\\ &= \int_{S^1}\int_0^\infty K_2(r,\phi|\tilde{r},\tilde{\phi})\big(1-\omega(\tilde{r})\big)g(\tilde{r})\tilde{r}d\tilde{r}d\tilde{\phi}\\ &= \tfrac{1}{4}r^2\big(1-\ln(r)\big)\underbrace{\int_0^\infty\big(1-\omega(\tilde{r})\big)g(\tilde{r})\tilde{r}d\tilde{r}}_{=:C_1}+\tfrac{1}{4}r^2\underbrace{\int_0^\infty \ln(\tilde{r})\big(1-\omega(\tilde{r})\big)g(\tilde{r})\tilde{r}d\tilde{r}}_{C_2}. \end{aligned} \tag{4.25}$$

Like for the Laplace operator, a calulation gives

$$\Delta_2\Delta_2\,\mathrm{Pf.}\,u = -\tfrac{C_1}{p_0}\,\delta, \tag{4.26}$$

which shows also in this case that u is a constant multiple of a fundamental solution of $\Delta_2\Delta_2$. The $\frac{1}{4}(C_1+C_2)r^2$ term in (4.25) represents a smooth function which belong to the kernel of the bi-Laplace operator and is irrelevant for a fundamental solution. It is only the $-\frac{1}{4}C_1r^2\ln(r)$ term which is required to correct the kernel function (4.24). According to (4.26) we have to add a counterterm $\frac{1}{4}r^2\ln(r)p_0$ to (4.20) which would yield, however, a non symmetric kernel function, therefore its better to add the symmetric expression

$$\tfrac{1}{4}\big(r^2\ln(r)+\tilde{r}^2\ln(\tilde{r})\big)p_0$$

where the second term maps onto a constant function and is irrelevant. Furthermore, we observe that the p_1 term in (4.20) is not symmetric with respect to a permutation of r and $\tilde{r}$. In order to restore this symmetry of the kernel, we have to add another correction term $A(r,\tilde{r})$ which has to satisfy

$$-\tfrac{1}{4}\Theta(r-\tilde{r})r\tilde{r}\ln\left(\frac{r}{\tilde{r}}\right)+A(r,\tilde{r}) = -\tfrac{1}{4}\Theta(\tilde{r}-r)r\tilde{r}\ln\left(\frac{\tilde{r}}{r}\right)+A(\tilde{r},r)$$

or equivalently

$$A(r,\tilde{r})-A(\tilde{r},r)=\tfrac{1}{4}r\tilde{r}\ln(r)-\tfrac{1}{4}r\tilde{r}\ln(\tilde{r}).$$

Among two possible choices for $A(r,\tilde{r})$, these are $-\frac{1}{4}r\tilde{r}\ln(\tilde{r})$ and $\frac{1}{4}r\tilde{r}\ln(r)$, it is only the first one which maps into a smooth function in the kernel of the bi-Laplace operator. According to Eq. (4.24) and the preceding discussion, we can therefore obtain al classical Green's function, for $2<\gamma<3$, by adding the Green operators

$$\mathcal{G}_0\, u = \int_0^\infty \tfrac{1}{4}\big(r^2\ln(r)+\tilde{r}^2\ln(\tilde{r})\big)P_0 u\,\tilde{r}d\tilde{r},$$

$$\mathcal{G}_1\, u = -\int_0^\infty \tfrac{1}{4}r\tilde{r}\ln(\tilde{r})P_1 u\,\tilde{r}^2 d\tilde{r},$$

whereby the kernel function of $\mathcal{P}+\mathcal{G}_0+\mathcal{G}_1$ becomes[9]

$$\begin{aligned}\tilde{K}_4(r,\phi|\tilde{r},\tilde{\phi}) &= \left(\frac{1}{4}\big(r_>^2+r_<^2\big)\ln(r_>)+\frac{1}{4}r_<^2\right)p_0-\left[\frac{1}{4}r_<r_>\ln(r_>)+\frac{1}{16}\frac{r_<^3}{r_>}\right]p_1\\ &\quad+\frac{1}{8}\sum_{\ell=2}^\infty\left(\frac{r_<}{r_>}\right)^\ell\left(\frac{r_>^2}{\ell(\ell-1)}-\frac{r_<^2}{\ell(\ell+1)}\right)p_\ell. \end{aligned} \tag{4.27}$$

Taking the limit $\lim_{\tilde{r}\to 0}$ in (4.27), one recovers the fundamental solution $\frac{1}{4}r^2\ln(r)p_0$, which means that we eventually obtained a Green's function $G_4(x,\tilde{x})$ of the bi-Laplace operator in a separable form given by a generalized Laplace expansion, i.e.,

$$G_4(\cdot,\tilde{x}) := \begin{cases} \mathrm{Pf.}\,\tilde{K}_4(\cdot|\tilde{r},\tilde{\phi}) & \text{for } \tilde{x}=\varphi(\tilde{r},\tilde{\phi})\neq 0 \\ \lim_{\tilde{r}\to 0}\mathrm{Pf.}\,\tilde{K}_4(\cdot|\tilde{r},\tilde{\phi}) & \text{for } \tilde{x}=0\end{cases}.$$

