C^0 INTERIOR PENALTY METHODS FOR AN ELLIPTIC DISTRIBUTED OPTIMAL CONTROL PROBLEM ON NONCONVEX POLYGONAL DOMAINS WITH POINTWISE STATE CONSTRAINTS

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Abstract. We design and analyze C^0 interior penalty methods for an elliptic distributed optimal control problem on nonconvex polygonal domains with pointwise state constraints.

Key words. elliptic distributed optimal control problems, pointwise state constraints, nonconvex domains, variational inequalities, discontinuous Galerkin methods

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1. Introduction. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. We will consider the following elliptic distributed optimal control problem with pointwise state constraints (cf. [29]): Find $(y,u) \in H_0^1(\Omega) \times L_2(\Omega)$ that minimizes the cost functional

$$\frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L_2(\Omega)}, \tag{1.1}$$

subjected to the constraints

$$(\nabla y, \nabla v) = (u, v) \qquad \forall\, v \in H_0^1(\Omega), \tag{1.2a}$$

$$y \le Y \qquad a.e. \text{ in } \Omega, \tag{1.2b}$$

where $(\cdot,\cdot)$ is the inner product of $L_2(\Omega)$ (or $[L_2(\Omega)]^2$), $y_d \in L_2(\Omega)$, $Y \in C^2(\Omega) \cap C(\bar\Omega)$ and $Y > 0$ on $\partial\Omega$. Here and below we will follow the standard notation for differential operators, function spaces and norms that can be found for example in [31, 19, 1].

If Ω is convex, then the optimal control problem can be reformulated as a fourth order variational inequality, and finite element methods based on this reformulation have been investigated in [53, 42, 24, 14, 18, 21]. Here we focus on the case where Ω is nonconvex.

Let the space $\mathring{E}(\Omega;\Delta)$ be defined by

$$\mathring{E}(\Omega;\Delta) = \{y \in H_0^1(\Omega) :\ \Delta y \in L_2(\Omega)\},$$

where Δy is understood in the sense of distributions. It is straightforward to check that $\mathring{E}(\Omega;\Delta)$ is a Hilbert space under the inner product

$$(y,z)_{\mathring{E}(\Omega;\Delta)} = (\Delta y, \Delta z) + (y,z).$$

According to the elliptic regularity theory for polygonal domains in [43, 33, 44, 55, 51], $\mathring{E}(\Omega;\Delta)$ is a subspace of $H^{1+\gamma}(\Omega)$ for some $\gamma \in (\frac{1}{2}, 1]$, where γ is determined

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by the interior angles at the corners of Ω. In particular, the functions in $\mathring{E}(\Omega;\Delta)$ belong to $C(\bar{\Omega})$ by the Sobolev embedding theorem (cf. [1]) and

$$\|v\|_{L_\infty(\Omega)} \le C_\Omega \|v\|_{\mathring{E}(\Omega;\Delta)} \qquad \forall\, v \in \mathring{E}(\Omega;\Delta). \tag{1.3}$$

Since (1.2a) is equivalent to $y \in \mathring{E}(\Omega;\Delta)$ and $-\Delta y = u$, we can rewrite the optimal control problem defined by (1.1)–(1.2) as follows:

$$\text{Find} \quad \bar{y} = \operatorname*{argmin}_{y\in\mathbb{K}} \Big[\frac{1}{2}\|y-y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|\Delta y\|^2_{L_2(\Omega)}\Big], \tag{1.4}$$

where $\mathbb{K} = \{y \in \mathring{E}(\Omega;\Delta) : y \le Y \text{ in } \Omega\}$.

C^0 interior penalty methods (cf. [37, 20, 13]) are discontinuous Galerkin methods based on P_k ($k \ge 2$) Lagrange finite element spaces originally designed for the biharmonic equation. They have been applied to fourth order variational inequalities in [23, 22, 24, 18]. Our goal is to extend these methods to (1.4) by taking advantage of the structure of $\mathring{E}(\Omega;\Delta)$ (cf. Section 2.1). We will show that the errors for our methods are $O(h^\alpha)$ on quasi-uniform meshes, where α is determined by the elliptic regularity of simply supported plates (cf. Section 2.4), and $O(h)$ on graded meshes. In the case where the free boundary is smooth, the errors on graded meshes can be improved to $O(h^{1+\delta})$ for any $\delta < 1/2$ if we use cubic or higher order Lagrange elements. As far as we can tell from the literature (cf. [35, 54, 46, 53, 42, 30, 56]), our methods are the only ones proven to be convergent for elliptic distributed optimal control problems with pointwise state constraints on arbitrary polygons. We note that an elliptic optimal control problem on general polygonal domains with pointwise constraints on the gradient of the state was investigated in [60, 61].

