HODGE DECOMPOSITION FOR TWO-DIMENSIONAL TIME-HARMONIC MAXWELL'S EQUATIONS: IMPEDANCE BOUNDARY CONDITION*

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Abstract. We extend the Hodge decomposition approach for the cavity problem of two-dimensional time-harmonic Maxwell's equations to include the impedance boundary condition, with anisotropic electric permittivity and sign changing magnetic permeability. We derive error estimates for a P_1 finite element method based on the Hodge decomposition approach and present results of numerical experiments that involve metamaterials and electromagnetic cloaking. The well-posedness of the cavity problem when both electric permittivity and magnetic permeability can change sign is also discussed.

1. Introduction

Let Ω be a polygonal domain in $\mathbb{R}^2$ whose boundary is the union of two disjoint closed subsets $\Gamma_{\rm pc}$ and $\Gamma_{\rm imp}$. We will consider the following cavity problem for the time-harmonic Maxwell equations: Find $\boldsymbol{u} \in H_{\rm imp}(\mathrm{curl};\Omega;\Gamma_{\rm imp}) \cap H_0(\mathrm{curl};\Omega;\Gamma_{\rm pc}) \cap H(\mathrm{div}^0;\Omega;\epsilon)$ such that

$$(\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v}) - k^2(\epsilon\boldsymbol{u},\boldsymbol{v}) - ik\langle\lambda\boldsymbol{n}\times\boldsymbol{u},\boldsymbol{n}\times\boldsymbol{v}\rangle_{\Gamma_{\rm imp}} = (\boldsymbol{f},\boldsymbol{v}) + \langle g,\boldsymbol{n}\times\boldsymbol{v}\rangle_{\Gamma_{\rm imp}} \tag{1.1}$$

for all $\boldsymbol{v} \in H_{\rm imp}(\mathrm{curl};\Omega;\Gamma_{\rm imp}) \cap H_0(\mathrm{curl};\Omega;\Gamma_{\rm pc}) \cap H(\mathrm{div}^0;\Omega;\epsilon)$. (The definitions of these function spaces will be given below.)

Here $\boldsymbol{u}$ is the electric field (in a transverse magnetic problem), ϵ is the electric permittivity, μ is the the magnetic permeability, $k > 0$ is the frequency, $i = \sqrt{-1}$, $\boldsymbol{f}$ is the electric current density, $1/\lambda$ is the impedance on $\Gamma_{\rm imp}$, g is a magnetic field intensity on $\Gamma_{\rm imp}$, $\boldsymbol{n}$ is the outward pointing unit normal along $\partial\Omega$, $(\cdot,\cdot)$ is the inner product for the complex function space $L_2(\Omega)$ (or $[L_2(\Omega)]^2$), and $\langle\cdot,\cdot\rangle_{\Gamma_{\rm imp}}$ is the inner product for the complex function space $L_2(\Gamma_{\rm imp})$. The impedance (respectively perfectly conducting) boundary condition is imposed on $\Gamma_{\rm imp}$ (respectively $\Gamma_{\rm pc}$). In the special case where $\Gamma_{\rm imp}$ is the outer boundary of Ω and $\Gamma_{\rm pc}$ is the inner boundary of Ω, the problem described by (1.1) corresponds to a truncated scattering problem where $\Gamma_{\rm pc}$ is the boundary of the scatterer(s) and $\Gamma_{\rm imp}$ is an artificial boundary where the impedance boundary condition acts as an absorbing boundary condition [1, 2].

1991 *Mathematics Subject Classification.* 65N30, 65N15, 35Q61.

Key words and phrases. time-harmonic Maxwell equations, Hodge decomposition, finite element method, metamaterials, electromagnetic cloaking.

*The work of the first and third authors was supported in part by the National Science Foundation under grant no. DMS-13-19172. The work of the second author was supported by a fellowship within the Postdoc-Program of the German Academic Exchange Service (DAAD).

Published in special issue of Math. Methods Appl. Sci., Volume 40, Issue 2, pages 370–390, 2017. Available online at wileyonlinelibrary.com, http://dx.doi.org/10.1002/mma.3398.

We assume that $\boldsymbol{f} \in [L_2(\Omega)]^2$, λ is a smooth strictly positive function defined on $\Gamma_{\rm imp}$, $g \in L_2(\Gamma_{\rm imp})$, and, for most of the paper, that ϵ is a smooth real symmetric positive-definite (SPD) 2×2 tensor field defined on $\bar{\Omega}$ and the real-valued functions μ and $1/\mu$ both belong to $L_\infty(\Omega)$. The conditions on ϵ and μ will be relaxed for the electromagnetic cloaking simulation in Section 4.5, and we will even allow ϵ to change sign in Section 5.

The spaces $H_{\rm imp}(\mathrm{curl};\Omega;\Gamma_{\rm imp})$, $H_0(\mathrm{curl};\Omega;\Gamma_{\rm pc})$ and $H(\mathrm{div}^0;\Omega;\epsilon)$ are defined as follows:

$$
\begin{aligned}
&H(\mathrm{curl};\Omega) = \{\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in [L_2(\Omega)]^2 : \nabla\times\boldsymbol{v} = \partial v_2/\partial x_1 - \partial v_1/\partial x_2 \in L_2(\Omega)\},\\
&H_{\rm imp}(\mathrm{curl};\Omega;\Gamma_{\rm imp}) = \{\boldsymbol{v} \in H(\mathrm{curl};\Omega) : \boldsymbol{n}\times\boldsymbol{v}|_{\Gamma_{\rm imp}} \in L_2(\Gamma_{\rm imp})\},\\
&H_0(\mathrm{curl};\Omega;\Gamma_{\rm pc}) = \{\boldsymbol{v} \in H(\mathrm{curl};\Omega) : \boldsymbol{n}\times\boldsymbol{v}|_{\Gamma_{\rm pc}} = 0\},\\
&H(\mathrm{div}^0;\Omega) = \{\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in [L_2(\Omega)]^2 : 0 = \nabla\cdot\boldsymbol{v} = \partial v_1/\partial x_1 + \partial v_2/\partial x_2\},\\
&H(\mathrm{div}^0;\Omega;\epsilon) = \{\boldsymbol{v} \in [L_2(\Omega)]^2 : \epsilon\boldsymbol{v} \in H(\mathrm{div}^0;\Omega)\},
\end{aligned}
$$

where $\boldsymbol{n} = [n_1\ n_2]^t$ is the outward pointing unit normal along $\partial\Omega$ and $\boldsymbol{n}\times\boldsymbol{v} = n_1v_2 - n_2v_1$ is the tangential component of $\boldsymbol{v}$.

We will use the Hodge/Helmholtz decomposition of divergence-free vector fields to reduce the problem (1.1) to several scalar elliptic boundary value problems, which can be solved numerically by standard finite element methods. This idea was first proposed in [3] for the time-harmonic Maxwell equations with the perfectly conducting boundary condition, and was exploited in [4] and [5] for the construction of adaptive and multigrid algorithms. In this paper, besides treating more general boundary conditions, we also consider more general ϵ and μ. In particular, μ is allowed to be both positive and negative in Ω throughout the paper, and ϵ is also allowed to change sign in the last section of the paper, which can occur if a part of Ω is occupied by certain metamaterials [6, 7]. Therefore the well-posedness of (1.1) does not follow immediately from the standard theory that can be found for example in [2].

We note that the well-posedness of the time-harmonic Maxwell equations for metamaterials has been investigated by many authors. For example, the well-posedness of three-dimensional cavity problems with lossy materials and the impedance boundary condition were studied in [8, 9], and an electromagnetic scattering problem with sign changing constants ϵ and μ in $\mathbb{R}^3$ was treated in [10]. In the case of the perfectly conducting boundary condition, the well-posedness for cavity problems for lossless materials in two and three dimensions was investigated in [11, 12, 13, 14]. We will show that the Hodge decomposition approach can provide yet another treatment of the well-posedness involving metamaterials.

The rest of the paper is organized as follows. We develop in Section 2 the Hodge decomposition approach and show that (1.1) is well-posed if the positive number k does not belong to a discrete subset of $(0,\infty)$. Error estimates for a P_1 finite element method based on the Hodge decomposition are derived in Section 3. Numerical results that involve metamaterials and electromagnetic cloaking are presented in Section 4. In Section 5, under stability conditions on the sesquilinear form $(\nabla\times\cdot,\epsilon^{-1}\nabla\times\cdot)$ (cf. (5.2) and (5.3)), the well-posedness result for (1.1) is extended to the general case where both the electric permittivity ϵ and the magnetic permeability

μ can change sign, provided that $\Gamma_{\text{imp}} \neq \emptyset$. We end with some concluding remarks in Section 6. Throughout the paper we will follow standard notation for differential operators, function spaces and norms that can be found for example in [15, 16].

To facilitate the presentation, we recall here that the tangential trace map $\boldsymbol{v} \longrightarrow \boldsymbol{n} \times \boldsymbol{v}$ can be extended from $[C^\infty(\bar{\Omega})]^2$ to $H(\text{curl};\Omega)$ such that $\boldsymbol{n} \times \boldsymbol{v} \in H^{-\frac{1}{2}}(\partial\Omega)$ and we have an integration by parts formula [17, Theorem 2.11]

$$(\zeta, \nabla \times \boldsymbol{v}) = (\nabla \times \zeta, \boldsymbol{v}) + \langle\langle \zeta, \boldsymbol{n} \times \boldsymbol{v} \rangle\rangle \qquad \forall\, \zeta \in H^1(\Omega),\ \boldsymbol{v} \in H(\text{curl};\Omega), \tag{1.2}$$

where $\nabla \times \zeta = [\partial\zeta/\partial x_2, -\partial\zeta/\partial x_1]^t$ and $\langle\langle \cdot, \cdot \rangle\rangle$ is the duality pairing of $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$.

In particular, we have the orthogonality relation

$$(\nabla \times \zeta, \nabla\eta) = 0 \quad \forall\, \zeta \in H^1(\Omega) \tag{1.3}$$

if the trace of $\eta \in H^1(\Omega)$ is a constant on each component of $\partial\Omega$, since $\nabla\eta \in H_0(\text{curl};\Omega) = \{\boldsymbol{w} \in H(\text{curl};\Omega) : \boldsymbol{n} \times \boldsymbol{w} = 0 \text{ on } \partial\Omega\}$ and we can take $\boldsymbol{v} = \nabla\eta$ in (1.2). The integration by parts formula below is a special case of (1.2):

$$(\zeta, \nabla \times \boldsymbol{v}) = (\nabla \times \zeta, \boldsymbol{v}) + \langle \zeta, \boldsymbol{n} \times \boldsymbol{v} \rangle_{\Gamma_{\text{imp}}} \qquad \forall\, \zeta \in H^1(\Omega), \tag{1.4}$$

$\boldsymbol{v} \in H_{\text{imp}}(\text{curl};\Omega;\Gamma_{\text{imp}}) \cap H_0(\text{curl};\Omega;\Gamma_{\text{pc}})$.

Throughout this paper we use C (with or without subscript) to denote a generic positive constant that does not depend on the mesh size but which may depend on ϵ, μ and k.

2. The Hodge Decomposition Approach

Let $m \geq 0$ be the Betti number of Ω ($m = 0$ if Ω is simply connected). We will denote the outer boundary of Ω by Γ_0 and the m components of the inner boundary of Ω by $\Gamma_1, \ldots, \Gamma_m$. The functions $\varphi_1, \ldots, \varphi_m \in H^1(\Omega)$ are determined by the following Dirichlet boundary value problems:

$$(\epsilon\nabla\varphi_j, \nabla v) = 0 \qquad\qquad \forall\, v \in H_0^1(\Omega), \tag{2.1a}$$

$$\varphi_j\big|_{\Gamma_0} = 0, \tag{2.1b}$$

$$\varphi_j\big|_{\Gamma_\ell} = \delta_{j\ell} = \begin{cases} 1 & j = \ell \\ 0 & j \neq \ell \end{cases} \qquad \text{for} \quad 1 \leq \ell \leq m. \tag{2.1c}$$

The space of ϵ-harmonic functions spanned by the functions $\varphi_1, \ldots, \varphi_m$ is denoted by $\mathcal{H}(\Omega;\epsilon)$.