Furthermore, we can impose homogeneous Dirichlet boundary conditions

$$\Delta_2\Delta_2 u = f,\quad u|_{\partial B_R}=0,$$

on an open ball $B_R(0)$ of radius R, by the method of images. By substituting $r\to R$ and $\tilde{r}\to\frac{r\tilde{r}}{R}$ in (4.27), the image kernel function is given by

$$\begin{aligned}I_4(r,\phi|\tilde{r},\tilde{\phi}) &= \left[\frac{1}{4}\left(\frac{r_<r_>}{R}\right)^2\big(\ln(R)+1\big)+\frac{1}{4}R^2\ln(R)\right]p_0-\left[\frac{1}{4}r_<r_>\ln(R)+\frac{1}{16}\left(\frac{r_<r_>}{R}\right)^3\frac{1}{R}\right]p_1(\phi,\tilde{\phi})\\ &\quad+\frac{1}{8}\sum_{\ell=2}^\infty\left(\frac{r_<r_>}{R^2}\right)^\ell\left(\frac{R^2}{\ell(\ell-1)}-\left(\frac{r_<r_>}{R}\right)^2\frac{1}{\ell(\ell+1)}\right)p_\ell(\phi,\tilde{\phi}).\end{aligned}$$

such that the kernel function

$$\hat{K}_4(r,\phi|\tilde{r},\tilde{\phi}) := \tilde{K}_4(r,\phi|\tilde{r},\tilde{\phi})-I_4(r,\phi|\tilde{r},\tilde{\phi})$$

satisfies

$$\hat{K}_4(r,\phi|R,\tilde{\phi})=\hat{K}_4(R,\phi|\tilde{r},\tilde{\phi})=0.$$

Additionally, we want to impose Neumann boundary conditions, such that the radial derivative of the kernel function vanishes. Taking the radial derivative of $\hat{K}_4(r,\phi|\tilde{r},\tilde{\phi})$ at $r=R$, we get

$$\partial_r\hat{K}_4(r,\phi|\tilde{r},\tilde{\phi})\Big|_{r=R}=\frac{1}{4R}\big(R^2-\tilde{r}^2\big)\big[1+\ln\big(R^2\big)\big]p_0-\big(R^2-\tilde{r}^2\big)\sum_{\ell=1}^\infty\frac{1}{4\ell R^{\ell+1}}\tilde{r}^\ell p_\ell(\phi,\tilde{\phi}).$$

[9] Comparison with (4.22) reveals, that this kernel function agrees with the Green's function G_4 up to the term $-\frac{1}{8}r_<r_>p_1$, which however is irrelevant, because it maps into a smooth function in the kernel of the bi-Laplace operator.

From this expression and taking into account the permutational symmetry of the kernel function with respect to r and $\tilde{r}$ one gets the term[10]

$$H_4(r,\phi|\tilde{r},\tilde{\phi}) := \frac{1}{8R^2}(R^2-r^2)(R^2-\tilde{r}^2)\big[1+\ln\big(R^2\big)\big]p_0 - \frac{1}{8R^2}(R^2-r^2)(R^2-\tilde{r}^2)\sum_{\ell=1}^{\infty}\Big(\frac{r_< r_>}{R^2}\Big)^{\ell}\frac{1}{\ell}p_\ell(\phi,\tilde{\phi}),$$

which has to be added to $\hat{K}(r,\phi|\tilde{r},\tilde{\phi})$ and eventually derive an kernel function

$$\check{K}_4(r,\phi|\tilde{r},\tilde{\phi}) := \hat{K}_4(r,\phi|\tilde{r},\tilde{\phi}) - I_4(r,\phi|\tilde{r},\tilde{\phi}) + H_4(r,\phi|\tilde{r},\tilde{\phi})$$

which satisfies

$$\check{K}_4(r,\phi|R,\tilde{\phi}) = \check{K}_4(R,\phi|\tilde{r},\tilde{\phi}) = 0 \text{ and } \partial_r\check{K}_4(r,\phi|\tilde{r},\tilde{\phi})\Big|_{r=R} = \partial_{\tilde{r}}\hat{K}_4(r,\phi|\tilde{r},\tilde{\phi})\Big|_{\tilde{r}=R} = 0.$$

and represents the unique Green's function for the bi-Laplace operator with Dirichlet and Neumann boundary conditions on $B(R)$.

The previous calculations for the Laplace and bi-Laplace operator demonstrate the feasibility of our approach to obtain classical Green's functions from a pseudo-differential calculus on the cone. However, in contrast to the cases in dimensions ≥ 3 considered in [10], our approach this time required some additional fine-tuning concerning the kernel functions of the parametrices. The reason behind are poles of order > 1 in the spectral decomposition of the operator valued symbols of the parametrices (4.5) and (4.17). Our treatment of such poles, i.e. by adding appropriate Green operators, could be performed within the pseudo-differential calculus, but nevertheless it asks for a better conceptual understanding. Actually, it demands for a theory which identifies relationships between classical Green's functions and pseudo-differential parametrices in a more general setting and not only by means of concrete examples.

