Efficient Deterministic Numerical Simulation of Stochastic Asset-Liability Management Models in Life Insurance

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Abstract

New regulations, stronger competitions and more volatile capital markets have increased the demand for stochastic asset-liability management (ALM) models for insurance companies in recent years. The numerical simulation of such models is usually performed by Monte Carlo methods which suffer from a slow and erratic convergence, though. As alternatives to Monte Carlo simulation, we propose and investigate in this article the use of deterministic integration schemes, such as quasi-Monte Carlo and sparse grid quadrature methods. Numerical experiments with different ALM models for portfolios of participating life insurance products demonstrate that these deterministic methods often converge faster, less erratic and produce more accurate results than Monte Carlo simulation even for small sample sizes and complex models if the methods are combined with adaptivity and dimension reduction techniques. In addition, we show by an analysis of variance (ANOVA) that ALM problems are often of very low effective dimension which provides a theoretical explanation for the success of the deterministic quadrature methods.

Keywords: numerical integration, quasi-Monte Carlo, sparse grids, effective dimension

JEL classification: C63; G22
MSC: IB10; IM22

1 Introduction

Much effort has been spent on the development of stochastic asset-liability management (ALM) models for life insurance companies in the last years, see, e.g., [3, 4, 5, 8, 13, 16, 22, 27, 28] and the references therein. Such models are becoming more and more important due to new accountancy standards, greater globalisation, stronger competition, more volatile capital markets and low interest rates. They are employed to simulate the medium and long-term development of all assets and liabilities. This way, the exposure of the insurance company to financial, mortality and surrender risks can be analysed. The results are used to support management decisions regarding, e.g., asset allocation, bonus declaration or the development of more profitable and competitive insurance products. The models are also applied to obtain market-based, fair value accountancy standards as required by Solvency II and the International Financial Reporting Standard.

Due to the wide range of path-dependencies, guarantees and option-like features of insurance products, closed-form representations of statistical target figures, like expected values or variances, which in turn yield embedded values or risk-return profiles of the company, are in general not available. Therefore, insurance companies have to resort to numerical methods for the simulation of ALM models. In practise, usually Monte Carlo methods are used which are based on the averaging

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of a large number of simulated scenarios. These methods are robust and easy to implement but suffer from an erratic convergence and relatively low convergence rates. In order to improve an initial approximation by one more digit precision, Monte Carlo methods require, on average, the simulation of a hundred times as many scenarios as have been used for the initial approximation. Since the simulation of each scenario requires to run over all relevant points in time and all policies in the portfolio of the company, often very long computing times are needed to obtain approximations of satisfactory accuracy. As a consequence, a frequent and comprehensive risk management, extensive sensitivity investigations or the optimisation of product parameters and management rules are often not possible.

In this article, we focus on approaches to speed up the simulation of ALM models. To this end, we rewrite the ALM simulation problem as a multivariate integration problem and apply quasi-Monte Carlo (see, e.g., [19, 31, 38]) and sparse grid methods (see, e.g., [10, 14, 15, 20, 33, 41]) in combination with adaptivity [15] and dimension reduction techniques [1, 30] for its numerical computation. Quasi-Monte Carlo and sparse grid methods are alternatives to Monte Carlo simulation, which can attain faster rates of convergence, can exploit the smoothness of the integrand and have deterministic upper bounds on the error. In this way, they have the potential to significantly reduce the number of required scenarios and computing times.

For many problems from mathematical finance, the efficiency of these deterministic quadrature methods has already been studied and numerical experiments have showed that they are indeed often faster and more accurate than Monte Carlo simulation. Examples are the pricing of bonds [2, 32], options [1, 17] and mortgage backed securities [11, 14, 15, 37]. To our knowledge, it is so far not known if deterministic methods can also be successfully applied to ALM simulations. Such simulations are usually much more complex and time-consuming than the examples mentioned above since many different model equations for the capital market, for the management of the insurance company and for the policyholder behaviour are involved. While there are many publications on modeling aspects, only very few focus on the numerical issues which arise in the simulation of ALM models in life insurance. Monte Carlo methods with antithetic variates, Faure low-discrepancy sequences[1] and different time discretisation methods were studied in [12]. Finite difference methods were considered in [6, 25, 43].

In order to assess the efficiency of deterministic simulation methods we here use the general ALM model framework developed in [16] as a benchmark. This model includes as special cases many other models (e.g., [3, 4, 22]) which have been proposed in the literature for the ALM of participating life insurance products. We show in numerical experiments based on different parameter setups how the accuracy of Monte Carlo, quasi-Monte Carlo and sparse grid integration depends on mathematical properties such as the variance and the smoothness of the corresponding integration problem. Our numerical results demonstrate that quasi-Monte Carlo methods based on Sobol sequences and dimension-adaptive sparse grids based on Gauss-Hermite quadrature formulas are often faster and more accurate than Monte Carlo simulation even for complex ALM models with many time steps.

It is known that the performance of these deterministic numerical methods is closely related to the effective dimension (see, e.g., [11, 44]) of the underlying integration problem. To this end, we determine the effective dimension in the truncation and in the superposition sense by means of an analysis of variance (ANOVA) decomposition of the integrand. The results indicate that ALM problems are often of very low effective dimension, which provides a theoretical explanation for the efficiency of the deterministic quadrature methods. In this context we also show that path generating methods for the capital market scenarios have a significant impact on the effective dimension and on the performance of the numerical methods. Thereby, we compare the random walk, the Brownian bridge and two principal component constructions.

The remainder of this article is as follows: First, in Section 2, we describe the framework of the ALM model and introduce our benchmark model. In Section 3, we then discuss the numerical simulation of ALM models by Monte Carlo and deterministic integration methods. In Section 4, we

[1] Faure sequences were found to be inferior to Monte Carlo simulation in [12] for an ALM problem similar to the one considered here. But, no dimension reduction techniques were considered there.

then present numerical results for several different parameter setups which illustrate the efficiency of the different numerical approaches as well as the numerical effects which arise due to different model components and path constructions. The article finally closes in Section 5 with concluding remarks.

2 Stochastic ALM Models in Life Insurance

In the context of ALM simulations, it is necessary to estimate the medium and long-term development of all assets and liabilities of an insurance company as well as the interactions between them in order to determine their sensitivity to financial, mortality and surrender risks. Thereby, the future development of the capital markets, the behaviour of the policyholders and the decisions of the company's management have to be taken into account. This can either be achieved by the computation of particular scenarios (stress tests) which are based on historical data, subjective expectations, and guidelines of regulatory authorities or by a stochastic modelling and simulation of the capital markets, of the policyholder behaviour and of all involved accounts. In the latter case, the simulation of such models can either be performed under the risk-neutral probability measure, which is appropriate for the fair valuation of embedded options and the identification of fair contract designs, or under the objective probability measure which is used for risk analyses. In the following, we restrict ourselves to the second approach. For numerical simulation both settings are equivalent, though.

In the following, we start with an abstract representation of ALM models in life insurance in terms of a general state space model. This representation reveals the different building blocks from a computational point of view and is used in the remainder of this article.

2.1 Overall Model Structure

Here, we focus on the situation where a stochastic capital market model is used, while all other model components are assumed to be deterministic. We model all terms in discrete time.[2] The simulation starts at time $t = 0$ and ends at time $t = T$. The time interval $[0, T]$ is decomposed into K periods $[t_{k-1}, t_k]$ with $t_k = k\,\Delta t$ and period length $\Delta t = T/K$. Thereby, a (Markov) multi-period model specifies how the different accounts evolve from one point in time to the next. Then, the overall structure of one time step of such ALM models can often be organized into different modules as illustrated in Figure 1.

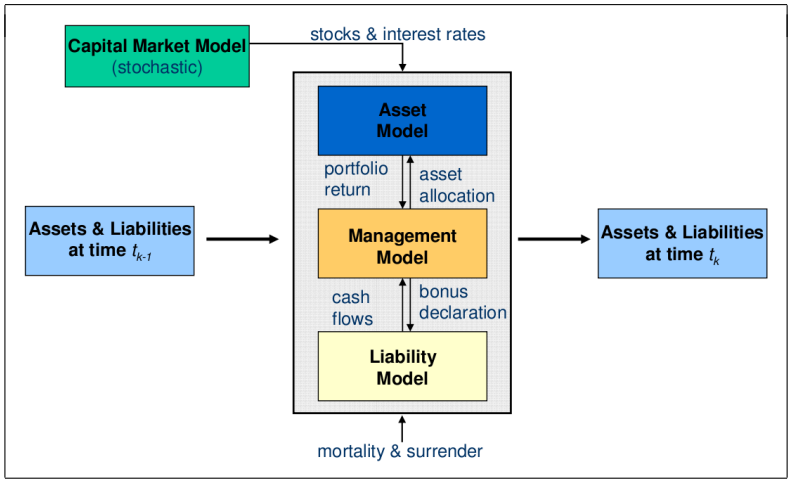

Figure 1: Overall structure of one time step of the ALM model.