Remark 2.1. Note that, for any $\varphi \in \mathcal{H}(\Omega;\epsilon)$, we have $\nabla\varphi \in H(\text{div}^0;\Omega;\epsilon)$ by (2.1a) and $\nabla\varphi \in H_0(\text{curl};\Omega)$ by (2.1b)–(2.1c). Therefore $\nabla\varphi$ belongs to $H_{\text{imp}}(\text{curl};\Omega;\Gamma_{\text{imp}}) \cap H_0(\text{curl};\Omega;\Gamma_{\text{pc}}) \cap H(\text{div}^0;\Omega;\epsilon)$ for any $\varphi \in \mathcal{H}(\Omega;\epsilon)$.

Remark 2.2. Let H^1_Γ be the subspace of $H^1(\Omega)$ whose members vanish on Γ_0 and have constant traces on $\Gamma_1, \ldots, \Gamma_m$. Then $\mathcal{H}(\Omega;\epsilon)$ is a subspace of H^1_Γ and $H^1_\Gamma = H^1_0(\Omega) \overset{\perp}{\oplus} \mathcal{H}(\Omega;\epsilon)$ with respect to the inner product $(\nabla\cdot, \nabla\cdot)$ on H^1_Γ.

2.1. **Reduction to scalar elliptic problems.** Assume for the moment that $\boldsymbol{u} \in H_{\mathrm{imp}}(\mathrm{curl};\Omega;\Gamma_{\mathrm{imp}}) \cap H_0(\mathrm{curl};\Omega;\Gamma_{\mathrm{pc}}) \cap H(\mathrm{div}^0;\Omega;\epsilon)$ is a solution of (1.1). By the Hodge decomposition for $H(\mathrm{div}^0;\Omega;\epsilon)$ [3, Lemma 2.3], we can write in a unique way

$$\boldsymbol{u} = \epsilon^{-1}\nabla \times \phi + \sum_{j=1}^{m} c_j \nabla\varphi_j, \tag{2.2}$$

where $\phi \in H^1(\Omega)$ satisfies $(\phi, 1) = 0$ and $c_1, \ldots, c_m$ are constants. Our goal is to derive elliptic boundary value problems that determine the function ϕ and the coefficients $c_1, \ldots, c_m$ in (2.2).

Let $[L_2(\Omega;\epsilon)]^2$ be the space $[L_2(\Omega)]^2$ equipped with the inner product $(\cdot,\cdot)_\epsilon$ defined by

$$(\boldsymbol{v}, \boldsymbol{w})_\epsilon = \int_\Omega \epsilon \boldsymbol{v} \cdot \bar{\boldsymbol{w}}\, dx$$

and $Q : [L_2(\Omega;\epsilon)]^2 \longrightarrow H(\mathrm{div}^0;\Omega;\epsilon)$ be the orthogonal projection with respect to the inner product $(\cdot,\cdot)_\epsilon$. Note that $H(\mathrm{div}^0;\Omega;\epsilon)$ is the orthogonal complement of $\nabla H_0^1(\Omega)$ in $[L_2(\Omega;\epsilon)]^2$, i.e., $[L_2(\Omega;\epsilon)]^2 = \nabla H_0^1(\Omega) \overset{\perp}{\oplus} H(\mathrm{div}^0;\Omega;\epsilon)$.

Lemma 2.3. *If* $\boldsymbol{u}$ *is a solution of* (1.1)*, then we have*

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{u}) - k^2\epsilon\boldsymbol{u} = \epsilon Q(\epsilon^{-1}\boldsymbol{f}) \tag{2.3}$$

in the sense of distributions.

Proof. Let $\boldsymbol{\zeta} \in [C_c^\infty(\Omega)]^2$ be a C^∞ vector field with compact support in Ω. Then we have $\boldsymbol{\zeta} \in H_0(\mathrm{curl};\Omega)$, $Q\boldsymbol{\zeta} \in H(\mathrm{div}^0;\Omega;\epsilon)$, $\boldsymbol{\zeta} - Q\boldsymbol{\zeta} \in \nabla H_0^1(\Omega) \subset H_0(\mathrm{curl};\Omega)$, and

$$Q\boldsymbol{\zeta} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega;\epsilon) \subset H_{\mathrm{imp}}(\mathrm{curl};\Omega;\Gamma_{\mathrm{imp}}) \cap H_0(\mathrm{curl};\Omega;\Gamma_{\mathrm{pc}}) \cap H(\mathrm{div}^0;\Omega;\epsilon).$$

Furthermore, since $\nabla \times \nabla H_0^1(\Omega) = \{0\}$ and $H(\mathrm{div}^0;\Omega;\epsilon)$ is orthogonal to $\nabla H_0^1(\Omega)$ in $[L_2(\Omega;\epsilon)]^2$, we have

$$\nabla \times (\boldsymbol{\zeta} - Q\boldsymbol{\zeta}) = 0 = (\boldsymbol{u}, \boldsymbol{\zeta} - Q\boldsymbol{\zeta})_\epsilon = (\epsilon\boldsymbol{u}, \boldsymbol{\zeta} - Q\boldsymbol{\zeta}),$$

which together with (1.1) implies

$$\begin{aligned}
&(\mu^{-1}\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{\zeta}) - k^2(\epsilon\boldsymbol{u}, \boldsymbol{\zeta}) \\
&\quad = \big(\mu^{-1}\nabla \times \boldsymbol{u}, \nabla \times (Q\boldsymbol{\zeta} + (\boldsymbol{\zeta} - Q\boldsymbol{\zeta})\big) - k^2\big(\epsilon\boldsymbol{u}, Q\boldsymbol{\zeta} + (\boldsymbol{\zeta} - Q\boldsymbol{\zeta})\big) \\
&\quad = (\mu^{-1}\nabla \times \boldsymbol{u}, \nabla \times Q\boldsymbol{\zeta}) - k^2(\epsilon\boldsymbol{u}, Q\boldsymbol{\zeta}) \\
&\quad = (\boldsymbol{f}, Q\boldsymbol{\zeta}) = (\epsilon^{-1}\boldsymbol{f}, Q\boldsymbol{\zeta})_\epsilon = \big(Q(\epsilon^{-1}\boldsymbol{f}), \boldsymbol{\zeta}\big)_\epsilon = \big(\epsilon Q(\epsilon^{-1}\boldsymbol{f}), \boldsymbol{\zeta}\big).
\end{aligned}$$

□

Remark 2.4. If $\boldsymbol{f} \in H(\mathrm{div}^0;\Omega)$, then $\epsilon^{-1}\boldsymbol{f} \in H(\mathrm{div}^0;\Omega;\epsilon)$ and hence $\epsilon Q(\epsilon^{-1}\boldsymbol{f}) = \epsilon(\epsilon^{-1}\boldsymbol{f}) = \boldsymbol{f}$. Therefore $\epsilon Q(\epsilon^{-1}\cdot)$ is a projection from $[L_2(\Omega)]^2$ onto $H(\mathrm{div}^0;\Omega)$.

Let the function ξ be defined by

$$\xi = \mu^{-1}\nabla \times \boldsymbol{u}. \tag{2.4}$$

Then $\xi \in H^1(\Omega)$ and

$$\nabla \times \xi - k^2\epsilon\boldsymbol{u} = \epsilon(Q\epsilon^{-1}\boldsymbol{f}) \quad \text{in } \Omega \tag{2.5}$$

by (2.3).

Let $\boldsymbol{v} \in H_{\mathrm{imp}}(\mathrm{curl};\Omega;\Gamma_{\mathrm{imp}}) \cap H_0(\mathrm{curl};\Omega;\Gamma_{\mathrm{pc}})$ be arbitrary. We have $Q\boldsymbol{v} \in H(\mathrm{div}^0;\Omega;\epsilon)$, $\boldsymbol{v}-Q\boldsymbol{v} \in \nabla H_0^1(\Omega) \subset H_0(\mathrm{curl};\Omega)$, $Q\boldsymbol{v} \in H_{\mathrm{imp}}(\mathrm{curl};\Omega;\Gamma_{\mathrm{imp}}) \cap H_0(\mathrm{curl};\Omega;\Gamma_{\mathrm{pc}}) \cap H(\mathrm{div}^0;\Omega;\epsilon)$, and by repeating the arguments in the proof of Lemma 2.3, we find

$$\begin{aligned}
&(\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v}) - k^2(\epsilon\boldsymbol{u},\boldsymbol{v})\\
&\quad= (\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times Q\boldsymbol{v}) - k^2(\epsilon\boldsymbol{u},Q\boldsymbol{v})\\
&\quad= (\boldsymbol{f},Q\boldsymbol{v}) + ik\langle\lambda\boldsymbol{n}\times\boldsymbol{u},\boldsymbol{n}\times Q\boldsymbol{v}\rangle_{\Gamma_{\mathrm{imp}}} + \langle g,\boldsymbol{n}\times Q\boldsymbol{v}\rangle_{\Gamma_{\mathrm{imp}}}\\
&\quad= (\epsilon Q(\epsilon^{-1}\boldsymbol{f}),\boldsymbol{v}) + ik\langle\lambda\boldsymbol{n}\times\boldsymbol{u},\boldsymbol{n}\times\boldsymbol{v}\rangle_{\Gamma_{\mathrm{imp}}} + \langle g,\boldsymbol{n}\times\boldsymbol{v}\rangle_{\Gamma_{\mathrm{imp}}}.
\end{aligned} \tag{2.6}$$

It then follows from (1.4), (2.3), (2.4) and (2.6) that

$$\langle\xi - ik\lambda\boldsymbol{n}\times\boldsymbol{u},\boldsymbol{n}\times\boldsymbol{v}\rangle_{\Gamma_{\mathrm{imp}}} = \langle g,\boldsymbol{n}\times\boldsymbol{v}\rangle_{\Gamma_{\mathrm{imp}}}$$

for all $\boldsymbol{v} \in H_{\mathrm{imp}}(\mathrm{curl};\Omega;\Gamma_{\mathrm{imp}}) \cap H_0(\mathrm{curl};\Omega)$ and hence

$$\boldsymbol{n}\times\boldsymbol{u} = \frac{i}{k}(g-\xi)/\lambda \quad \text{on } \Gamma_{\mathrm{imp}}. \tag{2.7}$$

Let $\psi \in H^1(\Omega)$ be arbitrary. Then $\epsilon^{-1}\nabla\times\psi$ belongs to $H(\mathrm{div}^0;\Omega;\epsilon)$ and it follows from (1.4), (2.5) and (2.7) that

$$\begin{aligned}
(\boldsymbol{f},\epsilon^{-1}\nabla\times\psi) &= \big(\epsilon Q(\epsilon^{-1}\boldsymbol{f}),\epsilon^{-1}\nabla\times\psi\big)\\
&= \big(\nabla\times\xi - k^2\epsilon\boldsymbol{u},\epsilon^{-1}\nabla\times\psi\big)\\
&= (\nabla\times\xi,\epsilon^{-1}\nabla\times\psi) - k^2\big[(\nabla\times\boldsymbol{u},\psi) - \langle\boldsymbol{n}\times\boldsymbol{u},\psi\rangle_{\Gamma_{\mathrm{imp}}}\big]\\
&= (\nabla\times\xi,\epsilon^{-1}\nabla\times\psi) - k^2(\mu\xi,\psi) + ik\langle(g-\xi)/\lambda,\psi\rangle_{\Gamma_{\mathrm{imp}}}.
\end{aligned}$$

Therefore ξ satisfies

$$\begin{aligned}
&(\nabla\times\xi,\epsilon^{-1}\nabla\times\psi) - k^2(\mu\xi,\psi) - ik\langle\xi/\lambda,\psi\rangle_{\Gamma_{\mathrm{imp}}}\\
&\qquad= (\boldsymbol{f},\epsilon^{-1}\nabla\times\psi) - ik\langle g/\lambda,\psi\rangle_{\Gamma_{\mathrm{imp}}} \quad \forall\,\psi \in H^1(\Omega),
\end{aligned} \tag{2.8}$$

which implies immediately that

$$(\mu\xi,1) - \frac{i}{k}\langle(g-\xi)/\lambda,1\rangle_{\Gamma_{\mathrm{imp}}} = 0. \tag{2.9}$$

Remark 2.5. The problem defined by (2.8) is the weak form of a scalar elliptic problem with a Robin boundary condition on Γ_{imp} and a homogeneous Neumann boundary condition on Γ_{pc}.