5 Numerical examples for the Laplace operator

In order to demonstrate possible improvements due to our approach in numerical calculations, we consider the Laplace operator and a boundary value problem of mixed type a) and b) on the bounded domain $\Omega := (0,1)\times(0,1)\subset\mathbb{R}^2$, i.e.,

$$\Delta_2 u = \kappa + \sum_{i=1}^{m}\delta(\cdot-\xi_i), \quad u|_{\partial\Omega} = g$$

with $\xi_i\in\Omega$ and $\kappa\in\mathbb{R}$. In our examples, we take the ansatz

$$\tilde{u}^{(0)} = \frac{1}{2\pi}\sum_{i=1}^{m}\ln|\cdot-\xi_i|,$$

as a global defect correction (2.14) in Ω.

In the first example we want to consider a boundary value problem where an exact solution is known. For this we choose $\kappa=4$, $\xi_1=(0.2,0.5)$, $\xi_2=(0.8,0.5)$ and the artificial boundary condition

$$g(x) = \big(\tilde{u}^{(0)}(x)+1+|x|^2\big)|_{\partial\Omega}, \tag{5.1}$$

such that the exact solution becomes $u^*(x)=\tilde{u}^{(0)}(x)+1+|x|^2$. In Fig. 1, we compare the relative errors with respect to u^* of the standard FEM solution and our improved defect corrected FEM

[10]Obviously, H_2 is consistent with homogeneous Dirichlet boundary conditions. It can be easily seen, that functions of the type $r^{\ell}p_\ell(\phi,\tilde{\phi})$ and $r^{2+\ell}p_\ell(\phi,\tilde{\phi})$ belong to the kernel of the bi-Laplace operator. Therefore H_2 can be added to the kernel function $\hat{K}$ without further complications.

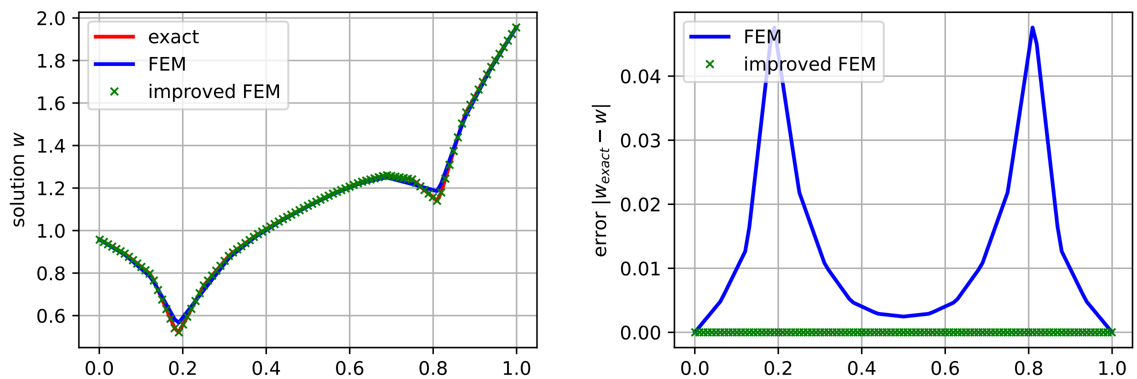

Figure 1: (left figure): Comparison of the standard FEM solution, improved defect corrected FEM solution and exact solution for the boundary value problem (5.1), with $\kappa = 4$, $\xi_1 = (0.2, 0.5)$, $\xi_2 = (0.8, 0.5)$, along a line parallel to the x-axes at $y = 0.5$. (right figure) Relative error with respect to the exact solution of the standard FEM solution and improved defect corrected FEM solution.

solution. It can be seen, that along a line parallel to the x-axes at $y = 0.5$, the errors at the singular points $\xi_{1,2}$ are significantly reduced.

Our second example corresponds to the first example with homogeneous Dirichlet boundary conditions. The difference between the standard FEM solution and our improved defect corrected FEM solution along a line parallel to the x-axes at $y = 0.5$ is shown in Fig. 2. A significant deviation between both solutions at the singular points $\xi_{1,2}$ can be clearly recognized.

6 Conclusions and outlook

Taking into account the asymptotic behaviour at singularities in numerical simulations provides an alternative to adaptive refinement schemes which require a posteriori error estimates and corresponding modifications of the underlying global grid. In particular for point-like isolated singularities one might not be willing to change the global grid and modify the corresponding underlying data structures of the numerical algorithm. The LDC approach offers an attractive alternative in such a case either by a global subtraction of the singular part of the solution or by a local subtraction with additional grid refinement in the vicinity of the singularity.

Within the present work, we tried to tackle the basic obstacle of such kind of approach which consists of a lack of knowledge concerning the asymptotic behaviour of the exact solution in the vicinity of the singularity. In particular this is true for models where fundamental solutions or Green's functions are required like for the plate theories discussed in the present work. Classical Green's functions are explicitly known in the literature only for a few special cases. The general scheme for the construction of Green's functions, based on methods from singular analysis, outlined in the present work might be a way out of this dilemma. For the models problems under consideration we could demonstrate the feasibility of our approach. This opens the possibility to construct Green's functions for models where explicit expressions for these functions are presently not available. A particularly promising application are plates of variable thickness, where Laplace and bi-Laplace like operators with varialble coefficients appear, see e.g. [3] for further details. Such kind of partial differential operators require an asymptotic parametrix construction [8] which has been already sucessfully applied to shifted Laplace and Schrödinger operators [10] in dimensions ≥ 3. This will be subject of our future work.