The rest of the paper is organized as follows. We consider the continuous problem in Section 2 and introduce a refined minimization problem equivalent to (1.4). The discrete problem, where the constraint (1.2b) is imposed at the vertices of the triangles, is constructed in Section 3, followed by the convergence analyses in Section 4 (quasi-uniform meshes) and Section 5 (graded meshes). Numerical results are presented in Section 6 and we end with some concluding remarks in Section 7.

Throughout the paper we will use C with or without subscripts to denote a generic positive constant independent of the mesh size.

2. The Continuous Problem. Since $\mathbb{K}$ is a nonempty closed convex subset of $\mathring{E}(\Omega;\Delta)$ and the symmetric bilinear form

$$(y,z) \mapsto \beta(\Delta y, \Delta z) + (y,z)$$

is bounded and coercive on $\mathring{E}(\Omega;\Delta)$, it follows from the classical theory (cf. [52, 49, 40, 36]) that the optimization problem (1.4) has a unique solution $\bar{y} \in \mathbb{K}$ characterized by the variational inequality

$$\beta(\Delta\bar{y}, \Delta(y-\bar{y})) + (\bar{y}-y_d, y-\bar{y}) \ge 0 \qquad \forall\, y \in \mathbb{K},$$

and hence

$$\beta(\Delta\bar{y}, \Delta\phi) + (\bar{y}-y_d, \phi) \le 0 \qquad \text{for any nonnegative } \phi \in C_c^\infty(\Omega). \tag{2.1}$$

It then follows from (2.1) and the Riesz Representation Theorem (cf. [58, Section 1.4] and [57, Chapter 2]) that

$$\beta(\Delta\bar{y}, \Delta y) + (\bar{y}-y_d, y) = \int_\Omega y\, d\mu \qquad \forall\, y \in \mathring{E}(\Omega;\Delta), \tag{2.2}$$

where $\mu \leq 0$ is a regular Borel measure.

Let $\mathfrak{C}$ be the active set for the constraint (1.2b) defined by $\mathfrak{C} = \{x \in \Omega : \bar{y}(x) = Y(x)\}$. Then we have, by the principle of virtual work,

$$\beta(\Delta\bar{y}, \Delta\phi) + (\bar{y} - y_d, \phi) = 0 \qquad \text{if } \phi \in C_c^\infty(\Omega) \text{ and } \operatorname{supp}\phi \cap \mathfrak{C} = \emptyset$$

and hence

$$\operatorname{supp}\mu \subseteq \mathfrak{C}. \tag{2.3}$$

Since $Y > 0$ and $\bar{y} = 0$ on $\partial\Omega$, the active set $\mathfrak{C}$ is a compact subset of Ω. Consequently μ is a finite measure.

2.1. Structure of $\mathring{E}(\Omega;\Delta)$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_J$ be the reentrant corners of Ω. We have the following decomposition of $\mathring{E}(\Omega;\Delta)$ by the elliptic regularity theory for polygonal domains in [43, 33, 44, 55, 51]:

$$\mathring{E}(\Omega;\Delta) = [H^2(\Omega) \cap H_0^1(\Omega)] \oplus \langle \phi_1, \ldots, \phi_J \rangle, \tag{2.4}$$

where the functions $\phi_1, \ldots, \phi_J$ can be chosen so that $\phi_1^* = -\Delta\phi_1, \ldots, \phi_J^* = -\Delta\phi_J$ form a basis of the orthogonal complement of $\Delta[H^2(\Omega) \cap H_0^1(\Omega)]$ in $L_2(\Omega)$.