[2] Our starting point is thus either a discrete-time model or the discretisation of a continuous-time model.

The stochastic component, the capital market model (or scenario generator), is usually defined by a system of stochastic differential equations for the involved market components (e.g., stocks and interest rates). It is usually based on an underlying multivariate Brownian motion. The deterministic part of the ALM model includes all model equations specified in the asset model, the management model and the liability model. In the asset-model, the market prices of the different asset classes, the return rate of the company's portfolio and the overall development of the capital are determined. In the liability model, the premium payments and all survival, death and surrender benefits are collected, which depend on the specific insurance product. In addition, the balance sheet items on the liability side (e.g., the actuarial reserve, the bonus reserve or the equity) are updated. In the management model, the capital allocation, the bonus declaration and the shareholder participation are specified by deterministic management rules which can depend on the capital markets, the cash flows, the reserves and all other accounts.

2.2 State Space Representation

To obtain a convenient and compact representation of stochastic ALM models, we assume that the entire state of the insurance company at time t_k can be represented by a state vector $\mathbf{X}_k \in \mathbb{R}^M$. This vector contains all information, e.g., balance sheet items and policyholder accounts, which are required to carry the simulation from one point in time to the next, thereby depending on the development of the capital markets. We further assume that the state of the capital markets at time t_k can be described by a vector $\mathbf{S}_k \in \mathbb{R}^D$. It contains, for instance, the current stock prices and interest rates at time t_k.

The initial state of the insurance company at the start of the simulation is given and denoted by $\mathbf{X}_0$. Its temporal development is then specified by the equation

$$\mathbf{X}_k \quad = \quad r(\mathbf{X}_{k-1}, \mathbf{S}_k) \tag{1}$$

in a recursive way for $k = 1, \ldots, K$, which mirrors the Markov property of the ALM model. For a given input vector $\mathbf{S}_k$, the state equation $r : \mathbb{R}^{M+D} \to \mathbb{R}^M$ thereby relates the state of the insurance company at time t_k to the state at time t_{k-1}. It includes all model equations specified in the asset model, in the management model and in the liability model. Most models proposed in the literature or used in practise can be written in the form (1) using a sufficiently complex function r.

The computation of one scenario of the model (1) then involves the computation of the vector

$$\mathbf{X} \quad = \quad (\mathbf{X}_1, \ldots, \mathbf{X}_K)$$

for the states of the insurance company at the different points in time. The states thereby depend on the vector

$$\mathbf{S} \quad = \quad (\mathbf{S}_1, \ldots, \mathbf{S}_K)$$

which describes the underlying capital market scenario.

2.3 Benchmark Model

As an example for the abstract model (1), we here consider the general ALM model framework from [16]. It describes the temporal development of a portfolio of participating life insurance policies. The balance sheet items at time t_k which are used in this model are shown in Table 1. The asset side consists of the market value C_k of the company's assets at time t_k. On the liability side, the first item is the book value of the actuarial reserve D_k.[3] The second item is the allocated bonus B_k which constitutes the part of the surpluses that have been credited to the policyholders via the profit participation. The free reserve F_k is a buffer account for future bonus payments. It consists of surpluses which have not yet been credited to the individual policyholder accounts, and is used to smooth capital market oscillations in order to achieve a stable and low-volatile return participation

[3] i.e., the guaranteed savings part of the policyholders after deduction of risk premiums and administrative costs.

of the policyholders. The last item, the equity Q_k, consists of the part of the surpluses which is kept by the shareholders of the company. The model parameters and model equations which are used to describe the temporal development of the different balance sheet items are summarized in Figure 2 and are briefly explained in the following.

Assets		Liabilities	
Capital	C_k	Actuarial reserve	D_k
		Allocated bonus	B_k
		Free reserve	F_k
		Equity	Q_k

Table 1: Items of the simplified balance sheet at time t_k.

The capital market model contains the price of a stock and the short interest rate. The temporal dynamics of the stock price is modelled by a geometric Brownian motion while the short interest rates are obtained from the Cox-Ingersoll-Ross (CIR) model which is coupled to the stock price model via a constant correlation factor ρ. This system, which is based on a two-dimensional Brownian motion, is then discretised according to the period length Δt with an explicit Euler-Maruyama discretisation yielding discrete stock prices s_k and short interest rates r_k for each period k, see the equations (C1) and (C2) in Figure 2.

In the asset model, the market prices $b_k(\tau)$ of the bonds of the company at time t_k are determined, which in turn depend on the short interest rates r_k and on their duration τ, see equation (A1). This way, the portfolio return rate p_k is specified (A2) which contributes to the development of the capital C_k of the company in each period (A3).

Next, we consider the management model which specifies the capital allocation, the bonus declaration and the shareholder participation. The asset allocation in stocks and bonds is dynamic. Thereby, the company aims to invest a constant percentage β of the total capital in stocks while the remaining part is invested in zero coupon bonds using a buy-and-hold trading strategy, see (M1) – (M3). For the declaration of the policyholder interest z_k, the mechanism from [22] is used (M4), which is based on the reserve situation of the company. To distribute the total surplus G_k in each period k among policyholders and shareholders, a fixed percentage α of the surplus is saved in the free reserve F_k while the remaining part is added to the equity Q_k, see (M5).

In the liability model, the actuarial reserve D_k^i and the allocated bonuses B_k^i for each policyholder i, $i = 1, \dots, m$, are updated, see (L2) and (L3). They depend on the biometry assumptions and on the specific insurance products under consideration. Mortality and surrender are thereby assumed to be deterministic. The probabilities q_k^i and u_k^i that the policyholder i dies or surrenders, respectively, in period k are taken from experience-based tables and determine the number δ_k^i of contracts in the portfolio (L1). The surplus G_k in period k, see (L4), consists of the interest surplus, which results from the spread $p_k - z_k$ of portfolio and policyholder interest, and of the surrender surplus, which depends on the surrender factor ϑ. Finally, the equity Q_k is obtained (L5) so that the sum of the assets equals the sum of the liabilities.

Next, we formulate this particular model in terms of the state space representation (1). The state space $\mathbf{X}_k$ at time t_k of this model consists of all accounts of the insurance company and of the policyholders. We use

$$\mathbf{X}_k = \left(B_k^1, \dots, B_k^m, D_k^1, \dots, D_k^m, \delta_k^1, \dots, \delta_k^m, n_k, \dots, n_{k-\tau+1}, C_k, F_k\right)$$

and thus have $M = 3m + \tau + 2$, where m is the number of policyholders and τ the duration of the bonds. From the state space $\mathbf{X}$ all remaining variables in the model can be derived. Since any term of the model is recursively defined, see Figure 2, a state equation of the form $\mathbf{X}_k = r(\mathbf{X}_{k-1}, \mathbf{S}_k)$ can be formulated. It includes all model equations from Figure 2 except Equations C1 and C2. The state space of the capital market model is two-dimensional and given by

$$\mathbf{S}_k = (s_k, r_k).$$

Capital market model:

Input parameters: κ, θ, σ_r (short interest rates), μ, σ_s (stock prices)

(C1) Short interest rates	r_k	$=$	$r_{k-1} + \kappa(\theta - r_{k-1})\Delta t + \sigma_r \sqrt{\lvert r_{k-1} \rvert}\, \Delta W_{k,1}$
(C2) Stock prices	s_k	$=$	$s_{k-1} \exp\{(\mu - \sigma_s^2/2)\Delta t + \sigma_s\, \Delta W_{k,2}\}$

Management model:

Input parameters: β (asset allocation), ω, γ (bonus declaration), α (shareholders)

(M1) New investment	N_k	$=$	$C_{k-1} + P_k - \sum_{j=1}^{\tau-1} n_{k-j}\, b_{k-1}(\tau - j)$
(M2) Investment in stocks	A_k	$=$	$\max\{\min\{N_k,\, \beta(C_{k-1} + P_k)\},\, 0\}$
(M3) Number of new bonds	n_k	$=$	$(N_k - A_k)/b_{k-1}(\tau)$
(M4) Policyholder interest	z_k	$=$	$\max\{z,\, \omega(F_{k-1}/(D_{k-1} + B_{k-1}) - \gamma)\}$
(M5) Free reserve	F_k	$=$	$\max\{F_{k-1} + \min\{G_k, \alpha\, G_k\},\, 0\}$

Asset model:

Input parameters: τ (bond duration)