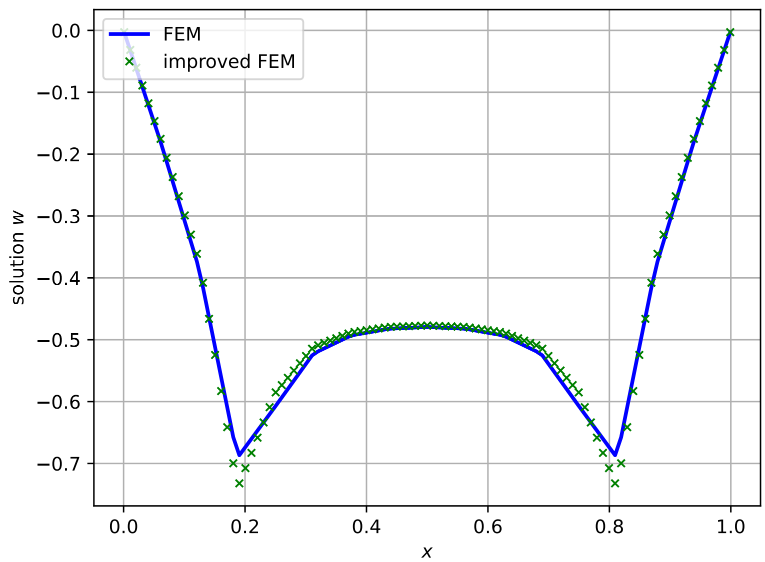

Figure 2: Comparison of the standard FEM solution and improved defect corrected FEM solution for homogeneous Dirichlet boundary conditions, with $\kappa = 4$, $\xi_1 = (0.2, 0.5)$, $\xi_2 = (0.8, 0.5)$, along a line parallel to the x-axes at $y = 0.5$.

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Appendices

A Laplace expansion for the (bi)-Laplace operator

Because of lack of an appropriate reference, we give a brief derivation of the corresponding expansion for the classical Green's function G_2 of the Laplace operator in $\mathbb{R}^2$, i.e.,

$$G_2(x,\tilde{x}) = \tfrac{1}{2\pi}\ln|x-\tilde{x}|$$

let $r_< := \min\{r,\tilde{r}\}$, $r_> := \max\{r,\tilde{r}\}$, $\Delta\phi := \phi - \tilde{\phi}$ and $h := r_</r_>$. Using polar coordinates (r,ϕ), we obtain

$$G_2(x,\tilde{x}) = \tfrac{1}{4\pi}\left[\ln(r_>^2) - 2\ln\big(1+h^2-2h\cos(\Delta\phi)\big)^{-\frac{1}{2}}\right]. \tag{A.1}$$

Next, we use the expansions, cf. [28, p. 303ff],

$$\ln\big(1+h^2-2h\cos(\Delta\phi)\big)^{-\frac{1}{2}} = \ln\big(1+\tfrac{1}{2}he^{i\Delta\phi}+\tfrac{1\cdot 3}{2\cdot 4}h^2e^{i2\Delta\phi}+\cdots\big)+\ln\big(1+\tfrac{1}{2}he^{-i\Delta\phi}+\tfrac{1\cdot 3}{2\cdot 4}h^2e^{-i2\Delta\phi}+\cdots\big)$$

and

$$\ln(1+a) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{a^k}{k}$$

for $a\in\mathbb{C}$ with $|a|<1$. One gets

$$\ln\big(1+\tfrac{1}{2}he^{i\Delta\phi}+\tfrac{1\cdot 3}{2\cdot 4}h^2e^{i2\Delta\phi}+\cdots\big) = \tfrac{1}{2}he^{i\Delta\phi}+\tfrac{1}{4}h^2e^{i2\Delta\phi}+\tfrac{1}{6}h^3e^{i3\Delta\phi}+\cdots$$

and its complex conjugate, which inserted into (A.1) yields the Laplace expansion

$$G_2(x,\tilde{x}) = \ln(r_>)\,p_0 - \frac{1}{2}\left(\frac{r_<}{r_>}\right)p_1 - \frac{1}{4}\left(\frac{r_<}{r_>}\right)^2 p_2 - \cdots \tag{A.2}$$

The previous caculation can be easily modified in order to obtain the corresponding expansion for the bi-Laplace operator in $\mathbb{R}^2$, where the fundamental solution is given by

$$G_2(x,\tilde{x}) = \tfrac{1}{8\pi}|x-\tilde{x}|^2\ln|x-\tilde{x}|.$$

In polar coordinates, one gets

$$G_2(x,\tilde{x}) = \tfrac{1}{8\pi}r_>^2\big(1+h^2-2h\cos(\Delta\phi)\big)\left[\ln(r_>^2) - 2\ln\big(1+h^2-2h\cos(\Delta\phi)\big)^{-\frac{1}{2}}\right]$$

which yields the generalized Laplace expansion for the bi-Laplace operator

$$\begin{aligned}
G_2(x,\tilde{x}) \;=\;& \left(\frac{1}{4}\big(r_>^2+r_<^2\big)\ln(r_>)+\frac{1}{4}r_<^2\right)p_0 \qquad\qquad (A.3)\\
&-\left(\frac{1}{4}r_{<}r_{>}\ln(r_>)+\frac{1}{8}r_{<}r_{>}+\frac{1}{16}\frac{r_<^3}{r_>}\right)p_1\\
&+\frac{1}{8}\left(\frac{r_<}{r_>}\right)^2\left(\frac{1}{2}r_>^2-\frac{1}{6}r_<^2\right)p_2\\
&+\frac{1}{8}\left(\frac{r_<}{r_>}\right)^3\left(\frac{1}{6}r_>^2-\frac{1}{12}r_<^2\right)p_3\\
&\vdots
\end{aligned}$$

