The Effective Dimension of Asset-Liability Management Problems in Life Insurance*

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1 ALM for participating life insurance policies

Much effort has been spent on the development of stochastic asset-liability management (ALM) models for life insurance policies in the last years, see, e.g., Bacinello (2001), Ballota et al. (2006), Briys, Varenne (1997), De Felice, Moriconi (2005), Gerstner et al. (2007), Grosen, Jorgensen (2000), Miltersen, Persson (2003) and the references therein. Such models are becoming more and more important due to new regulations and a stronger competition. They are employed by insurance companies to simulate the medium- and long-term development of all assets and liabilities of their insurance portfolios. This way, the exposure of the company to financial, mortality and surrender risks can be analysed. The results are used to support management decisions regarding, e.g., the asset allocation, the bonus declaration or the design of new profitable and competitive insurance products. The models are also applied to obtain market-based, fair value accountancy standards as required by Solvency II and the International Financial Reporting Standard.

Here, we consider the ALM model proposed in Gerstner et al. (2007). The main focus of this model is to simulate the temporal development of the most important balance sheet items for a portfolio of participating life insurance policies. Thereby, the time horizon $[0, T]$ is decomposed into M periods $(t_{k-1}, t_k]$, $0 \leq k \leq M$, with equal period length $\Delta t = T/M$. Within this discrete time framework all terms can then be calculated recursively and in a modular way which allows an easy implementation and efficient simulation of the model.

At time t_k, the asset side of the balance sheet is determined by the market value C_k of the assets of the company which are invested in stocks and bonds. The temporal dynamics of the stock prices are modelled by a geometric Brownian motion while the short interest rates are obtained from a one-factor mean-reversion model, the so-called Cox-Ingersoll-Ross (CIR) model which is coupled to the stock price model via a constant correlation factor. This system, which is based on two independent standard

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Brownian motions $W^s(t)$ and $W^r(t)$, is then discretized according to the period length Δt with an explicit Euler-Maruyama discretization yielding discrete stock prices and short interest rates for each period k. The asset allocation in stocks and bonds is dynamic. Here, the goal is to keep a certain percentage of the total capital invested in stocks while the remaining part is invested in a buy-and-hold trading strategy into zero coupon bonds whose market value is derived from the prevailing short interest rate. This way, the portfolio return rate is specified which determines the development of the asset base C_k of the company in each period.

On the liability side, the first item is the book value D_k of the actuarial reserve [1]. It depends on the development of the biometry and on the specific insurance products under consideration. Here, mortality and surrender are assumed to be deterministic and are modelled using experience-based tables. Different life insurance product specifics are incorporated into the model via premium, benefit and surrender characteristics in a fairly general and efficiently implementable framework. The second item on the liability side are the allocated bonuses B_k which constitute the part of the surpluses which have been credited to the policyholders. Their profit participation is determined by a bonus declaration mechanism which is based on the reserve situation of the company as proposed in Grosen, Jorgensen (2000). The next item, the free reserve F_k, is a buffer account for future bonus payments. It consists of surpluses which have not yet been credited to the individual policyholder accounts and is used to smooth capital market oscillations and to achieve a stable and low-volatile return participation of the policyholders. The last item is the company account Q_k which consists of the part of the surpluses which is kept by the shareholders of the company.

In a sensitivity analysis for example portfolios and parameters it was shown in Gerstner et al. (2007) that this model captures the most important behaviour patterns of the balance sheet development of life insurance products. Similar balance sheet models have also been considered in, e.g., Bacinello (2001), Grosen, Jorgensen (2000) and Miltersen, Persson (2003).

2 Deterministic simulation of the ALM model

The numerical simulation aspects of the ALM model from the previous section are discussed in detail in Gerstner, Griebel, Holtz (2007). The simulation of one scenario is based on the realisations of two Brownian motions $W^s(t_k)$ and $W^r(t_k)$ at the discrete times $t_k, 0 \leq k \leq M$. From these numbers, the development of the stock prices, the term structure, the asset allocation, the bonus declaration, the shareholder participation and the development of all involved accounts can then be derived. The balance sheet items C_M, B_M, F_M and Q_M at the end of period M can thus be regarded as deterministic functions $C_M(\mathbf{x})$, $B_M(\mathbf{x})$, $F_M(\mathbf{x})$, $Q_M(\mathbf{x})$: $\mathbb{R}^{2M} \to \mathbb{R}$ which depend on the realisations of $2\,M$ independent normally distributed random numbers $\mathbf{x} = (x_1, \ldots, x_{2M}) \sim N(0, \mathrm{Id})$. As a consequence, the expected values of the balance sheet items at the end of period M can be represented as $2\,M$-dimensional integrals,