Now we turn to the problem that will determine the function ϕ in the Hodge decomposition (2.2). Let $\psi \in H^1(\Omega)$ be arbitrary. We have, by (1.3), (1.4), (2.2), (2.4) and (2.7),

$$\begin{aligned}
(\nabla\times\phi,\epsilon^{-1}\nabla\times\psi) &= \Big(\nabla\times\phi + \epsilon\sum_{j=1}^{m} c_j\nabla\varphi_j,\epsilon^{-1}\nabla\times\psi\Big) = (\epsilon\boldsymbol{u},\epsilon^{-1}\nabla\times\psi)\\
&= (\nabla\times\boldsymbol{u},\psi) - \langle\boldsymbol{n}\times\boldsymbol{u},\psi\rangle_{\Gamma_{\mathrm{imp}}} = (\mu\xi,\psi) - \frac{i}{k}\langle(g-\xi)/\lambda,\psi\rangle_{\Gamma_{\mathrm{imp}}}.
\end{aligned}$$

Thus the function ϕ satisfies the equation

$$(\nabla\times\phi,\epsilon^{-1}\nabla\times\psi) = (\mu\xi,\psi) - \frac{i}{k}\langle(g-\xi)/\lambda,\psi\rangle_{\Gamma_{\mathrm{imp}}} \qquad \forall\,\psi \in H^1(\Omega) \tag{2.10}$$

and the constraint

$$(\phi,1) = 0. \tag{2.11}$$

Remark 2.6. Given $\xi \in H^1(\Omega)$, the problem defined by (2.10)–(2.11) is the weak form of a scalar elliptic problem with a nonhomogeneous Neumann boundary condition on Γ_{imp} and a homogeneous Neumann boundary condition on Γ_{pc}. It is uniquely solvable under the condition (2.9).

In the case where $m \geq 1$, the coefficients $c_1, \ldots, c_m$ in (2.2) are determined by the equations

$$\sum_{j=1}^{m} (\epsilon \nabla \varphi_j, \nabla \varphi_\ell) c_j = -\frac{1}{k^2}(\boldsymbol{f}, \nabla \varphi_\ell) \qquad \text{for} \quad 1 \leq \ell \leq m, \tag{2.12}$$

which are obtained from (1.1) by replacing $\epsilon \boldsymbol{u}$ with $\nabla \times \phi + \epsilon \sum_{j=1}^{m} c_j \nabla \varphi_j$ and by taking $\boldsymbol{v}$ to be $\nabla \varphi_k$ for $k = 1, \ldots, m$. Note that (2.12) is an SPD system since (2.1b) implies that $\varphi = 0$ is the only function in $\mathcal{H}(\Omega; \epsilon)$ that satisfies $(\epsilon \nabla \varphi, \nabla \varphi) = 0$.

Remark 2.7. Observe that the coefficients $c_1, \ldots, c_m$ depend only on the volume source term $\boldsymbol{f}$. In particular, if $\boldsymbol{f}$ equals $\mathbf{0}$, then we have $c_1 = \ldots = c_m = 0$ and the harmonic functions φ_j $(1 \leq j \leq m)$ do not contribute to the solution $\boldsymbol{u}$.

2.2. **Equivalence of the scalar problems with the original problem.** So far we have shown that if $\boldsymbol{u} \in H_{\text{imp}}(\text{curl}; \Omega; \Gamma_{\text{imp}}) \cap H_0(\text{curl}; \Omega; \Gamma_{\text{pc}}) \cap H(\text{div}^0; \Omega; \epsilon)$ satisfies (1.1), then the function ϕ and the coefficients $c_1, \ldots, c_m$ in the Hodge decomposition (2.2) are determined by (2.8), (2.10)–(2.11) and (2.12) (when $m \geq 1$). Conversely, we can show that if $\xi \in H^1(\Omega)$, $\phi \in H^1(\Omega)$ and $c_1, \ldots, c_m \in \mathbb{C}$ satisfy (2.8) and (2.10)–(2.12), and the vector field $\boldsymbol{u}$ is defined by (2.2), then $\boldsymbol{u}$ belongs to $H_{\text{imp}}(\text{curl}; \Omega; \Gamma_{\text{imp}}) \cap H_0(\text{curl}; \Omega; \Gamma_{\text{pc}}) \cap H(\text{div}^0; \Omega; \epsilon)$ and is a solution of (1.1).

Indeed (1.2) and (2.10) imply

$$\nabla \times (\epsilon^{-1} \nabla \times \phi) = \mu \xi \ \text{ in } \Omega \text{ and } \boldsymbol{n} \times (\epsilon^{-1} \nabla \times \phi) = \begin{cases} \frac{i}{k}(g - \xi)/\lambda & \text{on } \Gamma_{\text{imp}} \\ 0 & \text{on } \Gamma_{\text{pc}} \end{cases}. \tag{2.13}$$

Hence $\epsilon^{-1} \nabla \times \phi$ belongs to $H_{\text{imp}}(\text{curl}; \Omega; \Gamma_{\text{imp}}) \cap H_0(\text{curl}; \Omega; \Gamma_{\text{pc}}) \cap H(\text{div}^0; \Omega; \epsilon)$. In view of Remark 2.1, the vector field $\boldsymbol{u}$ defined by (2.2) also belongs to $H_{\text{imp}}(\text{curl}; \Omega; \Gamma_{\text{imp}}) \cap H_0(\text{curl}; \Omega; \Gamma_{\text{pc}}) \cap H(\text{div}^0; \Omega; \epsilon)$. Moreover we have

$$\nabla \times \boldsymbol{u} = \mu \xi \ \text{ in } \Omega \quad \text{and} \quad \boldsymbol{n} \times \boldsymbol{u} = \begin{cases} \frac{i}{k}(g - \xi)/\lambda & \text{on } \Gamma_{\text{imp}} \\ 0 & \text{on } \Gamma_{\text{pc}} \end{cases} \tag{2.14}$$

by (2.2) and (2.13).

Given any $\boldsymbol{v} \in H_{\text{imp}}(\text{curl}; \Omega; \Gamma_{\text{imp}}) \cap H_0(\text{curl}; \Omega; \Gamma_{\text{pc}}) \cap H(\text{div}^0; \Omega; \epsilon)$, we have a Hodge decomposition

$$\boldsymbol{v} = \epsilon^{-1} \nabla \times \psi + \nabla \varphi, \tag{2.15}$$

where $\psi \in H^1(\Omega)$ and $\varphi \in \mathcal{H}(\Omega; \epsilon)$. Then $\nabla \times \boldsymbol{v} = \nabla \times (\epsilon^{-1} \nabla \times \psi)$ in Ω, $\boldsymbol{n} \times \boldsymbol{v} = \boldsymbol{n} \times (\epsilon^{-1} \nabla \times \psi)$ on $\partial\Omega$, and it follows from (1.4), (2.14) and (2.15) that

$$\begin{aligned} (\mu^{-1} \nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) &= (\xi, \nabla \times \boldsymbol{v}) = (\xi, \nabla \times (\epsilon^{-1} \nabla \times \psi)) \\ &= (\nabla \times \xi, \epsilon^{-1} \nabla \times \psi) + \langle \xi, \boldsymbol{n} \times \boldsymbol{v} \rangle_{\Gamma_{\text{imp}}}. \end{aligned} \tag{2.16}$$

From (1.3), (1.4), (2.2), (2.8), (2.12), (2.14) and (2.15) we have

$$
\begin{aligned}
&(\nabla\times\xi,\epsilon^{-1}\nabla\times\psi)\\
&\quad= k^2(\mu\xi,\psi)+(\boldsymbol{f},\epsilon^{-1}\nabla\times\psi)-ik\langle(g-\xi)/\lambda,\psi\rangle_{\Gamma_{\mathrm{imp}}}\\
&\quad= k^2(\nabla\times\boldsymbol{u},\psi)+(\boldsymbol{f},\epsilon^{-1}\nabla\times\psi)-ik\langle(g-\xi)/\lambda,\psi\rangle_{\Gamma_{\mathrm{imp}}}\\
&\quad= k^2\big[(\boldsymbol{u},\nabla\times\psi)+\langle\boldsymbol{n}\times\boldsymbol{u},\psi\rangle_{\Gamma_{\mathrm{imp}}}\big]+(\boldsymbol{f},\epsilon^{-1}\nabla\times\psi)-ik\langle(g-\xi)/\lambda,\psi\rangle_{\Gamma_{\mathrm{imp}}}\\
&\quad= k^2(\epsilon\boldsymbol{u},\epsilon^{-1}\nabla\times\psi)+(\boldsymbol{f},\epsilon^{-1}\nabla\times\psi)\\
&\quad= k^2(\epsilon\boldsymbol{u},\boldsymbol{v})-k^2\Big(\nabla\times\phi+\epsilon\sum_{j=1}^{m}c_j\nabla\varphi_j,\nabla\varphi\Big)+(\boldsymbol{f},\epsilon^{-1}\nabla\times\psi)\\
&\quad= k^2(\epsilon\boldsymbol{u},\boldsymbol{v})-k^2\sum_{j=1}^{m}(\epsilon\nabla\varphi_j,\nabla\varphi)c_j+(\boldsymbol{f},\epsilon^{-1}\nabla\times\psi)\\
&\quad= k^2(\epsilon\boldsymbol{u},\boldsymbol{v})+(\boldsymbol{f},\epsilon^{-1}\nabla\times\psi+\nabla\varphi)\\
&\quad= k^2(\epsilon\boldsymbol{u},\boldsymbol{v})+(\boldsymbol{f},\boldsymbol{v}).
\end{aligned}
\tag{2.17}
$$

From (2.14) we also have

$$\langle\xi,\boldsymbol{n}\times\boldsymbol{v}\rangle_{\Gamma_{\mathrm{imp}}}=ik\langle\lambda\boldsymbol{n}\times\boldsymbol{u},\boldsymbol{n}\times\boldsymbol{v}\rangle_{\Gamma_{\mathrm{imp}}}+\langle g,\boldsymbol{n}\times\boldsymbol{v}\rangle_{\Gamma_{\mathrm{imp}}}. \tag{2.18}$$

Equation (1.1) follows from (2.16)–(2.18).

2.3. **A well-posedness result.**

We can now formulate a well-posedness result for (1.1).

Theorem 2.8. *There exists a discrete (possibly empty) subset $\mathcal{S}_+$ of $\mathbb{R}_+=(0,\infty)$ such that* (1.1) *has a unique solution for $k\in\mathbb{R}_+$ if and only if $k\notin\mathcal{S}_+$.*

Proof. Since (1.1) is equivalent to (2.8)–(2.12), it suffices to consider the unique solvability of the latter and, because (2.10)–(2.11) and (2.12) are always uniquely solvable, we only have to consider the well-posedness of the scalar elliptic problem (2.8).

We can write (2.8) as

$$a(\xi,\psi)=(\boldsymbol{f},\epsilon^{-1}\nabla\times\psi)-ik\langle g/\lambda,\psi\rangle_{\Gamma_{\mathrm{imp}}}\qquad\forall\,\psi\in H^1(\Omega), \tag{2.19}$$

where

$$a(\eta,\psi)=(\nabla\times\eta,\epsilon^{-1}\nabla\times\psi)-k^2(\mu\eta,\psi)-ik\langle\eta/\lambda,\psi\rangle_{\Gamma_{\mathrm{imp}}}\qquad\forall\,\eta,\psi\in H^1(\Omega). \tag{2.20}$$

Since ϵ^{-1} is uniformly SPD on $\overline{\Omega}$, the problem (2.8) is Fredholm for any $k\in\mathbb{C}$. Therefore, according to the analytic Fredholm theorem [18, Theorem VI.14], we only need to show that (2.8) is uniquely solvable for some $k\in\mathbb{C}$.

Suppose $\Gamma_{\mathrm{imp}}=\emptyset$, $k^2=i$ and $\eta\in H^1(\Omega)$ satisfies

$$0=a(\eta,\psi)=(\nabla\times\eta,\epsilon^{-1}\nabla\times\psi)-i(\mu\eta,\psi)\quad\forall\psi\in H^1(\Omega). \tag{2.21}$$

By taking $\psi=\eta$ in (2.21), we see that η must be a constant and hence

$$\eta(\mu,\psi)=0\quad\forall\psi\in H^1(\Omega).$$

In view of our assumptions on μ, this means $\eta=0$ and thus the problem (2.8) is uniquely solvable when $k^2=i$.