(A1) Bond prices	$b_k(\tau)$	$=$	$A(\tau)\, \exp\{-B(\tau)\, r_k\}$
(A2) Portfolio return rate	p_k	$=$	$(\Delta A_k + \sum_{j=0}^{\tau-1} n_{k-j}\, \Delta b_{k,j})/(C_{k-1} + P_k)$
(A3) Capital	C_k	$=$	$(1 + p_k)(C_{k-1} + P_k) - E_k - T_k - S_k$

Liability model:

Input parameters: q_k^i, u_k^i (mortality, surrender), $z, \vartheta, P_k^i, E_k^{i,G}, T_k^{i,G}, S_k^{i,G}$ (product)

(L1) Number of contracts	δ_k^i	$=$	$\left(1 - q_k^i - u_k^i\right) \delta_{k-1}^i$
(L2) Actuarial reserve	D_k^i	$=$	$((1 + z)(D_{k-1}^i + P_k^i) - q_k^i T_k^{i,G})/(1 - q_k^i) - E_k^{i,G}$
(L3) Allocated bonus	B_k^i	$=$	$(1 + z_k)\, B_{k-1}^i + (z_k - z)\, (D_{k-1}^i + P_k^i)$
(L4) Surplus	G_k	$=$	$p_k\, F_{k-1} + (p_k - z_k)\, (D_{k-1} + B_{k-1} + P_k) + (1/\vartheta - 1) S_k$
(L5) Equity	Q_k	$=$	$C_k - D_k - B_k - F_k$

Figure 2: Summary of the most important model parameters and equations. Lower indices k refer to the point in time t_k whereas upper indices i refer to the i-th policyholder. The values $\Delta W_{k,\ell} = W_\ell(t_k) - W_\ell(t_{k-1})$ for $\ell \in \{1, 2\}$, $\Delta A_k = A_k(s_k/s_{k-1} - 1)$ and $\Delta b_{k,j} = b_k(\tau - j - 1) - b_{k-1}(\tau - j)$ denote the increments of the underlying Brownian motion, the changes of the stock investments and the changes of the bond prices from the beginning to the end of period k, respectively. The terms $A(\tau)$ and $B(\tau)$ are constants which are defined in the CIR model. The values P_k, E_k, T_k and S_k denote the total premium income and the total survival, death, and surrender payments in period k. Like D_k and B_k, these values result by summation of the individual policyholder accounts.

3 Numerical Simulation

Due to the large variety of path-dependencies, guarantees and option-like features of insurance products and management rules in ALM models, closed-form representations of statistical target figures, such as expected values, are in general not available. Therefore, numerical methods have to be used for their simulation. In this section, we discuss the stochastic simulation of ALM models of type (1) and the arising computational costs. In order to reduce the computing times, we then rewrite the simulation problem as high-dimensional integration problem and apply deterministic quadrature methods for its numerical computation. To profit from low effective dimensions we finally combine these methods with a hierarchical construction of the capital market paths.

3.1 Stochastic Simulation

A single simulation run $\mathbf{X}$ of the ALM model (1) corresponds to a particular capital market scenario $\mathbf{S}$. It can be analysed by looking, e.g., at the balance sheet positions or at cross sections of the portfolio at certain times. Here, we focus on stochastic simulations of the ALM model (1). To this end, a large number of scenarios is generated and statistical performance figures such as expected values are considered and evaluated. These measures are based on the most important state variables, e.g. the equity or the investment return, and can be written in the form

$$P = \mathbb{E}[f_P(\mathbf{X})] \tag{2}$$

for some evaluation function $f_P : \mathbb{R}^{M \cdot K} \to \mathbb{R}$. A simple example for such a function f_P evaluates the equity from the state vector $\mathbf{X}_K$ at the time t_K.

Next, we formulate a Monte Carlo algorithm for the approximation of the performance figure P. Thereby, we assume that the capital market scenarios $\mathbf{S}$ result from the discretisation and simulation of a system of stochastic Itô differential equations which is based on a D-dimensional Brownian motion. Since K time steps are used in the discretisation this results in an $(D \cdot K)$-dimensional problem represented by

$$\begin{aligned} \mathbf{S} &= f_C(\mathbf{W}), \\ \mathbf{W} &= f_B(\mathbf{Y}), \end{aligned} \tag{3}$$

where $\mathbf{W} = (\mathbf{W}_1, \ldots, \mathbf{W}_K)$ denotes the discrete path which contains the values of the Brownian motion at times $t_1, \ldots, t_K$ and where $\mathbf{Y}$ denotes a vector which contains $D \cdot K$ independent standard normally distributed random numbers. The functions $f_C : \mathbb{R}^{D \cdot K} \to \mathbb{R}^{D \cdot K}$ and $f_B : \mathbb{R}^{D \cdot K} \to \mathbb{R}^{D \cdot K}$ represent the generation of the capital market scenarios $\mathbf{S}$ and the path construction of the underlying Brownian motion $\mathbf{W}$, respectively. For the specific ALM model from Section 2.3 the equation $\mathbf{S} = f_C(\mathbf{W})$ is explicitly given by Equations C1 and C2 in Figure 2.

The main steps of a standard Monte Carlo algorithm (see, e.g., [19]) using N scenarios for the approximation of the expected value (2) are summarised in Figure 3. Thereby, the function $f_I : \mathbb{R}^{D \cdot K} \to \mathbb{R}^{M \cdot K}$ denotes the explicit (i.e. non-recursive) representation of the recursion (1) and thus contains all equations of the ALM model.

3.2 Computational Costs

The overall computational complexity R which is needed by the algorithm in Figure 3 is given by

$$R = O(N \cdot K \cdot \Lambda),$$

where N is the number of scenarios which are generated, K is the number of periods which are simulated, and Λ describes the costs of one time step of the ALM model, i.e., the costs to evaluate the recursion (1). The required number N of scenarios[4] depends on the accuracy

[4] In practise, usually a fixed prescribed number (e.g., 1000) of scenarios is used. To compare the efficiency of numerical algorithms it is important that these costs (the number of scenarios) have to be compared to the benefits (the accuracy). In this article, we therefore aim to find algorithms which obtain a fixed accuracy with as few scenarios as possible or, conversely, algorithms which are as precise as possible for fixed costs.

For $i = 1, 2, \ldots, N$			
1) Generate normally distributed random numbers	$\mathbf{Y}^i \in \mathbb{R}^{D \cdot K}$		
2) Construct path of Brownian motion	$\mathbf{W}^i$	$=$	$f_B(\mathbf{Y}^i)$
3) Generate capital market scenario	$\mathbf{S}^i$	$=$	$f_C(\mathbf{W}^i)$
4) Evaluate ALM model equations	$\mathbf{X}^i$	$=$	$f_I(\mathbf{S}^i)$
5) Compute performance figure	P^i	$=$	$f_P(\mathbf{X}^i)$
Compute the average $P \approx \frac{1}{N}\left(P^1 + \ldots + P^N\right)$.			

Figure 3: Approximation of the expected value (2) by Monte Carlo integration.

requirements, the employed numerical method and on the model parameters as they affect several important mathematical properties of the problem, e.g., its variance, its effective dimension and its smoothness, see Section 3.4. The costs Λ depends, e.g., on the number of different asset classes, insurance contracts and management rules. Typically, it is proportional to the size of the state vectors $\mathbf{X}_k$ and $\mathbf{S}_k$, i.e.

$$\Lambda = O(M + D).$$

In models, where market prices of some asset classes have to be computed with numerical approximation methods, where model equations can not be evaluated in recursive form, or where assets are allocated based on numerical optimisation routines, the complexity of Λ can be significantly larger than $O(M + D)$, though.

Several approaches exist to reduce the overall computational complexity R. The costs Λ can be reduced by model simplifications on the asset or the liability side, e.g., by the use of few representative asset classes and policies (model points), or less complex management rules. Such simplifications are easy to realise, the resulting model errors are difficult to control, though. The number of periods K can be reduced by using a coarser time discretisation (e.g., yearly instead of monthly periods) which however increases the time discretisation error. Higher order [12, 26] or multilevel methods [18] might be used to reduce this effect. In Monte Carlo simulations, the number of required scenarios N can often be reduced using variance reduction techniques, like, e.g., antithetic variates, control variates or importance sampling, see, e.g., [19]. For complex ALM models, good control variates or importance sampling functions are difficult to obtain, though. Finally, one can resort to parallel computers to speed up the computation, e.g., by a simultaneous calculation of different scenarios on different processors.

In this article, we focus on a different approach to decrease the overall computational complexity by reducing the required number of scenarios. We rewrite the ALM simulation problem as an multivariate integration problem (see Section 3.3) and apply quasi-Monte Carlo and sparse grid methods (see Section 3.4) in combination with adaptivity and dimension reduction techniques (see Section 3.5) for its numerical computation. These approaches are often faster and more accurate than Monte Carlo simulation as we show in Section 4. They can be combined with the other approaches mentioned above to reduce the overall computational complexity in order to achieve additional speedups.