B Analytic continuation of Green's functions

In Section 4.1 we have seen that within our approach, the Laplace operator in 2 dimensions behaves differently from Laplace operators in dimensions ≥ 3. The special status of 2 dimensions can be also seen by considering the analytic continuation of the Green's function G_κ for the shifted Laplace operator $\Delta_2 - \kappa^2$. Taking the limit $\kappa \to 0$ of G_κ in dimensions ≥ 3 one recovers the Green's function of the corresponding Laplace operator, cf. [10]. Due to the non-analyticity of the fundamental solution, with respect to the parameter κ, at $\kappa = 0$ this is not the case for the Laplace operator in 2 dimensions. However as we will see in the following, this non-analyticity only affects the $\ell = 0$ term in the spectral resolution (4.3).

A fundamental solution of the shifted Laplace operator $\Delta - \kappa^2$, see e.g. Schwartz [25][Section II, §3], is given by

$$u_\kappa = \mathrm{Pf.}\left[-(2\pi)^{-1} K_0(\kappa r)\right], \tag{B.1}$$

where K_0 denotes the modified Bessel function of the second kind and the corresponding Green's function is given by

$$G_\kappa(x, \tilde{x}) = -(2\pi)^{-1} K_0(\kappa |x - \tilde{x}|).$$

We have, cf. [1],

$$K_0(\kappa r) = -\left[\ln(\tfrac{1}{2}\kappa r) + \gamma\right] I_0(\kappa r) + Q_0(\kappa r)$$

with γ the Euler constant and power series

$$I_0(z) = 1 + \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{\left(\frac{1}{4}z^2\right)^2}{(2!)^2} + \cdots,$$

$$Q_0(z) = \frac{\frac{1}{4}z^2}{(1!)^2} + \left(1 + \tfrac{1}{2}\right)\frac{\left(\frac{1}{4}z^2\right)^2}{(2!)^2} + \cdots$$

Due to the logarithm, the limit $\kappa \to 0$ doesn't exist. To gain some insight into this divergent behaviour, it is instructive to consider the generalized Laplace expansion in the shifted case. For this, we apply Mehler's formula, cf. [28][p. 383],

$$K_0(z) = \int_0^\infty \frac{t J_0(tz)}{1 + t^2}\, dt$$

with $\kappa > 0$ (J_0 Bessel functions of the first kind), and get

$$K_0(\kappa z) = \int_0^\infty \frac{s J_0(sz)}{\kappa^2 + s^2}\, ds.$$

Let us take $z = |x - \tilde{x}| = \sqrt{r^2 + \tilde{r}^2 - 2r\tilde{r}\cos(\theta)}$ and apply the expansion, cf. [28][p. 380], to get

$$J_0(sz) = J_0(sr)J_0(s\tilde{r}) + 2\sum_{n=1}^\infty J_n(sr)J_n(s\tilde{r})\cos(n\theta),$$

here J_n, $n = 1, 2, \ldots$ denote Bessel functions of the first kind. We get the generalized Laplace expansion of the Green's function

$$\begin{aligned} G_\kappa(x, \tilde{x}) &= \mathrm{Pf.}\left[-K_0(\kappa z)\right] \\ &= \mathrm{Pf.}\left[\int_0^\infty \frac{s J_0(sr)J_0(s\tilde{r})}{\kappa^2 + s^2}\, ds + 2\sum_{n=1}^\infty \int_0^\infty \frac{s J_n(sr)J_n(s\tilde{r})\cos(n\theta)}{\kappa^2 + s^2}\, ds\right]. \end{aligned} \tag{B.2}$$

Taking into account, $J_0(0) = 1$ and $J_n(z) \sim \left(\frac{1}{2}z\right)^n / \Gamma(n+1)$ for $z \to 0$ and

$$J_n(z) = \sqrt{\frac{2}{\pi z}} \cos(z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi) + \mathcal{O}(z^{-1}) \text{ for } z \to \infty$$

it can be seen that it is only the first integral, corresponding to $n = 0$, which diverges in the limit $\kappa \to 0$. Therefore we get

$$\lim_{\kappa\to 0} P_n G_\kappa(x,\tilde{x}) = \text{Pf.} \int_0^\infty \frac{J_n(sr)J_n(s\tilde{r})}{s}\, ds \cos(n\theta), \text{ for } n \geq 1$$

The remaining indefinite integral is of Weber and Schafheitlin type, cf. [27][Chap. XIII, Section 13.4], and yields

$$\begin{aligned} \int_0^\infty \frac{J_n(sr)J_n(s\tilde{r})}{s}\, ds &= \frac{1}{2n}\left(\frac{r_<}{r_>}\right)^n {}_2\mathrm{F}_1\big(n,0,n+1,\left(\tfrac{r_<}{r_>}\right)^2\big) \\ &= \frac{1}{2n}\left(\frac{r_<}{r_>}\right)^n \end{aligned}$$

were we used [1][15.2.1]. This shows that terms with $n \geq 1$ agree in the limit $\kappa \to 0$ with the corresponding terms in the expansion of the Laplace operator (4.12).