We now consider the case where $\Gamma_{\rm imp} \neq \emptyset$. Since $\mu \in L_\infty(\Omega)$, ϵ^{-1} is SPD and the positive function $1/\lambda$ is bounded away from 0, we have a Poincaré-Friedrichs inequality [19, Theorem 2.7.1]

$$\|\eta\|^2_{L_2(\Omega)} \leq C_{\rm PF}\big[(\nabla\times\eta, \epsilon^{-1}\nabla\times\eta) + \langle \eta/\lambda, \eta\rangle_{\Gamma_{\rm imp}}\big] \qquad \forall\,\eta\in H^1(\Omega). \tag{2.22}$$

Therefore for $t \in (0,1)$ sufficiently small, we have

$$t^2|(\mu\eta,\eta)| \leq \frac{1}{2}\big[(\nabla\times\eta, \epsilon^{-1}\nabla\times\eta) + t\langle \eta/\lambda, \eta\rangle_{\Gamma_{\rm imp}}\big] \qquad \forall\,\eta\in H^1(\Omega). \tag{2.23}$$

It follows from (2.22) and (2.23) that there exists a positive number c_t such that

$$\begin{aligned}
&\big|(\nabla\times\eta, \epsilon^{-1}\nabla\times\eta) + t^2(\mu\eta,\eta) + t\langle \eta/\lambda, \eta\rangle_{\Gamma_{\rm imp}}\big| \\
&\qquad \geq \frac{1}{2}\big[(\nabla\times\eta, \epsilon^{-1}\nabla\times\eta) + t\langle \eta/\lambda, \eta\rangle_{\Gamma_{\rm imp}}\big] \\
&\qquad \geq c_t\|\eta\|^2_{H^1(\Omega)} \quad \forall\,\eta\in H^1(\Omega),
\end{aligned}$$

provided $t \in (0,1)$ is sufficiently small. This means the sesquilinear form $a(\cdot,\cdot)$ is coercive for $k = it$ where $t \in (0,1)$ is sufficiently small. Since $a(\cdot,\cdot)$ is obviously bounded, we can apply the Lax-Milgram lemma [20, Theorem 6.6] to conclude that (2.8) has a unique solution for such k. □

Remark 2.9. Note that μ, which appears in a lower order term in the scalar equation (2.8) posed on $H^1(\Omega)$, does not play any role in the Fredholm property of (2.8). In the three-dimensional case the corresponding system is posed on a space that involves $H(\mathrm{div}^0;\Omega;\mu)$ and hence additional conditions on μ are needed if μ is allowed to change sign (cf. [14]).

Remark 2.10. Theorem 2.8 is valid under the assumptions that ϵ and ϵ^{-1} are SPD and bounded. In particular, ϵ can be piecewise smooth. However in that case the regularity of ξ, ϕ and $\varphi_1,\ldots,\varphi_m$ can be very low [21] and their numerical solutions would be very challenging.

Remark 2.11. There are additional conditions on μ that would imply $\mathcal{S}_+$ is the empty set. For example, if μ is a negative function, then $a(\cdot,\cdot)$ is coercive for $k > 0$ and $\mathcal{S}_+ = \emptyset$. If $\Gamma_{\rm imp} \neq \emptyset$ and μ is a piecewise Lipschitz function, then, under appropriate conditions on the subdomains where μ is Lipschitz, one can use unique continuation results for second order elliptic problems [22, Section 17.2] to show that, for any $k > 0$, $a(\eta,\psi) = 0$ for all $\psi \in H^1(\Omega)$ implies $\eta = 0$ (cf. the treatments in [23] and [2, Section 4.6]). Hence in this case we also have $\mathcal{S}_+ = \emptyset$. On the other hand, if μ is positive and $\Gamma_{\rm imp} = \emptyset$, then it is well known that $\mathcal{S}_+$ is an infinite set with ∞ as the only limit point.

Remark 2.12. For $k \in \mathcal{S}_+$, the Fredholm problem (1.1) is solvable if and only if the data satisfy a finite number of compatibility conditions and the solution is unique if an equal number of appropriate constraints are imposed.

Remark 2.13. The proof of Theorem 2.8 actually establishes the well-posedness of (1.1) for $k \in \mathbb{C} \setminus \mathcal{S}$, where $\mathcal{S}$ is a discrete subset of $\mathbb{C}$ and $\mathcal{S}_+ = \mathcal{S} \cap \mathbb{R}_+$.

2.4. **A numerical procedure.** From here on we assume that k belongs to $\mathbb{R}_+\setminus\mathcal{S}_+$. We can then solve (1.1) numerically by the following procedure.

Procedure 2.14.

- Solve the boundary value problem (2.8) numerically to find an approximate solution $\tilde{\xi}$ for ξ such that (2.9) holds with ξ replaced by $\tilde{\xi}$.
- Solve the boundary value problem (2.10) numerically under the constraint (2.11), with ξ replaced by $\tilde{\xi}$, to find an approximate solution $\tilde{\phi}$ for ϕ.
- If Ω is not simply connected, solve the boundary value problem(s) (2.1) numerically to find $\tilde{\varphi}_j$ that approximates φ_j for $1 \leq j \leq m$, and then solve (2.12) numerically with the φ_j's replaced by the $\tilde{\varphi}_j$'s to find $\tilde{c}_j$ that approximates c_j for $1 \leq j \leq m$.
- The approximation $\tilde{\boldsymbol{u}}$ for the solution $\boldsymbol{u}$ of (1.1) is given by

$$\tilde{\boldsymbol{u}} = \epsilon^{-1}\nabla \times \tilde{\phi} + \sum_{j=1}^{m} \tilde{c}_j \nabla \tilde{\varphi}_j.$$

A P_1 finite element method based on this procedure will be analyzed in the next section.

3. A P_1 Finite Element Method

Since the case of the perfectly conducting boundary condition has already been carried out in [3], we will focus on the case $\Gamma_{\mathrm{pc}} = \emptyset$ to simplify the presentation and we will denote $\langle \cdot, \cdot \rangle_{\Gamma_{\mathrm{imp}}} = \langle \cdot, \cdot \rangle_{\partial\Omega}$ by $\langle \cdot, \cdot \rangle$.

Let $\mathcal{T}_h$ be a triangulation of Ω and $V_h \subset H^1(\Omega)$ be the (complex-valued) P_1 finite element space associated with $\mathcal{T}_h$, where h represents the mesh size.

3.1. **The P_1 finite element method for (2.8).** The approximation $\xi_h \in V_h$ for ξ is defined by

$$a(\xi_h, v) = (\boldsymbol{f}, \epsilon^{-1}\nabla \times v) - ik\langle g/\lambda, v\rangle \qquad \forall\, v \in V_h, \tag{3.1}$$

where $a(\cdot, \cdot)$ is the sesquilinear form defined in (2.20).

The error analysis for ξ_h involves the adjoint problem of (2.8). By the Fredholm theory, the adjoint problem

$$a(\psi, \zeta) = (\psi, f) \qquad \forall\, \psi \in H^1(\Omega) \tag{3.2}$$

has a unique solution $\zeta \in H^1(\Omega)$ for any $f \in L_2(\Omega)$ under the assumption that $k \notin \mathcal{S}_+$. It then follows from the elliptic regularity theory for Robin/Neumann problems on polygonal domains [24, 25, 26] and a standard interpolation error estimate that

$$\inf_{v \in V_h} \|\zeta - v\|_{H^1(\Omega)} \leq Ch^{\beta}\|f\|_{L_2(\Omega)}, \tag{3.3}$$

where the index of elliptic regularity $\beta \in (\frac{1}{2}, 1]$ is given by

$$\beta = \min(1, \min_{1 \leq \ell \leq L} \frac{\pi}{\omega_\ell}) \tag{3.4}$$

and $\omega_1, \ldots, \omega_L$ are the angles at the corners of Ω. Note that $\beta = 1$ if and only if Ω is convex.

Lemma 3.1. *There exists $h_0 > 0$ such that* (3.1) *has a unique solution for $h \leq h_0$, in which case we have*

$$\|\xi - \xi_h\|_{L_2(\Omega)} \leq Ch^{\beta} \inf_{v \in V_h} \|\xi - v\|_{H^1(\Omega)}. \tag{3.5}$$

Proof. We follow the arguments of Schatz in [27]. Suppose $\xi_h \in V_h$ satisfies (3.1). We have the Galerkin orthogonality

$$a(\xi - \xi_h, v) = 0 \qquad \forall\, v \in V_h. \tag{3.6}$$

Let $\zeta \in H^1(\Omega)$ be defined by

$$a(\psi, \zeta) = (\psi, \xi - \xi_h) \qquad \forall\, \psi \in H^1(\Omega). \tag{3.7}$$

It follows from the estimate (3.3) that

$$\inf_{v \in V_h} \|\zeta - v\|_{H^1(\Omega)} \leq C h^\beta \|\xi - \xi_h\|_{L_2(\Omega)}. \tag{3.8}$$

We can then use (3.6), (3.7) and the boundedness of $a(\cdot,\cdot)$ to obtain

$$\begin{aligned}\|\xi - \xi_h\|_{L_2(\Omega)}^2 &= a(\xi - \xi_h, \zeta) = a(\xi - \xi_h, \zeta - v)\\ &\leq C\|\xi - \xi_h\|_{H^1(\Omega)}\|\zeta - v\|_{H^1(\Omega)} \quad \forall\, v \in V_h,\end{aligned}$$

which together with (3.8) implies

$$\|\xi - \xi_h\|_{L_2(\Omega)} \leq C h^\beta \|\xi - \xi_h\|_{H^1(\Omega)}. \tag{3.9}$$

Combining (3.6), (3.9) with the Gårding inequality

$$\|v\|_{H^1(\Omega)}^2 \leq \gamma_1\big(|a(v,v)| + \gamma_2\|v\|_{L_2(\Omega)}^2\big) \qquad \forall\, v \in H^1(\Omega) \tag{3.10}$$

that is valid for γ_1 and γ_2 sufficiently large, we find

$$\|\xi - \xi_h\|_{H^1(\Omega)}^2 \leq C\big|a(\xi - \xi_h, \xi - v)\big| \qquad \forall\, v \in V_h, \tag{3.11}$$

provided $h \leq h_0$ for a sufficiently small positive number h_0.

For the special case where $\boldsymbol{f} = \mathbf{0}$ and $g = 0$, we have $\xi = 0$ and then (3.11) (with $v = 0$) implies that $\xi_h = 0$, i.e.,

$$a(\xi_h, v) = 0 \quad \forall\, v \in V_h \quad \Rightarrow \quad \zeta_h = 0$$

provided $h \leq h_0$. Therefore the discrete problem (3.1) is uniquely solvable for $h \leq h_0$, in which case it follows from (3.11) that

$$\|\xi - \xi_h\|_{H^1(\Omega)} \leq C \inf_{v \in V_h} \|\xi - v\|_{H^1(\Omega)}. \tag{3.12}$$

The estimate (3.5) follows from (3.9) and (3.12). □

The proof of the following result is entirely analogous to Lemma 3.1 and hence omitted.

Lemma 3.2. *For any* $\delta \in (0, 1/2)$ *we have*

$$\|\xi - \xi_h\|_{H^{(1/2)+\delta}(\Omega)} \leq C_\delta h^{(1/2)-\delta} \inf_{v \in V_h} \|\xi - v\|_{H^1(\Omega)} \tag{3.13}$$

provided $h \leq h_0$.

The following corollary is an immediate consequence of (3.5), (3.12), (3.13) and the following trace inequalities:

$$\begin{aligned}\|\xi - \xi_h\|_{L_2(\partial\Omega)} &\leq C\|\xi - \xi_h\|_{L_2(\Omega)}^{\frac12}\|\xi - \xi_h\|_{H^1(\Omega)}^{\frac12},\\ \|\xi - \xi_h\|_{L_2(\partial\Omega)} &\leq C_\delta\|\xi - \xi_h\|_{H^{(1/2)+\delta}(\Omega)}.\end{aligned}$$

Corollary 3.3. *We have, for $h \le h_0$,*

$$\|\xi - \xi_h\|_{L_2(\partial\Omega)} \le C h^{1/2} \inf_{v \in V_h} \|\xi - v\|_{H^1(\Omega)}, \tag{3.14}$$

if Ω is convex (i.e., $\beta = 1$), and

$$\|\xi - \xi_h\|_{L_2(\partial\Omega)} \le C_\delta h^{(1/2)-\delta} \inf_{v \in V_h} \|\xi - v\|_{H^1(\Omega)}, \tag{3.15}$$

if Ω is nonconvex (i.e., $\beta < 1$).

Remark 3.4. If μ is a negative function, then $a(\cdot,\cdot)$ is coercive and the estimates (3.5), (3.14) and (3.15) are valid for any $h > 0$.