The Construction of ϕ_j. Let $\omega_j > \pi$ be the interior angle at the reentrant corner $\mathcal{C}_j$, and (r_j, θ_j) be the polar coordinates at $\mathcal{C}_j$ so that the two edges emanating from $\mathcal{C}_j$ are given by $\theta = 0$ and $\theta = \omega_j$. Let $\xi_j = \eta_j\big[r_j^{-\pi/\omega_j} \sin\big((\pi/\omega_j)\theta_j\big)\big]$, where the polynomial η_j is chosen so that (i) $\xi_j = 0$ on $\partial\Omega \setminus \{\mathcal{C}_j\}$, (ii) $\eta_j = 1$ at $\mathcal{C}_j$, and (iii) $\nabla\eta_j = \mathbf{0}$ at $\mathcal{C}_j$. For example we can take

$$\eta_j(x) = \Big[1 + \sum_{\ell=1}^{L} \big(a_\ell(x_1 - x_{j,1}) + b_\ell(x_2 - x_{j,2})\big)\Big] \prod_{\ell=1}^{L} \big[1 - a_\ell(x_1 - x_{j,1}) - b_\ell(x_2 - x_{j,2})\big],$$

where $(x_{j,1}, x_{j,2})$ are the coordinates of $\mathcal{C}_j$ and $1 - a_\ell(x_1 - x_{j,1}) - b_\ell(x_2 - x_{j,2}) = 0$ for $1 \leq \ell \leq L$ define the edges of Ω away from $\mathcal{C}_j$.

Let $\zeta_j \in H_0^1(\Omega)$ be defined by

$$(\nabla\zeta_j, \nabla v) = (\Delta\xi_j, v) \qquad \forall\, v \in H_0^1(\Omega). \tag{2.5}$$

Then the function ϕ_j^* is defined by

$$\phi_j^* = \zeta_j + \xi_j, \tag{2.6}$$

and the function $\phi_j \in H_0^1(\Omega)$ is given by

$$(\nabla\phi_j, \nabla v) = (\phi_j^*, v) \qquad \forall\, v \in H_0^1(\Omega). \tag{2.7}$$

Remark 2.1. Since the function $r_j^{-\pi/\omega_j} \sin\big((\pi/\omega_j)\theta_j\big)$ is a harmonic function, we have

$$\Delta\xi_j = \Delta\big[(\eta_j - 1) r_j^{-\pi/\omega_j} \sin\big((\pi_j/\omega_j)\theta_j\big)\big], \tag{2.8}$$

which implies $\Delta\xi_j \in L_2(\Omega)$ (because $\eta_j - 1 = O(r^2)$) and $\Delta\xi_j \in C^\infty(\bar{\Omega} \setminus \{\mathcal{C}_j\})$.

Remark 2.2. One can also replace the polynomial η_j by a cut-off function in order to enforce conditions (i)–(iii) in the construction of ξ_j, in which case the radius for the cut-off region has to be chosen carefully. The construction based on η_j is simpler.

Remark 2.3. It is well-known that

$$\phi_1, \dots, \phi_J \in H^{1+\gamma}(\Omega) \tag{2.9}$$

for any $\gamma < \pi/\omega_*$, where ω_* is the largest reentrant corner of Ω. It follows from (2.4) and (2.9) that $\mathring{E}(\Omega;\Delta)$ is a subspace of $H^{1+\gamma}(\Omega)$.

Remark 2.4. In view of Remark 2.1 and standard elliptic regularity [2], the function ζ_j belongs to $C^\infty(\bar{\Omega}\setminus\mathcal{C})$, where $\mathcal{C}$ is the set of the corners of Ω.

Remark 2.5. It follows from (2.6) and Remark 2.4 that $\phi_j^* \in C^\infty(\Omega)$, and consequently ϕ_j belongs to $C^\infty(\bar{\Omega}\setminus\mathcal{C})$ by standard elliptic regularity.