C Some explicit calculations

In the following we want to calculate integrals with respect to distributional derivatives of kernel pseudofunctions in $\mathbb{R}^2$. The kernels are given in terms of a spectral resolution with respect to eigenfunctions of Δ_{S^1}. Therefore it is necessary to consider appropriate subspaces of test functions

$$\mathcal{D}_\ell(\mathbb{R}^2) := \{u \in \mathcal{D}(\mathbb{R}^2) \mid P_\ell u = u\}$$

Proposition 2. *A test function $w \in \mathcal{D}_\ell(\mathbb{R}^2)$ can be represented in an open ball B_R of radius R, centred at the origin, by a Taylor approximation*

$$w(x) = \sum_{k=0}^{q} \frac{c_k}{\sqrt{2\pi}} r^{2k} + \sum_{|\beta|=2q+1} \mathcal{R}_\beta(x)x^\beta \quad \textit{for } \ell = 0, \tag{C.1}$$

and

$$w(x) = \sum_{k=0}^{q} r^{2k}\big(c_k^{(1)} r^\ell y^{(1)}(\ell\phi) + c_k^{(2)} r^\ell y^{(2)}(\ell\phi)\big) + \sum_{|\beta|=2q+\ell+1} \mathcal{R}_\beta(x)x^\beta \quad \textit{for } \ell \geq 1, \tag{C.2}$$

with $y^{(1)}(\cdot) := \frac{1}{\sqrt{\pi}}\cos(\cdot)$, $y^{(2)}(\cdot) := \frac{1}{\sqrt{\pi}}\sin(\cdot)$ and $\mathcal{R}_\beta \in C^\infty(B_R,\mathbb{R})$.

Proof. We consider only the case $\ell \geq 1$, the case $\ell = 0$ follows analogously. A function $w \in \mathcal{D}(\mathbb{R}^2)$ can be represented in B_R by a Taylor approximation

$$w(x) = \sum_{|\alpha|\leq 2q+\ell} \frac{\partial^\alpha w(0)}{\alpha!} x^\alpha + \sum_{|\beta|=2q+\ell+1} \tilde{\mathcal{R}}_\beta(x)x^\beta,$$

with $\mathcal{R}_\beta \in C^\infty(\mathbb{R}^2,\mathbb{R})$. Introducing polar coordinates $x = \big(r\cos(\phi), r\sin(\phi)\big)$, it can be converted into

$$w(x) = \sum_{k=0}^{2q+\ell}\left[\frac{c_{k,0}}{\sqrt{2\pi}} r^k + \sum_{m=1}^{k} r^{k-m}\big(c_{k,m}^{(1)} r^m y^{(1)}(m\phi) + c_{k,m}^{(2)} r^m y^{(2)}(m\phi)\big)\right] + \sum_{|\beta|=2q+\ell+1} \tilde{\mathcal{R}}_\beta(x)x^\beta, \tag{C.3}$$

where the coefficients $c_{k,0}$ are non zero only for k even, and similarly $c^{(1)}_{k,m}$, $c^{(2)}_{k,m}$ are non zero only for $k-m$ even, which follows from the smoothness of the Taylor polynomial. For $w \in \mathcal{D}_\ell(\mathbb{R}^2)$, we have to impose the additional constraint $P_\ell w = w$, which applied to (C.3) yields

$$w(x) = \sum_{k=\ell}^{2q+\ell} r^{k-\ell}\big(c^{(1)}_{k,\ell} r^\ell y^{(1)}(\ell\phi) + c^{(2)}_{k,\ell} r^\ell y^{(2)}(\ell\phi)\big) + \sum_{|\beta|=2q+\ell+1} P_\ell\big(\tilde{\mathcal{R}}_\beta(x)x^\beta\big),$$

with $c^{(1)}_{k,\ell}$, $c^{(2)}_{k,\ell}$ non zero only for $k-\ell$ even. The latter expression is obviously equivalent to (C.2). $\square$

Remark 1. *When we consider for $w \in \mathcal{D}_\ell(\mathbb{R}^2)$, $\ell = 0, 1, \ldots,$ expressions of the form $\Delta_2 w$ or $\Delta_2\Delta_2 w$ in integrals, we can replace them, after changing to polar coordinates, by the corresponding expressions $\tilde{\Delta}_2 w$ or $\tilde{\Delta}_2\tilde{\Delta}_2 w$. It follows from Proposition 2 that $\tilde{\Delta}_2 w(r,\phi)$ and $\tilde{\Delta}_2\tilde{\Delta}_2 w(r,\phi)$ are $\mathcal{O}(r^\ell)$.*