3.3 Integral Representation

Since the distribution of the vector $\mathbf{X} \in \mathbb{R}^{M \cdot K}$, which contains the states of the insurance company, depends on the normally distributed vector $\mathbf{Y} \in \mathbb{R}^d$ with $d = D \cdot K$, see (1) and (3), the performance figure (2) can be represented as a d-dimensional integral

$$P = \int_{\mathbb{R}^d} h(\mathbf{y})\, \varphi(\mathbf{y})\; d\mathbf{y} \tag{4}$$

where the function $h : \mathbb{R}^d \to \mathbb{R}$ is explicitly given by $h = f_P \circ f_I \circ f_C \circ f_B$, see Figure 3, and where $\varphi(\mathbf{y}) = (2\pi)^{-d/2} e^{-\mathbf{y}^T \mathbf{y}/2}$ denotes the Gaussian density function. We see that, for instance,

the computation of a performance figure in an ALM model with a two-factor capital market and a monthly discretisation for a time horizon of $T = 10$ years corresponds to $D = 2$ and $K = 120$ and thus results in a 240-dimensional integral.

In order to apply numerical integration methods, it is often necessary to transform the integral (4) into an integral over the d-dimensional unit cube $[0,1]^d$. To this end, we to use the substitution $y_i = \Phi^{-1}(x_i)$ for $i = 1, \ldots, d$, where Φ^{-1} denotes the inverse[5] of the cumulative normal distribution function. One obtains

$$P = \int_{[0,1]^d} f(\mathbf{x})\ d\mathbf{x} \tag{5}$$

with the integrand $f(\mathbf{x}) = h(\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_d))$.[6]

3.4 Numerical Methods for High-Dimensional Integrals

There is a wide range of methods available for multivariate numerical integration. Typically, the integral (5) is approximated by a weighted sum of N function evaluations

$$P_N = \sum_{i=1}^{N} w_i f(\mathbf{x}^i) \tag{6}$$

with weights $w_i \in \mathbb{R}$ and nodes $\mathbf{x}^i \in [0,1]^d$. Depending on the choice of the weights and nodes, different methods with varying properties are obtained, which are shortly reviewed in the next paragraphs. Here, one can distinguish between statistical methods (Monte Carlo), number theoretic methods (quasi-Monte Carlo) and methods based on polynomial exactness (e.g., sparse grids). While the methods of the first two classes use uniformly distributed point sets, the rules of the third class are designed to be exact for a certain set of polynomials and thus particularly well suited for the integration of smooth functions. Example node sets for these three approaches are shown in Figure 4. Note that the number N of nodes corresponds to the number of simulation runs and each point $\mathbf{x}^i$ corresponds to exactly one capital market scenario $\mathbf{S}^i$.

Monte Carlo The Monte Carlo (MC) method is the most widely used method for the simulation of stochastic models. Here, all weights equal $w_i = 1/N$ and uniformly distributed sequences of pseudo-random numbers $\mathbf{x}^i \in (0,1)^d$ are used as nodes.[7] The law of large numbers then ensures that the estimate P_N converges to P for $N \to \infty$ if f has finite variance $\sigma^2(f)$. The expected mean square error thereby equals $\sigma^2(f)\ N^{-1/2}$. The method thus suffers from a relative low probabilistic convergence rate of $1/2$, i.e.,

$$|P - P_N| = O(N^{-1/2}). \tag{7}$$

The convergence rate is, however, independent of the dimension d.

Quasi-Monte Carlo Quasi-Monte Carlo (QMC) methods are equal-weight rules like MC.[8] Instead of pseudo-random numbers, however, deterministic low-discrepancy sequences based either on digital nets (see, e.g., [19, 31]) or lattices (see, e.g., [38, 39]) are used as nodes. They are designed to yield a better uniformity than random samples. For a two-dimensional example, this property is illustrated by the grids in Figure 4 where 128 MC points and 128 QMC points are displayed. One can see that the pseudo-random points tend to cluster more and are not as evenly

[5] For the fast computation of $\Phi^{-1}(x_i)$, we use Moro's method [29].

[6] The integrand f which results from this transformation is unbounded on the boundary of the unit cube, which is undesirable from a numerical as well as theoretical point of view. We tested several alternative transformations to the unit cube (e.g. using the logistic distribution or polar coordinates), but in combination with (quasi-) Monte Carlo the transformation $y_i = \Phi^{-1}(x_i)$ turned out to be the most effective one, probably because it cancels the Gaussian weight. Sparse grid quadrature based on Gauss-Hermite rules can be applied directly to the untransformed integral (4) such that no transformation is required and the corresponding loss of regularity can be avoided.

[7] Algorithms which generate uniform pseudo-random numbers are reviewed, e.g., in [19, 31].

[8] QMC methods can therefore easily be integrated in existing software packages, which are based on MC simulation, by replacing all pseudo random MC samples by quasi-random ones.

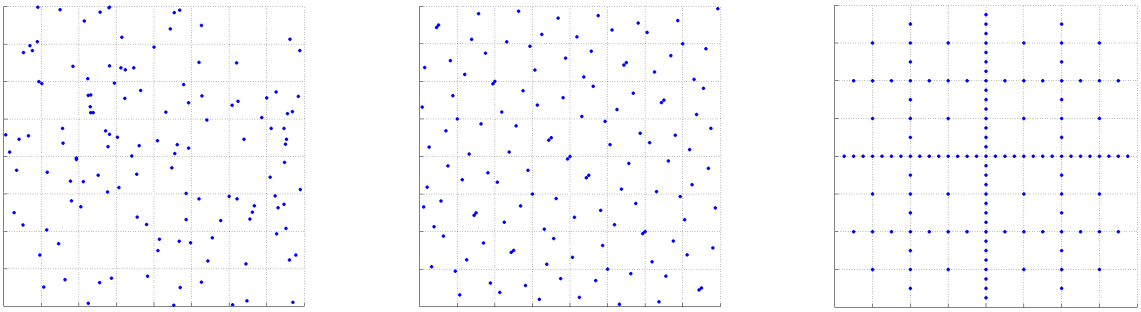

Figure 4: Sample points of the Monte Carlo method (left), Sobol quasi-Monte Carlo method (middle) and sparse grid method (right). The sparse grid is based on a trapezoidal rule.

distributed as the quasi-random ones. The most popular QMC methods are based on Halton, Faure, Sobol and Niederreiter-Xing sequences, or on lattice rules which result from Korobov or fast component-by-component constructions. From the Koksma-Hlawka inequality (see, e.g., [31]) it follows that the worst case error of a QMC method with N samples is given by

$$|P - P_N| = O(N^{-1}(\log N)^d) \tag{8}$$

for integrands of bounded variation, which is asymptotically better than the error bound (7) of the MC method but depends on the dimension d.

Sparse grid methods Sparse grid (SG) quadrature formulas are constructed using certain combinations of tensor products of one-dimensional quadrature rules, see, e.g., [10, 14, 20, 33, 41]. Depending on the one-dimensional quadrature rules, these methods integrate polynomials up to a certain degree exactly and can thus exploit the smoothness of the function f and also obtain convergence rates larger than one. They converge with order

$$|P - P_N| = O(N^{-s}(\log N)^{(d-1)(s-1)}) \tag{9}$$

for integrands which have bounded mixed derivatives up to order s. Sparse grid quadrature formulas come in various types depending on the one-dimensional basis integration routine, like the trapezoidal, the Clenshaw-Curtis, the Gauss-Patterson, the Gauss-Legendre or the Gauss-Hermite rule. In many cases, the performance of sparse grids can be enhanced by spatial adaptivity, see [7, 9], or by a dimension-adaptive refinement, see [15]. The latter algorithms allow for an adaptive detection of the important dimensions and adaptively refine in this respect guided by suitable error estimators. The costs of the method thus rather increase with the number of important dimensions than with the nominal dimension d, which, in particular, permits an efficient application to high-dimensional problems with few important dimensions.

3.5 Impact of the Dimension

The error bounds (8) and (9) indicate that the convergence rates of QMC and SG methods exhibit a logarithmic dependence on the dimension d. Furthermore, it is known that the implicit constants contained in the O-notation in these bounds also depend on d and often increase exponentially with the dimension, see, e.g., [20, 40, 46].[9] For problems with high nominal dimension d, it is thus not clear if the asymptotic advantages of the QMC and SG methods pay off for practical sample sizes N.