3.2. **The P_1 finite element method for (2.10)–(2.11).** For $h \le h_0$ so that (3.1) is well-posed, we define the approximation $\phi_h \in V_h$ for ϕ by

$$(\nabla \times \phi_h, \epsilon^{-1} \nabla \times v) = (\mu \xi_h, v) - \frac{i}{k}\langle (g - \xi_h)/\lambda, v\rangle \qquad \forall\, v \in V_h, \tag{3.16}$$

$$(\phi_h, 1) = 0. \tag{3.17}$$

Since (3.1) implies

$$(\mu \xi_h, 1) - \frac{i}{k}\langle (g - \xi_h)/\lambda, 1\rangle = 0, \tag{3.18}$$

the discrete Neumann problem (3.16)–(3.17) is uniquely solvable.

Lemma 3.5. *For $h \le h_0$, we have*

$$|\phi - \phi_h|_{H^1(\Omega)} \le C\big(h^{1/2} \inf_{v \in V_h} \|\xi - v\|_{H^1(\Omega)} + \inf_{v \in V_h} |\phi - v|_{H^1(\Omega)}\big), \tag{3.19}$$

if Ω is convex, and

$$|\phi - \phi_h|_{H^1(\Omega)} \le C_\delta h^{(1/2)-\delta} \inf_{v \in V_h} \|\xi - v\|_{H^1(\Omega)} + C \inf_{v \in V_h} |\phi - v|_{H^1(\Omega)}, \tag{3.20}$$

if Ω is nonconvex.

Proof. In view of (2.9), we can define $\tilde{\phi}_h \in V_h$ to be the unique solution of

$$(\nabla \times \tilde{\phi}_h, \epsilon^{-1} \nabla \times v) = (\mu \xi, v) - \frac{i}{k}\langle (g - \xi)/\lambda, v\rangle \qquad \forall\, v \in V_h, \tag{3.21}$$

$$(\tilde{\phi}_h, 1) = 0, \tag{3.22}$$

i.e., $\tilde{\phi}_h$ is the approximation of ϕ by the P_1 finite element method that uses the exact solution ξ of (2.8). By a standard argument based on the Galerkin orthogonality we have

$$|\phi - \tilde{\phi}_h|_{H^1(\Omega)} \le C \inf_{v \in V_h} |\phi - v|_{H^1(\Omega)}. \tag{3.23}$$

From (3.16)–(3.17) and (3.21)–(3.22), we find

$$\big(\nabla \times (\tilde{\phi}_h - \phi_h), \epsilon^{-1} \nabla \times v\big) = \big(\mu(\xi - \xi_h), v\big) - \frac{i}{k}\langle (\xi - \xi_h)/\lambda, v\rangle \quad \forall\, v \in V_h, \tag{3.24}$$

$$(\tilde{\phi}_h - \phi_h, 1) = 0. \tag{3.25}$$

Because of the constraint (3.25), we have a Poincaré-Friedrichs inequality

$$\|\tilde{\phi}_h - \phi_h\|_{L_2(\Omega)} \le C \|\nabla \times (\tilde{\phi}_h - \phi_h)\|_{L_2(\Omega)}. \tag{3.26}$$

Taking $v = \tilde{\phi}_h - \phi_h$ in (3.24), we obtain, by (3.26) and the trace theorem,

$$\begin{aligned}&\big(\nabla\times(\tilde{\phi}_h-\phi_h),\epsilon^{-1}\nabla\times(\tilde{\phi}_h-\phi_h)\big)\\&\quad\le C\big(\|\xi-\xi_h\|_{L_2(\Omega)}+\|\xi-\xi_h\|_{L_2(\partial\Omega)}\big)\|\nabla\times(\tilde{\phi}_h-\phi_h)\|_{L_2(\Omega)}\end{aligned}$$

which implies

$$|\tilde{\phi}_h-\phi_h|_{H^1(\Omega)}=\|\nabla\times(\tilde{\phi}_h-\phi_h)\|_{L_2(\Omega)}\le C\big(\|\xi-\xi_h\|_{L_2(\Omega)}+\|\xi-\xi_h\|_{L_2(\partial\Omega)}\big). \tag{3.27}$$

The estimate (3.19) and (3.20) follow from Lemma 3.1, Corollary 3.3, (3.23) and (3.27). □

3.3. **The P_1 finite element method for** (2.1)**.** The P_1 finite element method for (2.1) is to find $\varphi_{j,h} \in V_h$ $(1 \le j \le m)$ such that

$$(\epsilon\nabla\varphi_{j,h},\nabla v)=0 \qquad \forall\, v\in V_h\cap H^1_0(\Omega), \tag{3.28a}$$

$$\varphi_{j,h}\big|_{\Gamma_0}=0, \tag{3.28b}$$

$$\varphi_{j,h}\big|_{\Gamma_\ell}=\delta_{j\ell}=\begin{cases}1 & j=\ell\\ 0 & j\neq\ell\end{cases} \qquad \text{for}\quad 1\le\ell\le m. \tag{3.28c}$$

The approximation $c_{j,h}$ for the coefficient c_j in (2.2) is then obtained from the SPD system

$$\sum_{j=1}^{m}(\epsilon\nabla\varphi_{j,h},\nabla\varphi_{\ell,h})c_{j,h}=-\frac{1}{k^2}(\boldsymbol{f},\nabla\varphi_{\ell,h}) \qquad \text{for}\quad 1\le\ell\le m. \tag{3.29}$$

Since (3.28) and (3.29) do not involve the boundary condition of the time-harmonic Maxwell equations, their analysis is identical to the one in [3] and we have the following result (cf. [3, Lemma 4.6, Lemma 4.7 and Remark 4.8]).

Lemma 3.6. *The function $\varphi_{j,h}$ satisfies*

$$|\varphi_j-\varphi_{j,h}|_{H^1(\Omega)}\le Ch^\beta \qquad \textit{for}\quad 1\le j\le m, \tag{3.30}$$

and the coefficient $c_{j,h}$ satisfies

$$|c_j-c_{j,h}|\le Ch^\beta\|\boldsymbol{f}\|_{L_2(\Omega)} \qquad \textit{for}\quad 1\le j\le m. \tag{3.31}$$

3.4. **Error estimate for $\boldsymbol{u}_h$ and $\nabla\times\boldsymbol{u}_h$.** Following Procedure 2.14, we define the approximate solution $\boldsymbol{u}_h$ of (1.1) by

$$\boldsymbol{u}_h=\epsilon^{-1}\nabla\times\phi_h+\sum_{j=1}^{m}c_{j,h}\nabla\varphi_{j,h}. \tag{3.32}$$

The following theorem provides L_2 error estimates for $\boldsymbol{u}_h$.

Theorem 3.7. *The approximation $\boldsymbol{u}_h$ satisfies*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}\le C_\delta h^{(1/2)-\delta}\big(\|\boldsymbol{f}\|_{L_2(\Omega)}+\|g\|_{L_2(\partial\Omega)}\big) \tag{3.33}$$

for any $\delta\in(0,1/2)$.

If $\boldsymbol{f}$ belongs to $[H^1(\Omega)]^2$ and g belongs to $H^{\frac12}(E)$ for all $E\in\mathcal{E}(\Omega)$, where $\mathcal{E}(\Omega)$ denotes the set of the edges of Ω, then this error estimate can be improved to

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}\le Ch^\beta\big(\|\boldsymbol{f}\|_{H^1(\Omega)}+\sum_{E\in\mathcal{E}(\Omega)}\|g\|_{H^{\frac12}(E)}\big). \tag{3.34}$$

Proof. We have, by (2.2) and (3.32),

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)} \le \|\epsilon^{-1}\nabla\times(\phi-\phi_h)\|_{L_2(\Omega)} + \Big\|\sum_{j=1}^{m}(c_j\nabla\varphi_j - c_{j,h}\nabla\varphi_{j,h})\Big\|_{L_2(\Omega)}. \tag{3.35}$$

Using (3.30) and (3.31), the following estimate

$$\Big\|\sum_{j=1}^{m}(c_j\nabla\varphi_j - c_{j,h}\nabla\varphi_{j,h})\Big\|_{L_2(\Omega)} \le Ch^{\beta}\|\boldsymbol{f}\|_{L_2(\Omega)} \tag{3.36}$$

was established in [3, Theorem 4.9]. Therefore it only remains to estimate $|\phi - \phi_h|_{H^1(\Omega)}$.

Note that the well-posedness of (2.8) implies

$$\|\xi\|_{H^1(\Omega)} \le C\big(\|\boldsymbol{f}\|_{L_2(\Omega)} + \|g\|_{L_2(\partial\Omega)}\big) \tag{3.37}$$

and hence

$$\inf_{v\in V_h}\|\xi - v\|_{H^1(\Omega)} \le \|\xi\|_{H^1(\Omega)} \le C\big(\|\boldsymbol{f}\|_{L_2(\Omega)} + \|g\|_{L_2(\partial\Omega)}\big). \tag{3.38}$$

Moreover it follows from (2.10), (3.37), the elliptic regularity for Neumann problems on polygonal domains and a standard interpolation error estimate that

$$\inf_{v\in V_h}|\phi - v|_{H^1(\Omega)} \le C_\delta h^{(1/2)-\delta}\big(\|\boldsymbol{f}\|_{L_2(\Omega)} + \|g\|_{L_2(\partial\Omega)}\big). \tag{3.39}$$

The estimate (3.33) follows from Lemma 3.5, (3.36), (3.38) and (3.39).

If $\boldsymbol{f}$ belongs to $[H^1(\Omega)]^2$, then we can rewrite (2.8) as

$$\begin{aligned}
&(\nabla\times\xi, \epsilon^{-1}\nabla\times\psi) - k^2(\mu\xi,\psi) - ik\langle\xi/\lambda,\psi\rangle \\
&\quad = \big(\nabla\times(\epsilon^{-1}\boldsymbol{f}),\psi\big) - \langle\boldsymbol{n}\times(\epsilon^{-1}\boldsymbol{f}),\psi\rangle - ik\langle g/\lambda,\psi\rangle \quad \forall\,\psi\in H^1(\Omega),
\end{aligned}$$

and if $g \in H^{\frac12}(E)$ for all $E \in \mathcal{E}(\Omega)$, then we can apply the elliptic regularity theory of Neumann problems on polygonal domains [25, Corollary 4.4.4.14] and a standard interpolation error estimate to obtain

$$\inf_{v\in V_h}\|\xi - v\|_{H^1(\Omega)} \le Ch^{\beta}\big(\|\boldsymbol{f}\|_{H^1(\Omega)} + \sum_{E\in\mathcal{E}(\Omega)}\|g\|_{H^{\frac12}(E)}\big). \tag{3.40}$$

Similarly we have, in view of (2.10) and (3.37),

$$\inf_{v\in V_h}|\phi - v|_{H^1(\Omega)} \le Ch^{\beta}\big(\|\boldsymbol{f}\|_{L_2(\Omega)} + \sum_{E\in\mathcal{E}(\Omega)}\|g\|_{H^{\frac12}(E)}\big). \tag{3.41}$$

Combining Lemma 3.5, (3.40) and (3.41), we find

$$|\phi - \phi_h|_{H^1(\Omega)} \le Ch^{\beta}\big(\|\boldsymbol{f}\|_{H^1(\Omega)} + \sum_{E\in\mathcal{E}(\Omega)}\|g\|_{H^{\frac12}(E)}\big),$$

which together with (3.35) and (3.36) implies the estimate (3.34). $\square$

The following theorem provides L_2 error estimates for $\mu\xi_h$ as an approximation of $\nabla\times\boldsymbol{u}$.