[1] i.e., the guaranteed savings part of the policyholders after deduction of risk premium and administrative costs.

e.g.,

$$E[Q_M] = \int_{\mathbb{R}^{2M}} Q_M(\mathbf{x}) \frac{e^{-\mathbf{x}^T\mathbf{x}/2}}{(2\pi)^M} \, d\mathbf{x} \tag{1}$$

for the equity account. Often, monthly discretizations of the capital market processes are used. Then, typical values for the dimension $d = 2M$ range from $60 - 600$ depending on the time horizon of the simulation.

Due to the wide range of path-dependencies, guarantees and option-like features of the insurance products and management rules, closed-form solutions for the integral (1) are only available in special cases. As a consequence, in practice, the model is usually simulated by the Monte Carlo method. This method is independent of the dimension, robust and easy to implement but suffers from a relative low convergence rate $O(N^{-1/2})$ where N denotes the number of samples in the Monte Carlo algorithm. This often leads to very long simulation times in order to obtain approximations of satisfactory accuracy. Extensive sensitivity investigations or the optimisation of product or management parameters, which require a large number of simulation runs, are therefore often not possible.

In Gerstner, Griebel, Holtz (2007) the application of deterministic numerical integration schemes, such as quasi-Monte Carlo and sparse grid methods[2] as alternatives to Monte Carlo simulation is investigated. Like Monte Carlo, these methods can break the curse of dimension to some extent. They often achieve higher convergence rates, in particular, if the integrand is sufficiently smooth and of low effective dimension[3]. The error of quasi-Monte Carlo methods is bounded with the Koksma-Hlawka inequality by $O(N^{-1}(\log N)^d)$ for integrands of bounded variation. Sparse grids converge with order $O(N^{-s}(\log N)^{(d-1)(s-1)})$ if the integrand belongs to the space of functions which have bounded mixed derivatives of order s. Thus, sparse grids can much better exploit the smoothness of a problem than (quasi-) Monte Carlo methods and this way also obtain convergence rates larger than one. In many cases, a substantial reduction of the effective dimension and an improved performance of the deterministic integration schemes can be achieved by the Brownian bridge or the principal component (PCA) decompositions of the covariance matrix of the underlying multivariate Gaussian process as it was proposed in Moskowitz and Calfisch (1996) and Ackworth et al. (1997) for option pricing problems.

3 Effective dimension

For problems with high nominal dimension d, such as the ALM of life insurance products, the classical error bounds of the previous section have no practical use to control the numerical error of the approximation. However, it is known that many nominally high-dimensional integrals arising from the pricing of options or bonds are of low effective dimension, see, e.g., Moskowitz, Calfisch (1996), Wang, Fang (2003) and Wang, Sloan (2005). In contrast to Monte Carlo simulation, quasi-Monte Carlo

[2] See, e.g., Glasserman (2003) for an introduction to (quasi-) Monte Carlo methods. The employed sparse grid algorithms are based on Gerstner, Griebel (1998, 2003).

[3] Here, we focus on the effective dimension in the truncation sense defined in the next section.

and sparse grid methods can take advantage of low effective dimensions and this way produce substantially smaller errors than Monte Carlo methods even if the nominal dimension is high. Quasi-Monte Carlo methods profit from low effective dimensions by the fact that their integration points are usually more uniformly distributed in smaller dimensions. Sparse grid methods exploit different weightings of different dimensions by an dimension adaptive grid refinement, see Gerstner, Griebel (2003).