[9] This effect, called curse of dimension, is one of the main obstacles for the numerical treatment of high dimensional problems. For some function classes, e.g., Sobolov spaces with bounded mixed derivatives, it is known that the curse of dimension can not be avoided by deterministic methods of the form (6), see, e.g., [20, 21, 35]. Such problems are therefore called intractable.

One the other hand, numerical experiments showed that QMC methods are faster and more accurate than MC for certain financial problems even in hundreds of dimensions, like the pricing of bonds [32], options [1] and mortgage backed securities [11, 37], the valuation of catastrophe bonds [2] and the computation of the value at risk of asset portfolios [36]. Similar results have been observed for the SG methods, which were applied successfully to the valuation of performance-dependent options [17], the pricing of mortgage backed securities [14, 15] and to likelihood estimation [23].

The explanation of this success of QMC and SG methods is that application problems are often in different or smaller problem classes than the ones on which the classical error bounds are based. Important examples of such function classes are so-called weighted Sobolov spaces, for which it is shown in [34, 40] that the exponential dependence on the dimension can be avoided if the weights decay sufficiently fast. A closely related argumentation given in [11] states that many problems in finance have the property to be of low effective dimension which can be exploited by QMC and SG methods.

ANOVA decomposition and effective dimension The notion of effective dimension is based on the analysis of variance (ANOVA) decomposition of the integrand (5). In the ANOVA, a function $f : [0,1]^d \to \mathbb{R}$ is decomposed by

$$f(\mathbf{x}) = \sum_{\mathbf{u} \subseteq \mathcal{S}} f_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) \quad \text{with } f_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) = \int_{[0,1]^{d-|\mathbf{u}|}} f(\mathbf{x}) d\mathbf{x}_{\mathcal{S}\setminus\mathbf{u}} - \sum_{\mathbf{v} \subset \mathbf{u}} f_{\mathbf{v}}(\mathbf{x}_{\mathbf{v}})$$

into 2^d sub-terms $f_{\mathbf{u}}$ where $\mathcal{S} = \{1, \ldots, d\}$ and $\mathbf{x}_{\mathbf{u}}$ denotes the $|\mathbf{u}|$-dimensional vector containing those components of $\mathbf{x}$ whose indices belong to the set $\mathbf{u}$. Thereby, the sub-terms $f_{\mathbf{u}}$ describe the dependence of the function f on the dimensions $j \in \mathbf{u}$. The sub-terms satisfy $\int_0^1 f_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) dx_j = 0$ for all $j \in \mathbf{u}$ which insures the uniqueness of the decomposition and implies that the variance of the function f can be written as

$$\sigma(f) = \sum_{\mathbf{u} \subseteq \mathcal{S}} \sigma(f_{\mathbf{u}}),$$

where $\sigma(f_{\mathbf{u}})$ denotes the variance of $f_{\mathbf{u}}$.

This decomposition of variance is used to define the effective dimension of the function f as introduced in [11]. The effective dimension in the truncation sense (the truncation dimension) of the function f is defined as the smallest integer d_t, such that

$$\sum_{\mathbf{v} \subseteq \{1,\ldots,d_t\}} \sigma(f_{\mathbf{v}}) \geq 0.99 \; \sigma(f).$$

If the variables are ordered according to their importance, the effective dimension d_t roughly describes the number of important variables of the function f. The effective dimension in the superposition sense (the superposition dimension) is defined as the smallest integer d_s, such that

$$\sum_{|\mathbf{v}| \leq d_s} \sigma(f_{\mathbf{v}}) \geq 0.99 \; \sigma(f).$$

It roughly describes the highest order of important interactions between variables in the ANOVA decomposition.

For large d, it is no longer possible to compute all 2^d ANOVA sub-terms. The effective dimensions can still be computed in many cases, though. For details and an efficient algorithm for the computation of the truncation dimension we refer to [44]. For the more difficult problem to compute the superposition dimension, we use the recursive method described in [45].

The important point is that QMC and SG methods can take advantage of low effective dimensions. QMC methods profit from a low effective dimension by the fact that their nodes are usually more uniformly distributed in smaller dimensions than in higher ones. Lattice rules can be tailored to problems in which dimensions are of varying importance using the component-by-component

construction [39]. SG methods can automatically detect different weightings of the dimensions and exploit these by a dimension-adaptive grid refinement, see [15]. Thereby, SG methods profit from low truncation dimension as well as from low superposition dimension. In Section 4.2 we will give some indications that ALM problems in life insurance indeed belong to spaces of functions of low effective dimension, which explains the good convergence behaviour and justifies the use of deterministic methods.

Dimension reduction It is possible to change the effective dimension of the integral (2) without affecting its value and the distribution of the states $\mathbf{X}$ of the ALM model. To this end, we consider the function f_B from (3), which describes the construction of the discrete path $\mathbf{W}_1, \dots, \mathbf{W}_K$ of the underlying Brownian motion at the points in time $t_k = k\,\Delta t$, $k = 1, \dots, K$, with $t_0 = 0$ and $t_K = T$, see also Figure 3. Since the performance figures (2) are independent of the specific choice of f_B, this function is not part of the ALM model specification but a degree of freedom which can be used to optimize the numerical method which is used for their simulation

To approximate a univariate Brownian motion W at the points t_k typically a random walk (RW) discretisation

$$W_k = W_{k-1} + \sqrt{\Delta t}\, y_k$$

with $W_0 = 0$ and $y_k \sim N(0,1)$ is used. With respect to a low effective dimension and an optimal performance of the deterministic integration methods, much better results can often be achieved by alternative sampling methods, though. Here, we will investigate the application of the Brownian bridge (BB) and of the principal component (PCA) construction to ALM problems. For option pricing problems, these path generating methods have been proposed in [1, 30], see [19] for details.

The BB construction differs from the RW construction in that rather than constructing the increments sequentially, the path of the Gaussian process is constructed in a hierarchical way which has the effect that more importance is placed on the earlier variables than on the later ones. The PCA decomposition uses the eigenvalues and -vectors of the covariance matrix of the Brownian motion and maximises the concentration of the total variance of the Brownian motion in the first few dimensions.[10] The PCA construction requires $O(K^2)$ operations for the generation of one path instead of $O(K)$ operations which are needed for the RW or for the BB construction. For large K, this increases the run times of the simulation and can limit the practical use of the PCA construction.

In the multivariate case, a D-dimensional Brownian motion $\mathbf{W}$ with correlation matrix $\mathbf{\Sigma} \in \mathbb{R}^{D\times D}$ has to be constructed. To this end, one can first generate a discrete D-dimensional uncorrelated Brownian motion by applying RW, BB or PCA separately to each component of the Brownian motion and then respect the correlation in a second step by multiplication with either the Cholesky matrix or the eigenvalue decomposition of $\mathbf{\Sigma}$. Alternatively, one can perform an eigenvalue decomposition of the entire covariance matrix of $\mathbf{W}$. The latter approach, which we denote by EVD, leads in the multivariate case to the highest possible concentration of variance in the first variables, see [19]. We will investigate all four approaches, RW, BB, PCA and EVD, in Section 4.

4 Numerical Results

We now apply MC, QMC and SG methods for the simulation of several different parameter setups and compare the efficiency of the different numerical approaches. We determine the variances, the smoothness and the effective dimensions of the different setups and show how these properties relate to the convergence behaviour of MC, QMC and SG methods. Finally, we investigate the impact of the different path generating methods.

[10] Note that without further assumptions on the integrand f it is not clear which construction leads to the minimal effective dimension due to possibly non-linear dependencies of f on the underlying Brownian motion. As a partial remedy, also more complicated covariance matrix decompositions can be employed which take into account the linear part of the function f, see [24], which, however, are not considered here.

4.1 Parameter Setups

As remarked in Section 3, the efficiency of numerical integration methods depends on several mathematical properties of the integrand, such as its variance, its effective dimension and its smoothness, all of which are affected by the choice of the input parameters of the ALM model. To investigate the numerical issues which arise from the different choices of parameters and the influence of the different model components, we consider the following simple basic setup 1 and several extensions 2 – 10. Thereby, in each of the extensions, either one additional feature is added to the basic setup or one particular component of the basic setup is replaced by a different one. In setup 11, we then consider the combination of several extensions. All setups are special cases of the ALM model in Figure 2 and result from different choices of the input parameters.