Theorem 3.8. *The approximation $\mu\xi_h$ of $\nabla\times\boldsymbol{u}$ satisfies*

$$\|\nabla\times\boldsymbol{u} - \mu\xi_h\|_{L_2(\Omega)} \le Ch^{\beta}\big(\|\boldsymbol{f}\|_{L_2(\Omega)} + \|g\|_{L_2(\partial\Omega)}\big). \tag{3.42}$$

j	$\|\boldsymbol{u}-\boldsymbol{u}_j\|_{L_2(\Omega;\epsilon)}$	order	$\|\xi-\xi_j\|_{L_2(\Omega)}$	order
3	5.5277×10^{-1}	1.25	2.8653×10^{0}	0.81
4	2.8602×10^{-1}	0.95	1.2385×10^{0}	1.21
5	1.4190×10^{-1}	1.01	3.4716×10^{-1}	1.83
6	7.0696×10^{-2}	1.01	8.9453×10^{-2}	1.96
7	3.5311×10^{-2}	1.00	2.2539×10^{-2}	1.99
8	1.7651×10^{-2}	1.00	5.6458×10^{-3}	2.00

TABLE 1. Errors and convergence rates for the first experiment with inhomogeneous and anisotropic material

If $\boldsymbol{f}$ belongs to $[H^1(\Omega)]^2$ and $g\in H^{\frac{1}{2}}(E)$ for all $E\in\mathcal{E}(\Omega)$, then the error estimate can be improved to

$$\|\nabla\times\boldsymbol{u}-\mu\xi_h\|_{L_2(\Omega)}\le Ch^{2\beta}\big(\|\boldsymbol{f}\|_{H^1(\Omega)}+\sum_{E\in\mathcal{E}(\Omega)}\|g\|_{H^{\frac{1}{2}}(E)}\big). \tag{3.43}$$

Proof. In view of (2.4) and (3.5), the estimates (3.42) and (3.43) follow from (3.38) and (3.40) respectively. □

Remark 3.9. In view of the results in [3], Theorem 3.7 and Theorem 3.8 remain valid for (1.1) where $\Gamma_{\text{pc}}\neq\emptyset$

4. NUMERICAL EXPERIMENTS

We report in this section the results of several numerical experiments for the two-dimensional time-harmonic Maxwell equations with inhomogeneous and anisotropic permittivity ϵ, sign changing permeability μ, and the impedance boundary condition. We compare the Hodge decomposition approach with the lowest order edge element method and present numerical results for nonconvex domains. We use uniform triangulations with mesh size $h_j:=\sqrt{2}/2^j$ $(j=0,1,2,\ldots)$ in the computation. The corresponding finite element approximations are denoted by $\boldsymbol{u}_j:=\boldsymbol{u}_{h_j}$ and $\xi_j:=\xi_{h_j}$. We take the impedance $1/\lambda$ to be 1 in all the numerical experiments.

4.1. **Inhomogeneous and anisotropic material.** The first experiment (cf. [28]) involves the square domain $\Omega=(-1,1)^2$ with the following permittivity and permeability:

$$\epsilon=\begin{pmatrix}1+x^2 & xy\\ xy & 1+y^2\end{pmatrix}\qquad\text{and}\qquad\mu=(1+x^2+y^2)^{-1}.$$

For $k=1$, we take the exact solution of (1.1) to be $\boldsymbol{u}=(y/(x^2+y^2+0.02),-x/(x^2+y^2+0.02))^t$. It is easy to check that $\boldsymbol{u}\in H_{\text{imp}}(\text{curl};\Omega;\Gamma_{\text{imp}})\cap H(\text{div}^0;\Omega;\epsilon)$, $\boldsymbol{f}=\nabla\times(\mu^{-1}\nabla\times\boldsymbol{u})-\epsilon\boldsymbol{u}\in H(\text{div}^0;\Omega;\epsilon)\cap[H^1(\Omega)]^2$, and that $g=-i\boldsymbol{n}\times\boldsymbol{u}+\mu^{-1}\nabla\times\boldsymbol{u}$ satisfies $g\in H^{\frac{1}{2}}(E)$ for all $E\in\mathcal{E}(\Omega)$. The errors $\|\boldsymbol{u}-\boldsymbol{u}_j\|_{L_2(\Omega;\epsilon)}$ and $\|\xi-\xi_j\|_{L_2(\Omega)}$ are presented in Table 1, together with the computed order of convergence that is based on comparing the errors on two consecutive levels. We observe that the error $\|\boldsymbol{u}-\boldsymbol{u}_j\|_{L_2(\Omega;\epsilon)}$ converges asymptotically with order one and the error $\|\xi-\xi_j\|_{L_2(\Omega)}$ with order two, as predicted by the theory.

j	$\|\boldsymbol{u}-\boldsymbol{u}_j\|_{L_2(\Omega)}$	order	$\|\boldsymbol{u}-\boldsymbol{u}_j^{\mathrm{Nd}}\|_{L_2(\Omega)}$	order
3	3.9039×10^{-1}	1.14	5.5880×10^{-1}	1.30
4	1.8472×10^{-1}	1.08	2.6133×10^{-1}	1.10
5	9.0770×10^{-2}	1.03	1.2835×10^{-1}	1.03
6	4.5176×10^{-2}	1.01	6.3886×10^{-2}	1.01
7	2.2562×10^{-2}	1.00	3.1907×10^{-2}	1.00
8	1.1277×10^{-2}	1.00	1.5949×10^{-2}	1.00
j	$\|\nabla\times\boldsymbol{u}-\xi_j\|_{L_2(\Omega)}$	order	$\|\nabla\times(\boldsymbol{u}-\boldsymbol{u}_j^{\mathrm{Nd}})\|_{L_2(\Omega)}$	order
3	8.4800×10^{-1}	1.76	1.9478×10^{0}	1.61
4	2.2195×10^{-1}	1.93	$8.0880e\times10^{-1}$	1.27
5	5.6146×10^{-2}	1.98	3.7969×10^{-1}	1.09
6	1.4078×10^{-2}	2.00	1.8612×10^{-1}	1.03
7	3.5222×10^{-3}	2.00	9.2453×10^{-2}	1.01
8	8.8071×10^{-4}	2.00	4.6117×10^{-2}	1.00

TABLE 2. Results of the Hodge decomposition method and the lowest order edge element method for the second experiment with a plane wave exact solution

4.2. **Plane wave.** In the second numerical experiment we compare the Hodge decomposition approximations with the lowest order edge element [29] approximations, where

$$V_h^{\mathrm{Nd}} := \left\{ \boldsymbol{v}\in H(\mathrm{curl};\Omega) : (\boldsymbol{v}|_T)(x) = \begin{bmatrix} a_{T,1} \\ a_{T,2} \end{bmatrix} + b_T \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}, a_{T,1}, a_{T,2}, b_T \in \mathbb{R}, \forall T\in\mathcal{T}_h \right\}.$$

The domain is $\Omega = (-1,1)^2$, the electric permittivity and magnetic permeability are given by $\epsilon = 1$ and $\mu = 1$, and the frequency k is taken to be 5. The impedance boundary condition is defined by the plane wave solution

$$\boldsymbol{u} = \boldsymbol{p}\exp(ik\boldsymbol{d}\cdot x)$$

where $\boldsymbol{d} = (1,0)^t$ and $\boldsymbol{p} = (0,1)^t$, and $\boldsymbol{f} = \boldsymbol{0}$. The numerical results are reported in Table 2. For the edge element method the order of convergence is 1 for both errors $\|\boldsymbol{u}-\boldsymbol{u}_j^{\mathrm{Nd}}\|_{L_2(\Omega)}$ and $\|\nabla\times(\boldsymbol{u}-\boldsymbol{u}_j^{\mathrm{Nd}})\|_{L_2(\Omega)}$. Since the order of convergence for the error $\|\nabla\times\boldsymbol{u}-\xi_j\|_{L_2(\Omega)}$ is 2 in the Hodge decomposition approach, asymptotically these errors are much smaller than the ones for the edge element method. On the other hand the errors $\|\boldsymbol{u}-\boldsymbol{u}_j\|_{L_2(\Omega)}$ and $\|\boldsymbol{u}-\boldsymbol{u}_j^{\mathrm{Nd}}\|_{L_2(\Omega)}$ are comparable for both methods.

Note that the computation of ξ_j requires the solution of one scalar equation whose number of degree of freedoms (dofs) is roughly half the number of dofs for the lowest order edge element, while the computation of $\boldsymbol{u}_j$ requires the solution of two scalar equations whose combined number of dofs equals roughly the number of dofs for the lowest order edge element.

4.3. **L-shaped domain with sign changing μ.** In the third numerical experiment we consider the L-shaped domain $\Omega = (-1,1)^2\backslash[0,1]^2$ with sign changing μ. Such parameters occur in the study of metamaterials [6, 7]. For this experiment we chose $\epsilon = 1$, $\mu = 1$ in the 2nd quadrant, and $\mu = -1$ in the 3rd and 4th quadrants, and we take g to be 0, $\boldsymbol{f}$ to be $(1,1)^t$, and k to be 1. The error is computed by comparing the discrete solution to a reference solution that is computed on the fine

j	$\lVert \boldsymbol{u}_9 - \boldsymbol{u}_j \rVert_{L_2(\Omega)}$	order	$\lVert \xi_9 - \xi_j \rVert_{L_2(\Omega)}$	order
1	4.4279×10^{-1}	0.61	7.1024×10^{-2}	1.36
2	2.7456×10^{-1}	0.69	2.7601×10^{-2}	1.36
3	1.6899×10^{-1}	0.70	1.0699×10^{-2}	1.37
4	1.0432×10^{-1}	0.70	4.1407×10^{-3}	1.37
5	6.4455×10^{-2}	0.69	1.5913×10^{-3}	1.38
6	3.9513×10^{-2}	0.71	5.9878×10^{-4}	1.41

TABLE 3. Convergence history for the third experiment on an L-shaped domain with sign changing permeability

j	$\lVert \boldsymbol{u}_9 - \boldsymbol{u}_j \rVert_{L_2(\Omega)}$	order	$\lVert \xi_9 - \xi_j \rVert_{L_2(\Omega)}$	order
2	6.4948×10^{-1}	0.65	6.3263×10^{-2}	1.24
3	4.0680×10^{-1}	0.67	2.5837×10^{-2}	1.29
4	2.5398×10^{-1}	0.68	1.0361×10^{-2}	1.32
5	1.5804×10^{-1}	0.68	4.0826×10^{-3}	1.34
6	9.7298×10^{-2}	0.70	1.5631×10^{-3}	1.39

TABLE 4. Convergence history for the fourth experiment for a doubly connected domain with perfectly conducting boundary condition on the inner boundary and the impedance boundary condition on the outer boundary

j	$\lVert \boldsymbol{u}_9 - \boldsymbol{u}_j \rVert_{L_2(\Omega)}$	order	$\lVert \xi_9 - \xi_j \rVert_{L_2(\Omega)}$	order
3	1.5145×10^{0}	0.58	1.3411×10^{1}	0.67
4	7.0845×10^{-1}	1.10	5.6132×10^{0}	1.26
5	3.2102×10^{-1}	1.14	2.1270×10^{0}	1.40
6	1.5452×10^{-1}	1.05	7.8724×10^{-1}	1.43
7	8.1899×10^{-2}	0.92	2.9174×10^{-1}	1.43

TABLE 5. Convergence history for the fifth experiment pertaining to electromagnetic cloaking

mesh with mesh size $h_9 = \sqrt{2}/2^9$. Due to the nonconvex corner at the origin, the convergence is sub-optimal for both errors as shown in Table 3. The convergence of $\lVert \boldsymbol{u}_9 - \boldsymbol{u}_j \rVert_{L_2(\Omega)}$ and $\lVert \xi_9 - \xi_j \rVert_{L_2(\Omega)}$ are numerically close to 2/3 and 4/3, as predicted by Theorem 3.7 and Theorem 3.8 with $\beta = 2/3$.

4.4. **Doubly connected domain.** The computational domain for the fourth numerical experiment is the doubly connected domain $\Omega = (-1,1)^2 \backslash [-1/2, 1/2]$ so that $m = 1$ in (2.2). The impedance boundary condition is imposed on the outer boundary Γ_{imp} where $g = 0$, and the perfectly conducting boundary condition $\boldsymbol{n} \times \boldsymbol{u} = 0$ is imposed on the inner boundary Γ_{pc} (cf. Remark 3.9). We take ϵ, μ and k to be 1, and $\boldsymbol{f}$ to be $(e^{x_1}, e^{x_2})^t$. Thus we have to solve three scalar equations for this experiment.

Due to the four nonconvex corners with $\beta = 2/3$, the convergence of both errors are sub-optimal as demonstrated by the results in Table 4. Again the orders of convergence for $\lVert \boldsymbol{u}_9 - \boldsymbol{u}_j \rVert_{L_2(\Omega)}$ and $\lVert \xi_9 - \xi_j \rVert_{L_2(\Omega)}$ are numerically close to 2/3 and 4/3, as predicted by our theory.