The effective dimension can by analysed by the ANOVA decomposition. Here, a function $f : \mathbb{R}^d \to \mathbb{R}$ is decomposed into 2^d sub-terms f_u with $u \subseteq \{1, \ldots, d\}$ which only depend on variables x_j with $j \in u$. Thereby, the f_u describe the dependence of the function f on the dimensions $j \in u$. The effective dimension in the truncation sense[4] of a function $f : \mathbb{R}^d \to \mathbb{R}$ with variance $\sigma^2(f)$ is then defined as the smallest integer d_t, such that

$$\sum_{v \subseteq \{1,\ldots,d_t\}} \sigma_v^2(f) \geq 0.99\ \sigma^2(f)$$

where $\sigma_u^2(f)$ denote the variances of the sub-terms f_u from the ANOVA decomposition of the function f. For details and an efficient algorithm for the computation of the effective dimension we refer to Wang, Fang (2003)[5].

4 Numerical results

In this section, we determine the effective dimensions by the method described in Wang, Fang (2003) for the ALM problem based on a representative model portfolio of 500 endowment insurance policies which is described in detail in Gerstner et al. (2007). The policies are equipped with an interest rate guarantee of 3%. Typical parameters are assumed to specify the future behaviour of the policyholders and of the company's management. We consider two different capital market models:

- A geometric Brownian motion[6] (GBM) with drift $\mu = 0.08$ and volatility $\sigma = 0.2$.
- A Cox-Ingersoll-Ross process (CIR) with reversion rate $\kappa = 0.1$, mean reversion level $\theta = 0.04$ and volatility $\sigma = 0.05$.

In Figure 1, the effective dimensions d_t are displayed which arise in the cases GBM (left) and CIR (right) in the computation of the expected equity account (1) for the above model specification. Thereby, different nominal dimensions d and different discretizations of the underlying Gaussian process are considered.

One can see that the effective dimension d_t is in all cases almost as large as the nominal dimension d if the random walk discretization it used while it is much smaller

[4]The effective dimension in the truncation sense roughly describes the number of important variables. Note that other notions of effective dimension have also been introduced in the literature.

[5]For the problems to price Basket options, Asian options and bonds in the Black-Scholes and the Vasicek model, respectively, with path constructions based on PCA, the effective dimensions range from $1 - 6$ independent of the nominal dimension, see Wang, Fang (2003) and Wang, Sloan (2005).

[6]The same capital market model is used, e.g., in Bacinello (2001), Miltersen, Persson (2003), Ballotta et al. (2006), Grosen, Jorgensen (2000).

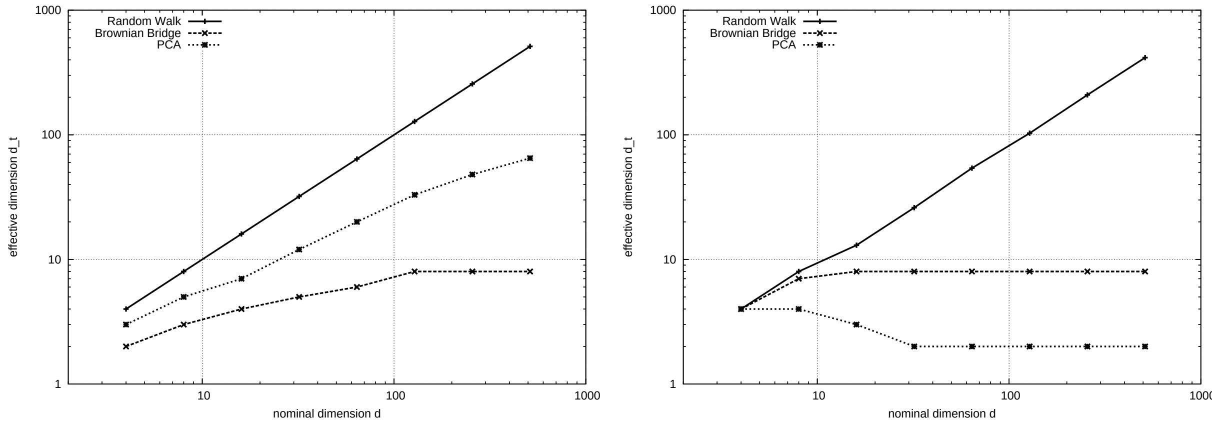

Figure 1: Effective dimensions d_t of the ALM problem with different nominal dimensions d and different covariance matrix decompositions for an underlying geometric Brownian motion (left) and a CIR process (right).