1 Basic model We start with a basic model which corresponds to the model from [22] for European-type participation contracts with conservative bonus declaration. Thereby, we consider a homogeneous portfolio of $50,000$ pure savings policies, which are exactly represented by one model point. The policyholders of this model point pay a constant monthly premium $P_k^i = 50$ and receive the guaranteed interest $z = 3\%$. The assets follow Equation C2 (Figure 2) with parameters $\mu = 5\%$ and $\sigma_s = 10\%$. This case is represented in the model framework by setting $m = 1$, $\beta = 1$, $\omega = 0$ and $q_k^i \equiv u_k^i \equiv 0$. We assume that the two accounts F_k and Q_k are merged into one single account, also denoted by Q_k, which is appropriate if the policyholders are also the owners of the company, see [22], and which corresponds to the case $\alpha = 1$. We further assume that the policies have not received any bonus payments before the start of the simulation, i.e., $B_0^i = 0$. It is finally assumed that the total initial reserves of the company are given by $Q_0 = \gamma_0 D_0$ with $\gamma_0 = 0.1$. We always choose a period length of $\Delta t = 1/12$ years, but consider different numbers of periods in the simulation.

2 Mortality and surrender As a representative for a more complex insurance product, we consider an endowment insurance with death benefit and surrender option with surrender factor $\vartheta = 90\%$. Thereby, the policyholders, which are assumed to be male with entry age 36, exit age 62 and current age 42, receive at maturity, with age 65, their guaranteed benefit D_k^i and the value of their bonus account B_k^i if they are still alive at this point in time. In case of death prior to maturity, the sum of all premium payments and the value of the bonus account is returned. In case of surrender, the policyholder receives $\vartheta(D_k^i + B_k^i)$. We take the probabilities q_k^i of death from the DAV 2004R mortality table and choose exponentially distributed surrender probabilities $u_k^i = 1 - e^{-0.03\Delta t}$.

3 Non-homogeneous portfolio We consider a more complex representative model portfolio with $50,000$ contracts which have been condensed into $m = 500$ equal-sized model points. The data of each model point i is generated according to the distribution assumptions that entry and exit age are normally distributed with mean 36 and 62 and variance 10 and 4, respectively. The current age at the start of the simulation is uniformly distributed between entry and exit age. The probability that the contracts of a model point belong to female policyholders is assumed to be 55%.

4 High volatility To illustrate the effect of more volatile capital markets, we here choose a volatility of $\sigma_s = 30\%$ instead of 10%.

5 and 6 Moderate and aggressive bonus payments To illustrate the effect of the bonus declaration mechanism, we choose $\omega = 25\%$ and $\omega = 100\%$, which correspond to the neutral and aggressive scenarios in [22], respectively. The target reserve rate is assumed to be $\gamma = 15\%$.

7 Shareholder participation In this setup, we choose $\alpha = 0.9$ which corresponds to a distribution of the surplus between free reserve F_k and equity Q_k according to the 90/10-rule. We assume

$Q_0 = 0$ which means that the shareholders will not make additional payments to the company to avoid a ruin. This way, $\mathbb{E}[Q_K]$ serves as a direct measure for the investment returns of the shareholders in the time interval $[0, t_K]$. The initial reserves of the company are collected in the free reserve, i.e., $F_0 = 0.1D_0$.

8 CIR model In this setup, we assume that the capital is only invested into bonds (i.e. $\beta = 0$) with a duration of $\tau = 1/12$ years. The short interest rates follow Equation C1 (Figure 2), where we use the parameters $\kappa = 0.1$, $\theta = 4\%$ and $\sigma_r = 5\%$. The bond prices then result from Equation A1. The terms $A(\tau)$ and $B(\tau)$ in this equation involve the market price of interest rate risk which we assume to be $\lambda_0 = -5\%$. At time t_0, we assume a uniform bond allocation, i.e., $n_j = (1-\beta)C_0 / \sum_{i=0}^{\tau-1} b_0(i)$ for $j = 1-\tau, \ldots, 0$.

9 CIR + GBM As a representative for a more complex capital market model, we consider a correlated system of geometric Brownian motion and square-root diffusion as in Equation C1 and C2 (Figure 2) with a correlation of $\rho = -0.1$. The parameters of the geometric Brownian motion and of the CIR model are as above, but with $\sigma_s = 5\%$. The capital allocation is performed with the target stock ratio $\beta = 10\%$.

10 CPPI strategy In this setup, we again use the correlated system of geometric Brownian motion and square-root diffusion as in the previous setup. In contrast to setup 9, we here replace $\beta(C_{k-1} + P_k)$ in Equation M2 by $\beta\, F_{k-1}$ with $\beta = 1$ such that the proportion of funds invested in stocks is linked to the current amount of reserves. This implements a CPPI (constant proportion portfolio insurance) capital allocation strategy.

11 Compound model We finally consider the simulation of a more complex ALM model which is obtained by a combination of the setups 1 – 3, 5 and 10. It models the development of a portfolio of endowment insurances with death benefit and incorporates the surrender of contracts, a reserve-dependent bonus declaration, a dynamic asset allocation and a two-factor stochastic capital market.

4.2 Properties of the Model Components

We here focus on the distribution of the equity account Q_K at time t_K in the setups 1 – 11. We compute the expected value of Q_K, which is given by the integral (2) with $f_P(\mathbf{X}) = Q_K$, and the variance of Q_K for all setups by numerical integration on a very fine simulation grid with 2^{20} QMC sample points.[11] The results for $K = 16$ and $K = 128$ time steps are shown in Table 2 and Table 3, respectively.

An detailed assessment of the impact of the different model parameters on the expected value of the equity account and on other performance figures such as, e.g., the default probabilities, can be found in [16]. Here, we merely focus on results which have an impact on the performance of the numerical methods. To this end, note that the small volatility and the mean-reverting property of the short interest rates in the CIR model result in rather small variances in the setups 8 – 11. The by far highest variance arises in the setup 4. A comparison of the cases $K = 16$ and $K = 128$ shows that longer time horizons with more periods lead to higher variances in all considered setups as expected. Setup 11 with $K = 128$ is of striking small variance as it combines several components which reduce the variance of the basic setup such as the decrement of contracts (setup 2), the bonus declaration (setup 5) and the CIR model (setup 10).

Note further that the integrands which correspond to the different setups not only differ in their variance but are also contained in different smoothness classes. This is indicated in Table 2

[11] For the setups 1 – 4 these values can also be derived analytically. For setup 1 and setup 4 one obtains $\mathbb{E}[Q_K] = \sum_{k=1}^{K} P_k^i \left(\exp\{\mu(K-k+1)\Delta t\} - P_k^i (1+z)^{K-k+1}\right) + \exp\{\mu K \Delta t\} C_0 - (1+z)^K D_0$, which we used as a first test to validate the correctness of the implementation and the accuracy of the numerical methods. The closed-form solution of a similar model but with bonus payments can be found in [3].

and Table 3 by the smoothness parameter s, which denotes the maximum number of continuous derivatives of the equity Q_K as function of the vector $\mathbf{Y} \in \mathbb{R}^d$. While Q_K is a C^∞-smooth function in the setups 1–4, it is only a C^0-smooth function in the other cases. This loss of regularity results from the maximum and minimum operators in the management rules M2, M4 and M5 for the capital allocation, the bonus declaration and the shareholder participation and from the absolute value in the model equation C1.

In addition to the expected values and variances, we also computed the truncation dimension of the function Q_K for all setups. We thereby used the Brownian bridge construction for the stock prices and short interest rates. The path generation does not affect the distribution or smoothness of Q_K, but has a significant influence on the truncation dimension as we will show in Section 4.4. One can see in Table 2 and in Table 3 that the truncation dimensions d_t are in all cases significantly smaller than the nominal dimensions d. In the setups 9 – 11 with $K = 128$, the nominal dimension is $d = 256$, while the truncation dimension is only $d_t = 15$. The highest truncation dimension appears in setup 6 where we have $d = 128$ and $d_t = 23$. In the setups 1 – 4 and 8, we observe that the truncation dimension is almost independent of the nominal dimension. It is in all cases smaller than eight and even only one in the setups 1 – 4. In setup 11 one can finally see that the combination of several extensions does not necessarily increase the effective dimension. It may even be smaller than the maximum effective dimension of the individual components. This indicates that also significantly more complex ALM models might be of low effective dimension.

Further numerical computations show that the considered ALM model problems are also of very low effective dimension d_s in the superposition sense. For $d = 32$ we obtain for almost all setups that the integral (2) is 'nearly' additive, i.e. $d_s = 1$, independent of d and independent of the chosen construction of the capital market paths.[12] Only for setup 7 we get the superposition $d_s = 2$.

A partial explanation for the low effective dimensions and a common feature of ALM problems is that the high nominal dimension mainly arises from the discretisation of an underlying continuous time process. The corresponding integrals can thus be written as an approximation to some infinite-dimensional integrals with respect to the Wiener measure. In these cases, the integrands are contained in some weighted function spaces whose weights are related to the eigenvalues of the covariance operator of the Wiener measure. The eigenvalues, sorted by their magnitude, are decaying proportional to j^{-2} (where j is the number of the eigenvalue) which induces strong tractability as shown in [40] and may explain the low effective truncation dimension.