2.2. A Refined Minimization Problem. According to (2.4), any $y \in \mathring{E}(\Omega;\Delta)$ can be written as $y = y_{\scriptscriptstyle R} + \phi_{\boldsymbol{\tau}}$, where $y_{\scriptscriptstyle R} \in H^2(\Omega)\cap H_0^1(\Omega)$, $\boldsymbol{\tau} = (\tau_1, \dots, \tau_J) \in \mathbb{R}^J$ and

$$\phi_{\boldsymbol{\tau}} = \sum_{j=1}^{J} \tau_j \phi_j. \tag{2.10}$$

In particular we have

$$\bar{y} = \bar{y}_{\scriptscriptstyle R} + \phi_{\bar{\boldsymbol{\tau}}} = \bar{y}_{\scriptscriptstyle R} + \sum_{j=1}^{J} \bar{\tau}_j \phi_j. \tag{2.11}$$

Remark 2.6. Let $y_{\scriptscriptstyle R} + \phi_{\boldsymbol{\tau}}$ be the decomposition of $y \in \mathring{E}(\Omega;\Delta)$. Then we have

$$\|y\|_{\mathring{E}(\Omega;\Delta)} \approx \|y_{\scriptscriptstyle R}\|_{H^2(\Omega)} + |\boldsymbol{\tau}|,$$

where $|\boldsymbol{\tau}|$ is the Euclidean norm of $\boldsymbol{\tau}$.

We will also use the shorthand notation

$$\phi_{\boldsymbol{\tau}}^* = \sum_{j=1}^{J} \tau_j \phi_j^* = -\Delta\phi_{\boldsymbol{\tau}}. \tag{2.12}$$

Note that both $\phi_{\boldsymbol{\tau}}$ and $\phi_{\boldsymbol{\tau}}^*$ depend linearly on $\boldsymbol{\tau}$.

Since $(\Delta y_{\scriptscriptstyle R}, \Delta\phi_{\boldsymbol{\tau}}) = 0$, the minimization problem (1.4) is equivalent to the following problem: Find

$$\begin{aligned}(\bar{y}_{\scriptscriptstyle R}, \bar{\boldsymbol{\tau}}) &= \operatorname*{argmin}_{(y_{\scriptscriptstyle R},\boldsymbol{\tau})\in K} \frac{1}{2}\big[\big(\beta\|\Delta y_{\scriptscriptstyle R}\|^2_{L_2(\Omega)} + \beta\|\phi^*_{\boldsymbol{\tau}}\|^2_{L_2(\Omega)} + \|(y_{\scriptscriptstyle R}+\phi_{\boldsymbol{\tau}}) - y_d\|^2_{L_2(\Omega)}\big] \\ &= \operatorname*{argmin}_{(y_{\scriptscriptstyle R},\boldsymbol{\tau})\in K} \frac{1}{2}\big[\big(\beta a(y_{\scriptscriptstyle R}, y_{\scriptscriptstyle R}) + \beta\|\phi^*_{\boldsymbol{\tau}}\|^2_{L_2(\Omega)} + \|(y_{\scriptscriptstyle R}+\phi_{\boldsymbol{\tau}}) - y_d\|^2_{L_2(\Omega)}\big],\end{aligned} \tag{2.13}$$

where

$$a(y,z) = \int_\Omega D^2 y : D^2 z\,dx = \int_\Omega \sum_{i,j=1}^{2} \Big(\frac{\partial^2 y}{\partial x_i \partial x_j}\Big)\Big(\frac{\partial^2 z}{\partial x_i \partial x_j}\Big) dx$$

and

$$K = \{(y_R, \boldsymbol{\tau}) \in [H^2(\Omega) \cap H_0^1(\Omega)] \oplus \mathbb{R}^J : y_R + \phi_{\boldsymbol{\tau}} \le Y \text{ on } \Omega\}.$$

Note that we have used the identity (cf. [44, Lemma 2.2.2])

$$\|\Delta y\|^2_{L_2(\Omega)} = \int_\Omega (\Delta y)(\Delta y)\, dx = \int_\Omega D^2 y : D^2 y\, dx = |y|^2_{H^2(\Omega)} \quad \forall\, y \in \mathring{E}(\Omega; \Delta).$$