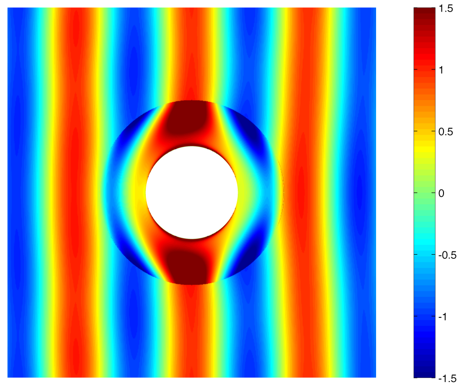

FIGURE 1. Cloaking effect illustrated by the real part of the second component of the reference solution $\boldsymbol{u}_9$

4.5. **Cloaking.** The fifth and last numerical experiment is concerned with a electromagnetic cloaking problem from [28, 30]. We consider a perfectly conduction cylinder with radius $R_1 = 0.25$ wrapped around by a cylindrical cloak with thickness $R_2 - R_1$, where $R_2 = 0.5$. Outside the radius R_2, we take both μ and ϵ to be 1. For $r \in (R_1, R_2)$, the inhomogeneous and anisotropic permittivity is given in polar coordinates by

$$\epsilon_{xx} = \left(\left(\frac{R_2 - R_1}{R_2}\right)^2 + \left(1 + 2\left(\frac{R_2 - R_1}{R_2}\right)^2 \frac{R_1}{r - R_1}\right)\sin^2\theta\right)\mu,$$

$$\epsilon_{xy} = \epsilon_{yx} = -\left(\left(1 + 2\left(\frac{R_2 - R_1}{R_2}\right)^2 \frac{R_1}{r - R_1}\right)\sin\theta\cos\theta\right)\mu,$$

$$\epsilon_{yy} = \left(\left(\frac{R_2 - R_1}{R_2}\right)^2 + \left(1 + 2\left(\frac{R_2 - R_1}{R_2}\right)^2 \frac{R_1}{r - R_1}\right)\cos^2\theta\right)\mu,$$

and the inhomogeneous permeability is given by

$$\mu = \left(\left(\frac{R_2 - R_1}{R_2}\right)^2 \frac{r}{r - R_1}\right)^{-1}.$$

For the resulting two-dimensional transverse magnetic problem, we impose an impedance boundary condition on the boundary of the square $(-1, 1)^2$ induced by the plane wave solution from the second experiment with $k = 10$, and we take $\boldsymbol{f}$ to be $\mathbf{0}$.

Therefore we have a time-harmonic Maxwell problem posed on the doubly connected domain $\Omega = (-1, 1)^2 \backslash D$, where D is the closed disc $\{x : |x| \leq R_1\}$, with an impedance boundary condition on the outer boundary and the perfectly conducting boundary condition on the inner boundary. Note that for this problem the permittivity ϵ is discontinuous on the circle with radius R_2 and it is positive semi-definite on the inner boundary, i.e. the circle with radius R_1, where the permeability μ also vanishes. Thus ϵ and μ do not satisfy the assumptions under which the Hodge decomposition approach is derived and analyzed. Nevertheless numerical results indicate that the Hodge decomposition approach still works for this problem. Note also that we only need to solve two scalar problems since $\boldsymbol{f} = \mathbf{0}$ (cf. Remark 2.7).

To preserve the symmetry of the mesh with respect to the circular obstacle, we consider a crisscross triangulation with mesh size $h_j := 2^{-j}$ and adjust the vertices of the inner boundary such that they match the radius R_1. The errors obtained by comparing the discrete solutions with the reference solution $\boldsymbol{u}_9$ are reported in Table 5. The order of convergence for the error $\|\xi_9 - \xi_j\|_{L_2(\Omega)}$ is roughly 1.5, sugessting that $\xi = \mu^{-1}\nabla \times \boldsymbol{u}$ belongs to $H^{(3/2)-\delta}$ near the circle with radius R_2. On the other hand, the order of convergence for the error $\|\boldsymbol{u}_9 - \boldsymbol{u}_j\|_{L_2(\Omega)}$ is roughly 1, suggesting that the solution $\boldsymbol{u}$ belongs to $[H^1(\Omega)]^2$.

A 2D surface plot of the real part of the second component of the reference solution $\boldsymbol{u}_9$ is presented in Figure 1, where the cloaking effect (i.e. the wave going through without being disturbed) is clearly visible.

5. Simultaneous sign changes in electric permittivity and magnetic permeability

Let $\mathbb{S}_{2\times 2}$ be the space of real-valued 2×2 symmetric nonsingular matrices. We consider in this section the well-posedness of (1.1) under the assumptions that

$$\epsilon \in L_\infty(\Omega, \mathbb{S}_{2\times 2}),\ \epsilon^{-1} \in L_\infty(\Omega, \mathbb{S}_{2\times 2}),\ \mu \in L_\infty(\Omega) \text{ and } \mu^{-1} \in L_\infty(\Omega). \tag{5.1}$$

We will also need two other assumptions on ϵ.

5.1. The space $\mathcal{H}(\Omega; \epsilon)$.

In order for (2.1) to be well defined, we assume that for any $w \in H_0^1(\Omega)$, there exists $\zeta_w \in H_0^1(\Omega)$ such that

$$(\epsilon\nabla\zeta_w, \nabla v) = (\nabla w, \nabla v) \quad \forall v \in H_0^1(\Omega). \tag{5.2}$$

Since the sesquilinear form $(\epsilon\nabla\cdot, \nabla\cdot)$ is Hermitian, the function ζ_w is unique and the map $\zeta_w \to w$ defines an isomorphism T_0^ϵ of $H_0^1(\Omega)$ by the open mapping theorem.

Let $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_m \in H^1(\Omega)$ satisfy the boundary conditions (3.28b)–(3.28c). For $1 \le j \le m$, the unique solution of (2.1) is then given by $\varphi_j = \tilde{\varphi}_j + \zeta_j$, where $\zeta_j \in H_0^1(\Omega)$ is determined by

$$(\epsilon\nabla\zeta_j, \nabla v) = -(\epsilon\nabla\tilde{\varphi}_j, \nabla v) = -(\nabla T_0^\epsilon\tilde{\varphi}_j, \nabla v) \quad \forall v \in H_0^1(\Omega).$$

The space $\mathcal{H}(\Omega; \epsilon)$ is spanned by $\varphi_1, \ldots \varphi_m$.

5.2. Hodge decomposition.

In order to construct a Hodge decomposition for $\boldsymbol{u} \in H(\mathrm{div}^0; \Omega; \epsilon)$, we assume that for any $w \in H_\Gamma^1$, there exists $\zeta_w \in H_\Gamma^1$ such that

$$(\epsilon\nabla\zeta_w, \nabla v) = (\nabla w, \nabla v) \quad \forall v \in H_\Gamma^1, \tag{5.3}$$

where H_Γ^1 is the space introduced in Remark 2.2. Since the sequilinear form $(\epsilon\nabla\cdot, \nabla\cdot)$ is Hermitian, the function ζ_w is unique and the map $\zeta_w \to w$ defines an isomorphism T_Γ^ϵ of H_Γ^1 by the open mapping theorem.

Remark 5.1. For a simply connected Ω, the conditions (5.2) and (5.3) are identical since in this case $H_\Gamma^1 = H_0^1(\Omega)$.

Lemma 5.2. *Let φ belong to $\mathcal{H}(\Omega; \epsilon)$. Then $(\epsilon\nabla\varphi, \nabla\rho) = 0$ for all $\rho \in \mathcal{H}(\Omega; \epsilon)$ if and only if $\varphi = 0$.*

Proof. Since $H^1_\Gamma = \mathcal{H}(\Omega;\epsilon) \oplus H^1_0(\Omega)$, if $\varphi \in \mathcal{H}(\Omega;\epsilon)$ satisfies $(\epsilon\nabla\varphi, \nabla\rho) = 0$ for all $\rho \in \mathcal{H}(\Omega;\epsilon)$, then definition (2.1) implies

$$0 = (\epsilon\nabla\varphi, \nabla v) = (\nabla(T^\epsilon_\Gamma\varphi), \nabla v) \quad \forall v \in H^1_\Gamma$$

and hence $T^\epsilon_\Gamma\varphi = 0$. □

Lemma 5.3. *Let $\varphi \in \mathcal{H}(\Omega;\epsilon)$. Then $\int_{\Gamma_j} \epsilon\nabla\varphi \cdot \boldsymbol{n}\, ds = 0$ for $1 \le j \le m$ if and only if $\varphi = 0$.*

Proof. If $\varphi \in \mathcal{H}(\Omega;\epsilon)$ satisfies the m conditions, then it follows from the Green's formula [17, (2.17)] that

$$(\epsilon\nabla\varphi, \nabla\varrho) = \sum_{j=1}^m \overline{\varrho}|_{\Gamma_j} \int_{\Gamma_j} \epsilon\nabla\varphi \cdot \boldsymbol{n}\, ds = 0 \quad \forall \varrho \in \mathcal{H}(\Omega;\epsilon)$$

and hence $\varphi = 0$ by Lemma 5.2. □

We can now apply the same arguments in [3, Lemma 2.3] to conclude that any $\boldsymbol{u} \in H(\mathrm{div}^0;\Omega;\epsilon)$ has a unique Hodge decomposition

$$\boldsymbol{u} = \epsilon^{-1}\nabla \times \phi + \sum_{j=1}^m c_j \nabla\varphi_j$$

where $\phi \in H^1(\Omega)$ satisfies $(\phi, 1) = 0$.

5.3. **A condition equivalent to** (5.3)**.** Let $\hat{H}^1(\Omega) = \{v \in H^1(\Omega) : (v, 1) = 0\}$ be the space of H^1 functions with zero means. The well-posedness of (2.8) under (5.1) will require the following condition: for any $w \in \hat{H}^1(\Omega)$, there exists $\zeta_w \in \hat{H}^1(\Omega)$ such that

$$(\nabla \times v, \epsilon^{-1}\nabla \times \zeta_w) = (\nabla \times v, \nabla \times w) \quad \forall v \in \hat{H}^1(\Omega). \tag{5.4}$$

Again ζ_w is unique since the sesquilinear form $(\nabla \times \cdot, \epsilon^{-1}\nabla \times \cdot)$ is Hermitian and the map $\zeta_w \to w$ defines an isomorphism $\hat{T}^\epsilon$ on $\hat{H}^1(\Omega)$.

It turns out that condition (5.4) is equivalent to condition (5.3), which is a consequence of the following relation between the spaces $\nabla \times \hat{H}^1(\Omega)$ and ∇H^1_Γ (cf. (1.3) and [17, Theorem 3.2]):

$$[L_2(\Omega)]^2 = \nabla \times \hat{H}^1(\Omega) \overset{\perp}{\oplus} \nabla H^1_\Gamma. \tag{5.5}$$

Lemma 5.4. *The condition* (5.4) *is equivalent to the condition* (5.3)*.*

Proof. We begin by showing (5.4) implies (5.3). Given any $w \in H^1_\Gamma$, we want to show that the equation in (5.3) is solvable. By (5.1) and the Riesz representation theorem, there exists $\rho_w \in \hat{H}^1(\Omega)$ such that $(\nabla \times v, \nabla \times \rho_w) = (\nabla \times v, \epsilon^{-1}\nabla w)$ for all $v \in \hat{H}^1(\Omega)$. It then follows from (5.4) that there exists $z \in \hat{H}^1(\Omega)$ such that

$$(\nabla \times v, \epsilon^{-1}\nabla \times z) = (\nabla \times v, \nabla \times \rho_w) = (\nabla \times v, \epsilon^{-1}\nabla w) \quad \forall v \in \hat{H}^1(\Omega),$$

which, in view of (5.5), implies that

$$\epsilon^{-1}\nabla w - \epsilon^{-1}\nabla \times z = \nabla\zeta_w$$

for some $\zeta_w \in H^1_\Gamma$. Hence we have, again by (5.5),

$$(\epsilon\nabla\zeta_w, \nabla v) = (\nabla w - \nabla \times z, \nabla v) = (\nabla w, \nabla v) \quad \forall v \in H^1_\Gamma.$$

The proof that (5.3) implies (5.4) is similar. Given any $w \in \hat{H}^1(\Omega)$, we want to show that the equation in (5.4) is solvable. By (5.1), (5.3) and the Riesz representation theorem, there exists z in H^1_Γ such that

$$(\epsilon \nabla z, \nabla v) = (\epsilon \nabla \times w, \nabla v) \quad \forall v \in H^1_\Gamma,$$

which, in view of (5.5), implies that

$$\epsilon \nabla \times w - \epsilon \nabla z = \nabla \times \zeta_w$$

for some $\zeta_w \in \hat{H}^1(\Omega)$. It follows that, by (5.5),

$$(\nabla \times v, \epsilon^{-1} \nabla \times \zeta_w) = (\nabla \times v, \nabla \times w - \nabla z) = (\nabla \times v, \nabla \times w) \quad \forall v \in \hat{H}^1(\Omega). \square$$

Remark 5.5. Lemma 5.4 is an analog of [13, Theorem 4.6].