In summary, one can see that the use of the different model components leads to comparably small changes in the expected value but it significantly affects the variance, the smoothness and the effective dimension of the function Q_K. A low effective dimension is thereby a property which ALM problems in life insurance seem to have in common.

4.3 Impact of the Model Components

We now apply MC, QMC and SG methods to compute the integral (2) for all eleven setups. Thereby, we compare

- MC simulation,
- QMC integration based on Sobol point sets (see [42]) and
- dimension-adaptive SG integration (see [15]) based on the Gauss-Hermite rule.

In various preliminary numerical experiments, the latter methods turned out to be the most efficient representatives of several QMC variants (we compared Halton, Faure and Sobol low discrepancy point sets and three different lattice rules[13] with and without randomisation) and of several SG

[12] It is more difficult and expensive to compute the superposition than the truncation dimension so that we here have to restrict the maximum dimension to $d = 32$.

[13] The lattice rules from [39] sometimes yield even more precise results than Sobol points if good weights are used in their CBC construction. The selection of good weights is a priori not clear, however, and is thus not further investigated here.

no.	1	2	3	4	5	6	7	8	9	10	11
sample setup	basic setup	mort. surr.	non-homog.	high vola.	moder. bonus	aggr. bonus	share-holder	CIR model	GBM+ CIR	CPPI strat.	comp. model
$\mathbb{E}$	5.9	6.1	6.6	5.9	5.8	5.6	5.8	4.6	4.9	4.7	5.7
σ	38	37	46	359	36	33	22	0.1	0.4	0.2	0.5
s	∞	∞	∞	∞	0	0	0	0	0	0	0
d	16	16	16	16	16	16	16	16	32	32	32
d_t	1	1	1	1	1	1	4	7	7	9	7
MC method											
rate	0.52	0.49	0.45	0.5	0.41	0.49	0.53	0.49	0.52	0.51	0.49
const	0.63	0.73	0.52	1.88	0.32	0.64	0.59	0.05	0.1	0.08	0.1
QMC method											
rate	0.77	0.8	0.75	0.8	0.79	0.7	0.77	0.71	0.8	0.8	0.79
const	0.5	0.53	0.45	2.26	0.5	0.25	0.64	0.02	0.05	0.05	0.04
SG method											
rate	3.01	2.33	2.37	2.26	0.35	-0.2	0.09	1.22	1.31	1.63	1.62
const	0.52	0.07	0.1	3.4	0.01	2e-3	0.11	2e-4	4e-3	0.01	0.01

Table 2: Results for the eleven setups with $K = 16$ periods: Expected values $\mathbb{E}$, variances σ, smoothness parameters s, nominal dimensions d and effective dimensions d_t of the equity Q_K. Moreover, convergence rates (rate) and constants (const) of the MC, QMC and SG method.

variants (we compared the trapezoidal, Clenshaw-Curtis, Gauss-Patterson, Gauss-Legendre and Gauss-Hermite rules and different grid refinement strategies), respectively. The MC method is applied without variance reduction techniques since good control variates or importance sampling functions are difficult to obtain for complicated and recursively defined integrands as in our examples. Further preliminary numerical experiments, see also [12], indicate that significant variance reductions can be achieved by the use of antithetic variates. In this way, the error constant of the MC and the QMC method can be improved but not their convergence rates. Antithetic variates have no effect on the SG method as sparse grids are antithetic by construction.

To demonstrate the impact of the nominal dimension d on the performance of the numerical methods, we again distinguish the two cases $K = 16$ and $K = 128$. They correspond to integrals, where the nominal dimension ranges from 16 to 256, see Table 2 and Table 3. Here, we again use the Brownian bridge path construction for the stock prices and short interest rates to obtain low effective dimensions avoiding the additional computational costs of the PCA and EVD constructions. To measure the accuracy of the three different numerical approaches we proceeded as follows: We approximate the integral (2) with $n = 1, 2, 4, \ldots, 2^{18}$ MC and QMC samples. As the considered SG method determines the required number of nodes automatically, we here successively refine the approximation until the grid size exceeded 2^{18} nodes. By a comparison of these results with reference solutions, the convergence rates and the error constants of the numerical methods are then computed using a least square fit. To eliminate the influence of the initial seed in the MC method, we show the average convergence rates and constants which are obtained after twenty independent runs of the MC method with different seeds.

For the eleven setups and the three numerical methods, the convergence rates and the constants are displayed in Table 2 and Table 3 for the cases $K = 16$ and $K = 128$, respectively. One can see that the MC method on average converges with the rate $1/2$ independently of the selection of the model parameters and of the number of time steps while the average constant of the approximation significantly varies from setup to setup. This is explained by the different variances σ of the considered setups and the fact that the expected MC error is proportional to the ratio $\sigma/\sqrt{n}$. For instance, the constants in the setups 8 – 11, which are of comparably small variance, are considerably smaller than, e.g., the constant in setup 4 which is of particular high variance. This

no.	1	2	3	4	5	6	7	8	9	10	11
sample setup	basic setup	mort. surr.	non-homog.	high vola.	moder. bonus	aggr. bonus	share-holder	CIR model	GBM+ CIR	CPPI strat.	comp. model
$\mathbb{E}$	26.5	26	22.8	26.5	14.2	6.6	24.2	10.9	15.3	12.7	10.2
σ	1233	937	791	16627	596	437	490	155	138	160	17
s	∞	∞	∞	∞	0	0	0	0	0	0	0
d	128	128	128	128	128	128	128	128	256	256	256
d_t	1	1	1	1	8	23	16	8	15	15	15
MC method											
rate	0.5	0.51	0.53	0.48	0.51	0.51	0.5	0.48	0.48	0.52	0.49
const	0.79	0.84	1.01	2.45	1.42	2	0.57	0.64	0.42	0.7	0.22
QMC method											
rate	0.81	0.8	0.71	0.69	0.81	0.71	0.72	0.73	0.75	0.75	0.78
const	0.93	0.82	0.6	3.08	0.6	0.93	0.54	0.5	0.44	0.44	0.14
SG method											
rate	1.65	1.64	1.56	1.5	0.14	0.21	0.1	0.63	0.29	0.33	0.15
const	2.5	1.58	0.99	70.7	0.28	3.31	1.36	0.07	0.37	0.04	0.01

Table 3: Results for the eleven setups with $K = 128$ periods: Expected values $\mathbb{E}$, variances σ, smoothness parameters s, nominal dimensions d and effective dimensions d_t of the equity Q_K. Moreover, convergence rates (rate) and constants (const) of the MC, QMC and SG method.

means that also the number of scenarios, which have to be simulated to obtain a prescribed precision, varies from setup to setup according to its variance. Computational results not displayed in Table 2 and Table 3 show that with 500 scenarios on average only one digit accuracy, i.e., a relative error of about 10%, can be expected.[14] For two digits precision, about $20,000$ sample points are required on average. Three digits accuracy are only attained in very few cases with the considered maximum number of sample points of 2^{18}.

One can further see that the QMC method outperforms MC simulation in all setups. It converges faster with a convergence rate of $0.7 - 0.8$ and has also smaller or comparably large error constants. It is therefore also more accurate than MC even for small sample sizes. On average it suffices to generate about 100 scenarios for one digit, $1,000$ scenarios for two digits and $10,000$ scenarios for three digits precision, i.e., low accuracy requirements of two digits precision are obtained by the QMC method about twenty times faster than by MC simulation. For higher accuracy requirements, the advantage of the QMC method is even more pronounced. Furthermore, an inspection of the high-dimensional case $K = 128$ shows that the QMC convergence rate as well as the constant of the approximation is almost independent of the dimension. These results can not be explained by the classical QMC convergence theory but by the low effective dimension of the ALM problems, which we reported in Section 4.2. We further see that the fast convergence behaviour of the QMC method is not affected by the smoothness parameter s. This shows on the other hand that the QMC method can hardly profit from setups with a high degree of smoothness $s = \infty$.

One can finally see that the performance of the SG method varies significantly from setup to setup. We observe that it is the by far most efficient method with a very high convergence rate of up to three for all setups which lead to smooth integrands with $s = \infty$. In the moderately high dimensional case $K = 16$ also the constants (except in setup 4) are clearly lower than the constants of the MC and QMC approximations. In these cases three digits precision (and more) are already attained with only about 50 points if $s = \infty$. The convergence rates and the constants slightly deteriorate with rising dimensions, though, showing that the curse of dimension can not be completely avoided by the SG method. Higher variances seem not to affect the convergence rates

[14] Note that we show the average results of 20 independent MC runs. Single runs of the MC method may be much better as well as much worse than the reported results.

of the SG method but lead to increasing constants of the approximation, see, e.g., setup 4. With respect to the smoothness of the integrands, we see that the low degree of regularity has a much more pronounced impact on the SG convergence in the setups 5 – 7 than in the setups 8 – 11. This is explained by the fact that in the setups 8 – 11 the arising maximum and minimum operators in the model equation M2 and the absolute value in the model equation C1 only apply in very rare cases (e.g. if the discrete version of the CIR model produces negative interest rates). In the setups 5 – 7, the non-smooth model equations are of higher importance such that we consequently observe only a very slow or even no convergence. To ensure a satisfactory performance of the SG method in these cases, the smoothness must first be recovered by suitable smoothing transformations (e.g., by a smoothing of maximum and minimum operators or by a decomposition of the integration domain into domains where the integrand is smooth), which are, however, not investigated in this article.