We can write (2.13) in a concise form:

$$\text{Find} \quad (\bar{y}_R, \bar{\boldsymbol{\tau}}) = \operatorname*{argmin}_{(y_R, \boldsymbol{\tau}) \in K} \Big[\frac{1}{2}\mathcal{A}((y_R, \boldsymbol{\tau}), (y_R, \boldsymbol{\tau})) - (y_d, y_R + \phi_{\boldsymbol{\tau}})\Big], \tag{2.14}$$

where

$$\mathcal{A}((y_R, \boldsymbol{\tau}), (z_R, \boldsymbol{\rho})) = \beta a(y_R, z_R) + \beta(\phi^*_{\boldsymbol{\tau}}, \phi^*_{\boldsymbol{\rho}}) + (y_R + \phi_{\boldsymbol{\tau}}, z_R + \phi_{\boldsymbol{\rho}}). \tag{2.15}$$

First Order Optimality Conditions. It follows from (2.14) that we can rewrite (2.2) and (2.3) as

$$\mathcal{A}((\bar{y}_R, \bar{\boldsymbol{\tau}}), (z_R, \boldsymbol{\rho})) - (y_d, z_R + \phi_{\boldsymbol{\rho}}) = \int_\Omega (z_R + \phi_{\boldsymbol{\rho}}) d\mu \tag{2.16}$$

for all $(z_R, \boldsymbol{\rho}) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times \mathbb{R}^J$, where $\mu \le 0$ is a finite Borel measure and

$$\int_\Omega (\bar{y}_R + \phi_{\bar{\boldsymbol{\tau}}} - Y) d\mu = 0. \tag{2.17}$$

2.3. Regularity of $\bar{y}_R$. In view of (2.13), we have

$$\bar{y}_R = \operatorname*{argmin}_{y_R \in \mathcal{K}} \Big[\frac{1}{2}\big(\beta\|\Delta y_R\|^2_{L_2(\Omega)} + \|y_R\|^2_{L_2(\Omega)}\big) - (y_d - \phi_{\bar{\boldsymbol{\tau}}}, y_R)\Big],$$

where $\mathcal{K} = \{v \in H^2(\Omega) \cap H_0^1(\Omega) : v \le Y - \phi_{\bar{\boldsymbol{\tau}}}\}$, i.e., $\bar{y}_R \in H^2(\Omega) \cap H_0^1(\Omega)$ is the solution of an obstacle problem for simply supported plates.

Since $Y - \phi_{\bar{\boldsymbol{\tau}}} \in C^2(\Omega) \cap C(\bar{\Omega})$ and $Y - \phi_{\bar{\boldsymbol{\tau}}} > 0$ on $\partial\Omega$, we can conclude from the results in [50, 11, 38, 39, 27] that

$$\bar{y}_R \in C^2(\Omega) \cap H^3_{loc}(\Omega) \cap H^{2+\alpha}(\Omega) \cap H_0^1(\Omega), \tag{2.18}$$

where the index of elliptic regularity $\alpha \in (0, 1)$ for the biharmonic equation with the boundary conditions of simply supported plates is determined by the angles at the corners of Ω (cf. Section 2.4).

Remark 2.7. In fact, by standard elliptic regularity [2], $\bar{y}_R$ belongs to H^4 away from the active set and the corners of Ω.

2.4. Index of Elliptic Regularity. Let the function $\alpha_\dagger$ be defined by

$$\alpha_\dagger(\omega) = \begin{cases} (\pi/\omega) - 1 & \text{if } \omega \in (0, \pi/2) \cup (\pi/2, \pi), \\ 2 & \text{if } \omega = \pi/2, \\ 1 - (\pi/\omega) & \text{if } \pi < \omega \le (3\pi/2), \\ 2(\pi/\omega) - 1 & \text{if } (3\pi/2) \le \omega < 2\pi. \end{cases} \tag{2.19}$$

Note that

$$\alpha_\dagger(\omega) < \pi/\omega \quad \text{for} \quad \omega \in (0, 2\pi) \setminus \{\pi/2\}. \tag{2.20}$$

According to [11, 33], the solution of the biharmonic equation with the boundary condition of simply supported plates belongs to $H^{2+\alpha_\dagger(\omega)-\epsilon}$ near a corner of Ω whose interior angle is ω ($\epsilon > 0$ is arbitrary).

Let α_* be the minimum of $\alpha_\dagger(\omega)$ over all the interior angles ω at the corners of Ω. Then we can take the index of elliptic regularity α in (2.18) to be $\alpha_* - \epsilon$ for any $\epsilon > 0$.