C.1 Laplace operator

Let us check Eqs. (4.9) and (4.10), for $\tilde{x} \neq 0$, by an explicit calculation. For this purpose let $u \in \mathcal{D}(\mathbb{R}^2)$ and $P_0 u = u$, i.e., the test functions depend on the radial variable only. Depending on the choice of γ, we get for $\frac{1}{2} < \gamma < 1$

$$\begin{aligned}
\int_{\mathbb{R}^2} u(x)\Delta_x \,\mathrm{Pf.}\, K_2(x,\tilde{x})\,dx &= \int_{\mathbb{R}^2} \big(\Delta_x u(x)\big) K_2(x,\tilde{x})\,dx \\
&= -\int_0^\infty \big(\tfrac{1}{r^2}(-r\partial_r)^2 u(r)\big)\,\Theta(\tilde{r}-r)\ln\left(\frac{r}{\tilde{r}}\right) r dr \\
&= \int_0^{\tilde{r}} \partial_r\big((-r\partial_r)u(r)\big)\ln\left(\frac{r}{\tilde{r}}\right)\,dr \\
&= \int_0^{\tilde{r}} \partial_r u(r)\,dr \\
&= u(\tilde{r}) - u(0)
\end{aligned}$$

and for $1 < \gamma < \frac{3}{2}$

$$\begin{aligned}
\int_{\mathbb{R}^2} u(x)\Delta_x \,\mathrm{Pf.}\, K_2(x,\tilde{x})\,dx &= \int_{\mathbb{R}^2} \big(\Delta_x u(x)\big) K_2(x,\tilde{x})\,dx \\
&= \int_0^\infty \big(\tfrac{1}{r^2}(-r\partial_r)^2 u(r)\big)\,\Theta(r-\tilde{r})\ln\left(\frac{r}{\tilde{r}}\right) r dr \\
&= -\int_0^\infty \partial_r\big((-r\partial_r)u(r)\big)\ln\left(\frac{r}{\tilde{r}}\right)\,dr + \int_0^{\tilde{r}} \partial_r\big((-r\partial_r)u(r)\big)\ln\left(\frac{r}{\tilde{r}}\right)\,dr \\
&= -\ln(\tilde{r})\int_0^\infty \partial_r\big((r\partial_r)u(r)\big)dr + \int_0^\infty \partial_r\big((r\partial_r)u(r)\big)\ln(r)dr + u(\tilde{r}) - u(0) \\
&= u(\tilde{r})
\end{aligned}$$

C.2 Bi-Laplace operator

Let $\hat{u} \in \mathcal{D}(\mathbb{R}^2)_0$, i.e., $P_0\hat{u} = \hat{u}$, and $u(r) := \hat{u}(x)$, for $r = |x|$, with $u \in C^\infty(\mathbb{R}_+, \mathbb{R})$. For $1 < \gamma < 3$, we get

$$
\begin{aligned}
\int_{\mathbb{R}^2} \hat{u}(x)\Delta_x\Delta_x \,\mathrm{Pf.}\, K_2(x,\tilde{x})\, dx &= \int_{\mathbb{R}^2} \big(\Delta_x\Delta_x\hat{u}(x)\big) K_2(x,\tilde{x})\, dx \\
&= \int_0^{\tilde{r}} \left[\frac{1}{r^2}\left(-r\frac{\partial}{\partial r}\right)^2 \frac{1}{r^2}\left(-r\frac{\partial}{\partial r}\right)^2 u(r)\right] \frac{1}{4}r^2 \left[1-\ln\left(\frac{r}{\tilde{r}}\right)\right] rdr \\
&\quad \int_{\tilde{r}}^{\infty} \left[\frac{1}{r^2}\left(-r\frac{\partial}{\partial r}\right)^2 \frac{1}{r^2}\left(-r\frac{\partial}{\partial r}\right)^2 u(r)\right] \frac{1}{4}\tilde{r}^2 \left[1-\ln\left(\frac{\tilde{r}}{r}\right)\right] rdr \\
&= -\int_0^{\tilde{r}} \left[\frac{\partial}{\partial r}\left(-r\frac{\partial}{\partial r}\right) \frac{1}{r^2}\left(-r\frac{\partial}{\partial r}\right)^2 u(r)\right] \frac{1}{4}r^2 \left[1-\ln\left(\frac{r}{\tilde{r}}\right)\right] dr \\
&\quad -\int_{\tilde{r}}^{\infty} \left[\frac{\partial}{\partial r}\left(-r\frac{\partial}{\partial r}\right) \frac{1}{r^2}\left(-r\frac{\partial}{\partial r}\right)^2 u(r)\right] \frac{1}{4}\tilde{r}^2 \left[1-\ln\left(\frac{\tilde{r}}{r}\right)\right] dr \\
&= \int_0^{\tilde{r}} \left[\left(-r\frac{\partial}{\partial r}\right) \frac{1}{r^2}\left(-r\frac{\partial}{\partial r}\right)^2 u(r)\right] \frac{1}{4}r \left[1-2\ln\left(\frac{r}{\tilde{r}}\right)\right] dr \\
&\quad +\int_{\tilde{r}}^{\infty} \left[\left(-r\frac{\partial}{\partial r}\right) \frac{1}{r^2}\left(-r\frac{\partial}{\partial r}\right)^2 u(r)\right] \frac{1}{4}\tilde{r}^2 \left[\frac{1}{r}\right] dr \\
&= \int_0^{\tilde{r}} \left[\frac{1}{r^2}\left(-r\frac{\partial}{\partial r}\right)^2 u(r)\right] \left[-r\ln\left(\frac{r}{\tilde{r}}\right)\right] dr \\
&= \int_0^{\tilde{r}} \left[\frac{\partial}{\partial r}\left(-r\frac{\partial}{\partial r}\right) u(r)\right] \ln\left(\frac{r}{\tilde{r}}\right) dr \\
&= -\int_0^{\tilde{r}} \left[\left(-r\frac{\partial}{\partial r}\right) u(r)\right] \frac{1}{r} dr \\
&= \int_0^{\tilde{r}} \partial_r u(r) dr \\
&= u(\tilde{r}) - u(0) \\
&= \hat{u}(\tilde{x}) - u(0)
\end{aligned}
$$