4.4 Impact of the Path Generating Methods

We now demonstrate that the path generating method f_B in (3) has a significant influence on the effective dimension and on the performance of the deterministic numerical methods. We here consider the most complex parameter setup, the compound model of setup 11.

We compare the component-wise application of the random walk (RW), the Brownian bridge (BB), and the principal component construction (PCA) and, in addition, the eigenvalue decomposition (EVD), see Section 3.5. The truncation dimensions d_t which result from these four different constructions of the short interest rates and stock prices are shown in Table 4 for different nominal dimensions d.

	Truncation dimension d_t				QMC convergence rates				SG convergence rates			
d	RW	BB	PCA	EVD	RW	BB	PCA	EVD	RW	BB	PCA	EVD
16	16	3	10	9	0.62	0.83	0.84	0.81	2.1	1.98	1.61	1.39
32	32	7	12	12	0.59	0.79	0.80	0.89	1.51	1.62	1.28	1.79
64	64	8	16	15	0.49	0.75	0.89	0.82	0.13	0.49	0.57	0.26
128	126	11	22	20	0.47	0.84	0.87	0.92	0.08	0.13	0.19	0.18

Table 4: Truncation dimensions d_t of the ALM model in setup 11 and the convergence rates of the QMC and SG method using different path constructions.

One can see that the BB, PCA and EVD path constructions all lead to effective dimensions d_t which are much smaller than the nominal dimensions d and are only slightly increasing with increasing d. If instead the RW discretisation is used then the effective dimension is nearly equal to the nominal dimension. PCA and EVD lead to almost the same results which is explained by the rather small correlation $\rho = -0.1$ between the two underlying Brownian motions. The lowest effective dimensions are achieved by the BB construction. Similar results hold for the setups 1–10.

We further show in Table 4 the impact of the path construction on the convergence rates of the QMC and SG method. The convergence behaviour of the MC method is not affected by the path construction since the total variance of the problems remains unchanged and is thus not displayed. One can see that the QMC method achieves higher convergence rates than MC if the paths are generated with BB, PCA or EVD. In these cases the rates are almost identical ranging from 0.75 to 0.92 and show almost no dependence on the nominal dimension d. If the random walk construction is used instead, then the convergence rates of the QMC method deteriorate with increasing d and no longer outperform the MC rate of 1/2 for $d \geq 64$. This effect illustrates the importance of the path generating method if QMC methods are applied to ALM simulations. It even more significantly affects other QMC points sets, like, e.g., the Halton sequence, whose points are in higher dimensions not as uniformly distributed as the Sobol points. One can finally see that the behaviour of the SG method is less clearly related to the truncation dimension d_t and to the path construction[15] but is rather affected by the nominal dimension d. While the SG method

[15]The reason are two different interacting effects. The BB, PCA and EVD constructions lead to integrands of

attains high convergence rates larger than one in the moderately high dimensional case $d \leq 32$, the rates deteriorate with increasing d and only a very slow or even no convergence is observed in the higher dimensional cases.

4.5 Simulation of the Compound Model

We finally consider the simulation of the most complex parameter setup, the compound model of setup 11 in more detail. To this end, we display the convergence behaviour of the MC, QMC and SG method for this setup in Figure 5. There, the number of function evaluations is shown which is required by each of the three numerical methods to obtain a fixed accuracy. We again use the Brownian bridge path construction and consider the two cases $K = 16$ and $K = 128$ which correspond to integration problems with nominal dimensions $d = 32$ and $d = 256$ and effective dimensions $d_t = 7$ and $d_t = 15$, respectively, see Table 2 and Table 3.

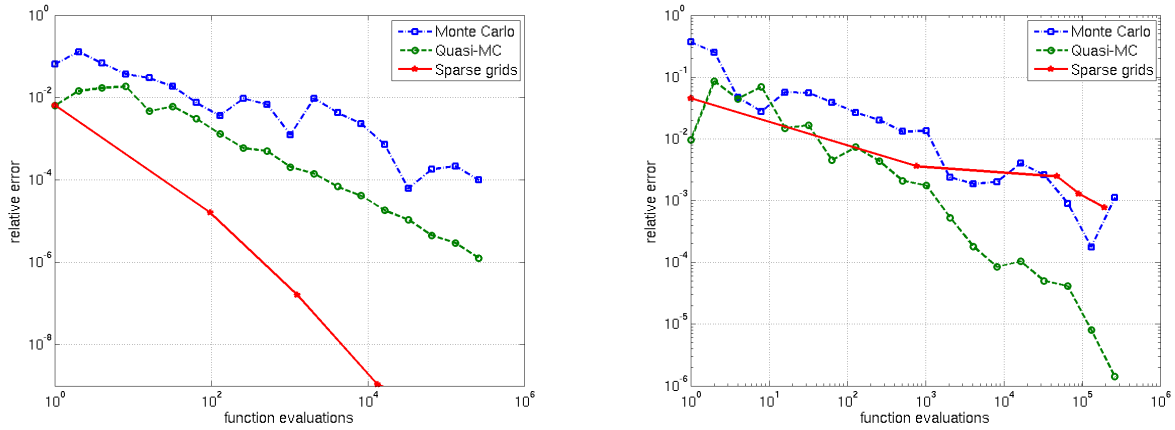

Figure 5: Relative errors and required number of function evaluations of the different numerical approaches to compute the expected value (2) with $d = 32$ (left) and $d = 256$ (right) for setup 11.

Here, the QMC method clearly outperforms MC simulation with a convergence rate close to one and independent of the dimension. If $1,000$ function evaluations are used, the QMC method is about 10 times as accurate as the MC method with the same number of scenarios. One can further see that the convergence of the QMC method is also less erratic than the convergence of the MC method. For $d = 32$, the SG method is the by far most efficient method for this setup. With $1,000$ function evaluations already an relative accuracy of 10^{-7} is achieved, which is $1,000$ times more accurate than the result of the QMC method. The performance of the SG method deteriorates for very high dimensions, though, and is not competitive to QMC for $d = 256$.

5 Concluding Remarks

In this article, we showed that deterministic integration schemes, such as quasi-Monte Carlo and sparse grid methods can be very efficiently applied to the numerical simulation of ALM models in life insurance. As a benchmark model for the different numerical approaches, we used a general model framework for the asset-liability management of portfolios of life insurance products. The model incorporates fairly general product characteristics, a surrender option, a reserve-dependent bonus declaration, a dynamic capital allocation and a two-factor stochastic capital market model. Numerical experiments with several different parameter setups demonstrate that Sobol quasi-Monte

low truncation dimension but with kinks which are not parallel to the coordinate axes. In the RW construction the integrands are of high truncation dimension but some of the kinks are axis-parallel. SG profit from low truncation dimension but suffer from kinks which are not axis-parallel.

Carlo and dimension-adaptive sparse grid methods often significantly outperform Monte Carlo simulation. The quasi-Monte Carlo method in combination with the Brownian bridge path construction converges nearly independently of the dimension, converges faster and less erratic than Monte Carlo and produces more precise results even for high dimensions and complex models.

The results further indicate that sparse grid integration is not suited as a black box method for the numerical simulation of ALM models in life insurance as its performance is comparably sensitive to different choices of model parameters and components which affect the smoothness of the integrand. For ALM models which are sufficiently smooth or which can be transformed such that they are sufficiently smooth, the sparse grid method constitutes an extremely efficient simulation approach and can outperform Monte Carlo and quasi-Monte Carlo methods by several orders of magnitude, though.

To explain the efficiency of the deterministic methods we computed the effective dimension of various ALM problems with and without dimension reduction techniques. The results indicate that a low effective dimension is a common feature of ALM problems in life insurance in the truncation as well as in the superposition sense. We believe that this property is maintained also in more complex ALM models and can be exploited to speed up the numerical simulation of such models in practise. In this article, we focused on a reduction of the overall run times of ALM simulations by the use of deterministic integration methods. Variance reduction techniques, different time discretisation schemes and model approximations will be the topic of future research.

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