2.5. Regularity of μ. Since μ is a finite measure supported on $\mathfrak{C}$, we have an obvious estimate

$$\Big| \int_\Omega y \, d\mu \Big| \le C \|y\|_{L_\infty(\mathfrak{C})} \qquad \forall \, y \in C(\Omega). \tag{2.21}$$

Other estimates involving μ can be derived from the interior regularity of $\bar{y}$.

Let G be an open neighborhood of $\mathfrak{C}$ with a smooth boundary such that $\bar{G}$ is a compact subset of Ω, and let Φ be a C^∞ function supported in G such that $0 \le \Phi \le 1$ and $\Phi = 1$ on $\mathfrak{C}$. We have, by (2.2), (2.3) and integration by parts,

$$\begin{aligned} \int_\Omega y \, d\mu = \int_\Omega (\Phi y) \, d\mu &= -\beta\big(\nabla(\Delta \bar{y}), \nabla(\Phi y)\big) + (\bar{y} - y_d, \Phi y) \\ &= B(\bar{y}, y) - (y_d, \Phi y) \qquad \forall \, y \in \mathring{E}(\Omega; \Delta), \end{aligned} \tag{2.22}$$

where $B(\zeta, \chi) = -\beta\big(\nabla(\Delta\zeta), \nabla(\Phi\chi)\big) + (\zeta, \Phi\chi)$.

We have

$$|B(\zeta, \chi)| \le C_G \|\zeta\|_{H^3(G)} \|\chi\|_{H^1(G)} \qquad \forall \, \zeta \in H^3(G), \, \chi \in H^1(G), \tag{2.23}$$

and also, through another integration by parts,

$$|B(\zeta, \chi)| \le C_G \|\zeta\|_{H^4(G)} \|\chi\|_{L_2(G)} \qquad \forall \, \zeta \in H^4(G), \, \chi \in H^1(G). \tag{2.24}$$

Since $\bar{y}$ belongs to $H^3_{loc}(\Omega)$ by Remark 2.5, (2.11) and (2.18), we can combine (2.23) and (2.22) to conclude

$$\Big| \int_\Omega y \, d\mu \Big| \le (C_G \|\bar{y}\|_{H^3(G)} + \|y_d\|_{L_2(G)}) \|y\|_{H^1(G)} \le C \|y\|_{H^1(\Omega)} \tag{2.25}$$

for all $y \in \mathring{E}(\Omega; \Delta)$. Consequently we can treat μ as a member of $H^{-1}(\Omega)$.

In the case where $\bar{y}$ belongs to $H^{3+\delta}_{loc}(\Omega)$ for some $\delta \in (0, 1)$, we can further improve the regularity of μ. Observe that (2.23), (2.24) and the interpolation of bilinear forms on Sobolev spaces [8, Section 4.4] imply that B can be extended to $H^{3+\delta}(G) \times H^{1-\delta}(G)$ such that

$$|B(\zeta, \chi)| \le C_{G,\delta} \|\zeta\|_{H^{3+\delta}(G)} \|\chi\|_{H^{1-\delta}(G)} \qquad \forall \, \zeta \in H^{3+\delta}(G) \text{ and } \chi \in H^{1-\delta}(G),$$

which together with (2.22) implies

$$\Big| \int_\Omega y \, d\mu \Big| \le C_\delta \|y\|_{H^{1-\delta}(\Omega)} \qquad \forall \, y \in \mathring{E}(\Omega; \Delta). \tag{2.26}$$

Therefore in this case we can treat μ as a member of $H^{-1+\delta}(\Omega)$.

3. C^0 Interior Penalty Methods. Let $\mathcal{T}_h$ be a shape regular triangulation of Ω, $\mathcal{E}_h^i$ be the set of the edges of $\mathcal{T}_h$ interior to Ω, $V_h \subset H_0^1(\Omega)$ be the P_k ($k \geq 2$) Lagrange finite element space associated with $\mathcal{T}_h$, and $a_h(\cdot,\cdot)$ be the bilinear form

$$a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e \Big(\Big\{\!\!\Big\{\frac{\partial^2 w}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big] + \Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial w}{\partial n}\Big]\!\!\Big]\Big)ds$$
$$+ \sum_{e\in\mathcal{E}_h^i}\frac{\sigma}{|e|}\int_e \Big[\!\!\Big[\frac{\partial w}{\partial n}\Big]\!\!\Big]\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]\,ds,$$

where $|e|$ is the length of the edge e, σ is a positive penalty parameter, and the jumps and averages of the normal derivatives for the functions in V_h are defined as follows.