5.4. **A reduced problem.** Consider the following problem related to (1.1): Find $\tilde{\boldsymbol{u}} \in H_{\text{imp}}(\text{curl}; \Omega; \Gamma_{\text{imp}}) \cap H_0(\text{curl}; \Omega; \Gamma_{\text{pc}}) \cap (\epsilon^{-1} \nabla \times \hat{H}^1(\Omega))$ such that

$$(\mu^{-1} \nabla \times \tilde{\boldsymbol{u}}, \nabla \times \boldsymbol{v}) - k^2 (\epsilon \tilde{\boldsymbol{u}}, \boldsymbol{v}) - ik \langle \lambda \boldsymbol{n} \times \tilde{\boldsymbol{u}}, \boldsymbol{n} \times \boldsymbol{v} \rangle_{\Gamma_{\text{imp}}} = (\boldsymbol{f}, \boldsymbol{v}) + \langle g, \boldsymbol{n} \times \boldsymbol{v} \rangle_{\Gamma_{\text{imp}}} \tag{5.6}$$

for all $\boldsymbol{v} \in H_{\text{imp}}(\text{curl}; \Omega; \Gamma_{\text{imp}}) \cap H_0(\text{curl}; \Omega; \Gamma_{\text{pc}}) \cap (\epsilon^{-1} \nabla \times \hat{H}^1(\Omega))$.

Remark 5.6. For a simply connected Ω, the two problems (1.1) and (5.6) are identical since $H(\text{div}^0; \Omega; \epsilon) = \epsilon^{-1} \nabla \times \hat{H}^1(\Omega)$ in this case.

For this reduced problem we have $\tilde{\boldsymbol{u}} = \epsilon^{-1} \nabla \times \phi$ for some $\phi \in \hat{H}^1(\Omega)$, and (5.6) is equivalent to the two scalar problems (2.8) and (2.10)–(2.11). The derivation of this equivalence is identical to the derivation given in Section 2.1 and Section 2.2 for (1.1), with one modification. The projection Q in Lemma 2.3 is now defined by $Q : [L_2(\Omega)]^2 \to \epsilon^{-1} \nabla \times \hat{H}^1(\Omega)$ and

$$(Q\boldsymbol{\zeta}, \epsilon \boldsymbol{v}) = (\boldsymbol{\zeta}, \epsilon \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \epsilon^{-1} \nabla \times \hat{H}^1(\Omega). \tag{5.7}$$

Note that, for $\boldsymbol{\zeta} = \epsilon^{-1} \nabla \times \zeta$, $\boldsymbol{v} = \epsilon^{-1} \nabla \times v$ and $\zeta, v \in \hat{H}^1(\Omega)$, we have

$$(\boldsymbol{\zeta}, \epsilon \boldsymbol{v}) = (\epsilon^{-1} \nabla \times \zeta, \nabla \times v).$$

Therefore condition (5.4) implies that the sesquilinear form $(\cdot, \epsilon \cdot)$ is nonsingular on the space $\epsilon^{-1} \nabla \times \hat{H}^1(\Omega)$ and hence (5.7) is well-defined. By replacing the orthogonal decomposition $[L_2(\Omega; \epsilon)]^2 = \nabla H^1_0(\Omega) \overset{\perp}{\oplus} H(\text{div}^0; \Omega; \epsilon)$ with the orthogonal decomposition (5.5), the proof of Lemma 2.3 remains the same and the rest of the results in Section 2.1 and Section 2.2 are valid for the reduced problem (5.6).

We now turn to the well-posedness of (5.6) under assumptions (5.1) and (5.4). Since the unique solvability of (2.10) is guaranteed by condition (5.4) directly, it only remains to consider the well-posedness of (2.8).

Lemma 5.7. *Let $T_\epsilon : H^1(\Omega) \to H^1(\Omega)$ be defined by*

$$(T_\epsilon v, w)_{H^1(\Omega)} = (\nabla \times v, \epsilon^{-1} \nabla \times w) + (v, 1)(1, w) \quad \forall v \in H^1(\Omega), \tag{5.8}$$

where $(z, w)_{H^1(\Omega)} = (\nabla \times z, \nabla \times w) + (z, w) = (\nabla z, \nabla w) + (z, w)$ is the inner product for $H^1(\Omega)$. Then T_ϵ is an isomorphism.

Proof. Clearly T_ϵ is a bounded linear operator under the assumption (5.1). Since T_ϵ is self-adjoint with respect to $(\cdot, \cdot)_{H^1(\Omega)}$, it suffices to show that

$$||v||_{H^1(\Omega)} \le C ||T_\epsilon v||_{H^1(\Omega)} \quad \forall v \in H^1(\Omega) \tag{5.9}$$

for some positive constant C, which implies T_ϵ is injective and has a closed range.

Given any $v \in H^1(\Omega)$, we define the function $\hat{v} \in \hat{H}^1(\Omega)$ by $\hat{v} = v - (v,1)/|\Omega|$. It follows from condition (5.4) that there exists $w \in \hat{H}^1(\Omega)$ such that

$$(\nabla \times \hat{v}, \epsilon^{-1}\nabla \times w) = (\nabla \times \hat{v}, \nabla \times \hat{v}) \quad \text{and} \quad |w|_{H^1(\Omega)} \leq C|v|_{H^1(\Omega)}.$$

Putting this w in (5.8) we find

$$|v|_{H^1(\Omega)} \leq C||T_\epsilon v||_{H^1(\Omega)}. \tag{5.10}$$

On the other hand, by taking $w = 1$ in (5.8), we have

$$|(v,1)| \leq C||T_\epsilon v||_{H^1(\Omega)}. \tag{5.11}$$

The estimate (5.9) follows from (5.10), (5.11) and a Poincaré-Friedrichs inequality. □

Theorem 5.8. *Under the assumptions* (5.1), (5.3) *(or equivalently* (5.4)*) and* $\Gamma_{imp} \neq \emptyset$*, there exists a discrete (possibly empty) subset* $\mathcal{S}_+$ *of* $\mathbb{R}_+ = (0,\infty)$ *such that* (5.6) *has a unique solution for* $k \in \mathbb{R}_+$ *if and only if* $k \notin \mathcal{S}_+$*.*

Proof. It follows from Lemma 5.7, elliptic regularity and compact embeddings of Sobolev spaces that we can interpret (2.8) as a Fredholm equation in $H^1(\Omega)$. Therefore, by the analytic Fredholm theorem, it suffices to show that (2.8) is uniquely solvable for some $k \in \mathbb{C}$, or equivalently, that for some $k \in \mathbb{C}$ the only solution in $H^1(\Omega)$ of

$$a(\eta,\psi) = 0 \quad \forall \psi \in H^1(\Omega) \tag{5.12}$$

is $\eta = 0$, where the sequilinear form $a(\cdot,\cdot)$ is defined in (2.20).

First we observe that (2.20) and (5.12) imply $\eta = 0$ on Γ_{imp} as long as $k \in \mathbb{R}\backslash\{0\}$. Therefore for $k \in \mathbb{R}\backslash\{0\}$, the homogeneous problem is equivalent to $\eta \in H^1(\Omega)$,

$$\eta|_{\Gamma_{\text{imp}}} = 0, \tag{5.13}$$

and

$$\tilde{a}(\eta,\psi) = 0 \quad \forall \psi \in H^1(\Omega), \tag{5.14}$$

where

$$\tilde{a}(\eta,\psi) = (\nabla \times \eta, \epsilon^{-1}\nabla \times \psi) - k^2(\mu\eta,\psi). \tag{5.15}$$

There exists, in view of (5.4), a function $\psi \in \hat{H}^1(\Omega)$ such that

$$(\nabla \times \eta, \epsilon^{-1}\nabla \times \psi) = (\nabla \times \eta, \nabla \times \eta) \quad \text{and} \quad |\psi|_{H^1(\Omega)} \leq C|\eta|_{H^1(\Omega)}. \tag{5.16}$$

It follows from (5.1), (5.13)–(5.16) and Poincaré-Friedrichs inequalities that

$$|\eta|^2_{H^1(\Omega)} = k^2(\mu\eta,\psi) \leq k^2 C||\eta||_{L_2(\Omega)}|\psi|_{H^1(\Omega)} \leq Ck^2|\eta|^2_{H^1(\Omega)}.$$

Therefore $\eta = 0$ if $k \in \mathbb{R}\backslash\{0\}$ is sufficiently small. □

5.5. **The well-posedness of** (1.1)**.** We can deduce a well-posedness result for (1.1) from Theorem 5.8 by including the assumption (5.2).

Theorem 5.9. *Under the assumptions* (5.1)*,* (5.2)*,* (5.3) *(or equivalently* (5.4)*) and* $\Gamma_{\rm imp} \neq \emptyset$*, there exists a discrete (possibly empty) subset* $\mathcal{S}_+$ *of* $\mathbb{R}_+ = (0, \infty)$ *such that* (1.1) *has a unique solution for* $k \in \mathbb{R}_+$ *if and only if* $k \notin \mathcal{S}_+$*.*

Proof. Under assumptions (5.2) and (5.3) we have a Hodge decomposition (2.2) for $\boldsymbol{u}$ (cf. Section 5.2). Let $\mathcal{S}_+$ be the discrete subset of $\mathbb{R}_+$ from Theorem 5.8, $k \notin \mathcal{S}_+$ and $\tilde{\boldsymbol{u}} \in H_{\rm imp}({\rm curl};\Omega;\Gamma_{\rm imp}) \cap H_0({\rm curl};\Omega;\Gamma_{\rm pc}) \cap (\epsilon^{-1}\nabla \times \hat{H}^1(\Omega))$ be the solution of the reduced problem (5.6), then a straightforward computation shows that

$$\boldsymbol{u} = \tilde{\boldsymbol{u}} + \sum_{j=1}^{m} c_j \nabla \varphi_j$$

is the unique solution of (1.1) if the coefficients $c_1, \ldots, c_m$ are determined by (2.12), whose unique solvability is guaranteed by Lemma 5.2. □

Remark 5.10. In the case where $\epsilon = \tilde{\epsilon} I_{2\times 2}$ and $\tilde{\epsilon}$ is a real-valued function, conditions on $\tilde{\epsilon}$ that imply (5.2) and (5.4) are discussed in [12, 31].

Remark 5.11. It follows from Theorem 5.9 that under conditions (5.2) and (5.4) (or equivalently condition (5.3)) the cavity problem (1.1) is well-posed for $k \in \mathbb{R}_+$ outside a discrete subset when both ϵ and μ change sign simultaneously, provided that $\Gamma_{\rm imp} \neq \emptyset$. The condition that $\Gamma_{\rm imp} \neq \emptyset$ plays a key role because it implies the solution of the homogeneous problem satisfies (5.13) for $k \in \mathbb{R}\backslash\{0\}$ so that a Poincaré-Friedrichs inequality can be applied. The well-posedness of the cavity problem with the perfectly conducting boundary condition and simultaneous sign changes in ϵ and μ remains an open problem (cf. [13, Remark 3.8]). However, as pointed out in [13, Remark 3.8], if either $\mu(x) \geq \delta > 0$ for all $x \in \Omega$ or $\mu(x) \leq \gamma < 0$ for all $x \in \Omega$, then the conclusion of Theorem 5.9 remains valid even if $\Gamma_{\rm imp} = \emptyset$, since in this case (5.12) implies $\eta = 0$ when $k^2 = i$.

6. Concluding Remarks

The two-dimensional time-harmonic Maxwell equations considered in this paper and [3] are associated with the transverse magnetic problem. But the Hodge decomposition approach can also be applied to the transverse electric problem under various boundary conditions.

For problems on nonconvex domains (cf. Sections 4.3 and 4.4) or problems with discontinuity in the material properties (cf. Section 4.5), optimal convergence can be restored by adaptive algorithms based on the Hodge decomposition approach (cf. [4] for the case of the perfectly conducting boundary condition). For example, the plot in Figure 1 is the result of solving scalar problems with roughly two million dofs. Preliminary results indicate that a similar but slightly better plot can be obtained by an adaptive computation that only involves roughly one hundred thousand dofs.

The investigation of the adaptive versions of the P_1 finite element method in this paper with applications to problems with sign changing electric permittivity and magnetic permeability is an ongoing project.

Acknowledgements

The authors would like to thank two anonymous referees for many helpful suggestions and, in particular, for suggesting that the case of simultaneous sign changes in electric permittivity and magnetic permeability should also be considered.

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