than the nominal dimension in case of the Brownian bridge or PCA path construction. The largest dimension reduction is achieved by the Brownian bridge construction[7] in the geometric Brownian motion (GBM) case and by the PCA discretizations in the CIR case. One can further see that for the Brownian bridge construction the effective dimension is almost insensitive to the nominal dimension and is bounded by $d_t = 8$ even for very large dimensions. In the CIR case with PCA decomposition the effective dimension is only $2 - 4$. It even decreases slightly for increasing d. The effective dimension is affected by several parameters of the ALM model. Numerical tests not displayed in this paper indicate that a higher smoothing of the policyholder interest rates usually leads to a lower effective dimension. Furthermore, the presence of mortality and surrender or a high correlation between stock and bond market reduces the effective dimension.

The low effective dimension of the ALM problem is in agreement with the numerical results in Gerstner, Griebel, Holtz (2007) where it is shown that quasi-Monte Carlo and sparse grid methods outperform Monte Carlo simulation for the ALM of participating life insurance products even for long simulation horizons up to 10 years and longer. Furthermore, quasi-Monte Carlo methods converge even nearly independently of the dimension and the dimension reduction techniques in fact lead to an improved performance of the quasi-Monte Carlo and sparse grid methods while they do not have any effect on the Monte Carlo scheme. Let us finally note that in many cases the efficiency of the numerical methods does not deteriorate very much even if the effective dimension is high in the truncation sense. This indicates that ALM models are also of low effective dimensions in the superposition sense.

[7] This is not in contradiction to the well-known fact that the PCA discretizations is optimal with respect to a concentration of variance of a standard Brownian motion because of the non-linear dependence of the integrand from the realisations of the Brownian motion.

References

Ackworth, P., Broadie, M. and Glasserman, P. (1997). A comparison of some Monte Carlo and quasi Monte Carlo techniques for option pricing. *in: P. Hellekallek, H. Niederreiter (Eds.), Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing*, Springer, 1-18.

Bacinello, A. (2001). Fair pricing of life insurance participating contracts with a minimum interest rate guaranteed. *Astin Bulletin* **31,2**, 275-297.

Ballotta, L., Haberman, S. and Wang, N. (2006). Guarantees in with-profit and unitized with-profit life insurance contracts: Fair valuation problem in presence of the default option. *Insurance: Math. Ecomonics* **73,1**, 97-121.

Briys, E. and Varenne, F. (1997). On the risk of life insurance liabilities: Debunking some common pitfalls. *J. Risk and Insurance* **64**, 37-57.

De Felice, M. and Moriconi, F. (2005). Market based tools for managing the life insurance company. *Astin Bulletin* **1, 35**, 79-111.

Gerstner T. and Griebel, M. (1998). Numerical integration using sparse grids. *Numerical Algorithms* **18**, 209–232.

Gerstner, T. and Griebel, M. (2003). Dimension–adaptive tensor–product quadrature. *Computing* **71, 1**, 65-87.

Gerstner, T., Griebel, M. and Holtz, M. (2007). Deterministic simulation for asset-liability management problems in life insurance. *Working paper*, in preparation.

Gerstner, T., Griebel, M., Holtz, M., Goschnick, R. and Haep, M. (2007). A general asset-liability management model for the efficient simulation of portfolios of life insurance policies, *Insurance: Math. Ecomonics*, submitted.

Glasserman, P. (2003). Monte Carlo Methods in Financial Engineering. *Springer*.

Grosen, A. and Jorgensen, P. (2000). Fair valuation of life insurance liabilities: The impact of interest rate guarantees, surrender options and bonus policies. *Insurance: Math. Economics* **26,1**, 37-57.

Miltersen, K. and Persson, S. (2003). Guaranteed investment contracts: Distributed and undistributed excess returns. *Scand. Actuarial J.* **23**, 257-279.

Moskowitz, B. and Calfisch, R. (1996). Smoothness and dimension reduction in quasi-Monte Carlo methods. *J. Math. Comp. Modeling* **23**, 37-54.

Wang, X. and Fang, K.-T. (2003). The effective dimension and quasi-Monte Carlo integration. *J. Complexity* **19, 2**, 101-124.

Wang, X. and Sloan, I. (2005). Why are high-dimensional finance problems often of low effective dimension? *SIAM J. Sci. Comput.* **27, 1**, 159-183.