Let $\hat{u} \in \mathcal{D}(\mathbb{R}^2)_1$, i.e., $P_1\hat{u} = \hat{u}$, and w.l.o.g. assume $\hat{u}(x) = \frac{1}{\sqrt{\pi}}u(r)\cos(\phi)$, with $u \in C^\infty(\mathbb{R}_+, \mathbb{R})$.

For $1 < \gamma < 2$, we get

$$
G_2(x,\tilde{x}) - K_2(x,\tilde{x})|_{\ell=1} = \mathrm{Pf.}\big(-\tfrac{1}{4}\ln(r) - \tfrac{1}{8}\big) r\tilde{r}p_1
$$

which gives

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \hat{u}(x)\Delta_x\Delta_x \,\mathrm{Pf.}\big(G_2(x,\tilde{x}) - K_2(x,\tilde{x})\big)\, dx \\
&= \int_{\mathbb{R}^2} \big(\Delta_x\Delta_x\hat{u}(x)\big)\big(G_2(x,\tilde{x}) - K_2(x,\tilde{x})\big)\, dx \\
&= -\int_0^\infty \left\{ \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r)\right\} \left[\frac{1}{4}\ln(r) + \frac{1}{8}\right] r^2\, dr\, \frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&= \int_0^\infty \left\{ \left(r\frac{\partial}{\partial r}\right) \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r)\right\} \left[\frac{1}{4}\ln(r) + \frac{3}{8}\right] dr\, \frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&\quad + \int_0^\infty \left\{ \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r)\right\} \left[\frac{1}{4}\ln(r) + \frac{1}{8}\right] dr\, \frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&= -\int_0^\infty \left\{ \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r)\right\} \left[\frac{1}{4}\ln(r) + \frac{5}{8}\right] dr\, \frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&\quad + \int_0^\infty \left\{ \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r)\right\} \left[\frac{1}{4}\ln(r) + \frac{1}{8}\right] dr\, \frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&= -\frac{1}{2}\int_0^\infty \left\{ \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r)\right\} dr\, \frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&= -\frac{1}{2}\int_0^\infty \left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\left(\frac{u(r)}{r}\right) + 2\frac{\partial}{\partial r}\left(\frac{u(r)}{r}\right)\right] dr\, \frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&= \lim_{r\to 0}\left(\frac{u(r)}{r}\right)\frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&= \lim_{x\to 0}\left(\frac{\hat{u}(x)}{|x|}\right)\frac{1}{\sqrt{\pi}}\tilde{x}
\end{aligned}
$$

where we have used in the second last line the identity

$$
\frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] w(r) = \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\left(\frac{w(r)}{r}\right) + 2\frac{\partial}{\partial r}\left(\frac{w(r)}{r}\right)
$$

For $2 < \gamma < 3$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \hat{u}(x)\Delta_x\Delta_x \,\mathrm{Pf.}\big(G_2(x,\tilde{x}) - K_2(x,\tilde{x})\big)\,dx \\
&= \int_{\mathbb{R}^2} \big(\Delta_x\Delta_x \hat{u}(x)\big)\big(G_2(x,\tilde{x}) - K_2(x,\tilde{x})\big)\,dx \\
&= -\int_0^\infty \left\{ \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r) \right\} r^2\,dr\ \left[\frac{1}{4}\ln(\tilde{r}) + \frac{1}{8}\right]\frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&= \int_0^\infty \left\{ \left(r\frac{\partial}{\partial r}\right)\frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r) \right\} dr\ \left[\frac{1}{4}\ln(\tilde{r}) + \frac{1}{8}\right]\frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&\quad + \int_0^\infty \left\{ \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r) \right\} dr\ \left[\frac{1}{4}\ln(\tilde{r}) + \frac{1}{8}\right]\frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&= -\int_0^\infty \left\{ \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r) \right\} dr\ \left[\frac{1}{4}\ln(\tilde{r}) + \frac{1}{8}\right]\frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&\quad + \int_0^\infty \left\{ \frac{1}{r^2}\left[\left(-r\frac{\partial}{\partial r}\right)^2 - 1\right] u(r) \right\} dr\ \left[\frac{1}{4}\ln(\tilde{r}) + \frac{1}{8}\right]\frac{1}{\sqrt{\pi}}\tilde{r}\cos(\tilde{\phi}) \\
&= 0
\end{aligned}
$$

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