Let $e \in \mathcal{E}_h^i$ be the common edge of $T_e^{\pm} \in \mathcal{T}_h$ and n_e be the unit normal of e pointing from T_e^- to T_e^+. We define on e

$$\Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\} = \frac{1}{2}\Big(\frac{\partial^2 v_+}{\partial n_e^2}\Big|_e + \frac{\partial^2 v_-}{\partial n_e^2}\Big|_e\Big) \quad \text{and} \quad \Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big] = \frac{\partial v_+}{\partial n_e}\Big|_e - \frac{\partial v_-}{\partial n_e}\Big|_e \qquad \forall\, v \in V_h,$$

where $v_\pm = v\big|_{T_e^\pm}$.

Remark 3.1. The bilinear form $a_h(\cdot,\cdot)$ is the one that appears in C^0 interior penalty methods for simply supported plates in [16, 13, 24]. It is independent of the choices of $T_e^\pm$.

For σ sufficiently large, we have (cf. [20, 13])

$$a_h(y_h,y_h) \geq C_\sharp \|y_h\|_h^2 \qquad \forall\, y_h \in V_h, \tag{3.1}$$

where the mesh-dependent norm $\|\cdot\|_h$ is defined by

$$\|y_h\|_h^2 = \sum_{T\in\mathcal{T}_h}|y_h|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\Big\|\Big[\!\!\Big[\frac{\partial y_h}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}^2.$$

Note that we have the following discrete Poincaré-Friedrichs and Sobolev inequalities [26, 17]

$$\|y_h\|_{L_2(\Omega)} + |y_h|_{H^1(\Omega)} + \|y_h\|_{L_\infty(\Omega)} \leq C\|y_h\|_h \qquad \forall\, y_h \in V_h. \tag{3.2}$$

We will use $a_h(\cdot,\cdot)$ in a discrete analog of (2.13)/(2.14), for which we also need the discrete analogs of the singular functions.

3.1. Discrete Singular Functions. Let $\zeta_{j,h} \in V_h$ be defined by

$$(\nabla\zeta_{j,h},\nabla v) = (\Delta\xi_j, v) \qquad \forall\, v \in V_h, \tag{3.3}$$

where ξ_j is the function from Section 2.1. In other words we have $\zeta_{j,h} = R_h\zeta_j$, where ζ_j is defined by (2.5) and $R_h : H_0^1(\Omega) \longrightarrow V_h$ is the Ritz projection.

We then define $\phi_{j,h}^* \in L_2(\Omega)$ by

$$\phi_{j,h}^* = \zeta_{j,h} + \xi_j, \tag{3.4}$$

and $\phi_{j,h} \in V_h$ by

$$(\nabla\phi_{j,h},\nabla v) = (\phi_{j,h}^*, v) \qquad \forall\, v \in V_h. \tag{3.5}$$

Remark 3.2. The constructions of $\zeta_{j,h} \in V_h$, $\phi^*_{j,h} \in L_2(\Omega)$ and $\phi_{j,h} \in V_h$ given by (3.3)–(3.5) are the discrete analogs of the constructions of $\zeta_j \in H^1_0(\Omega)$, $\phi^*_{j,h} \in L_2(\Omega)$ and $\phi_j \in H^1_0(\Omega)$ given by (2.5)–(2.7). Other constructions can be found for example in [32], where $\phi^*_{j,h}$ is approximated by the sum of a finite element function and the exact dual singular function and $\phi_{j,h}$ is approximated by the sum of a finite element function and a multiple of the exact singular function. A numerical scheme based on the constructions in [32] would likely have similar performance.

3.2. The Discrete Problem. The discrete problem for (2.13) is to find $\bar{y}_{R,h} \in V_h$ and $\bar{\boldsymbol{\tau}}_h = (\bar{\tau}_{h,1}, \dots, \bar{\tau}_{h,J}) \in \mathbb{R}^J$ such that

$$(\bar{y}_{R,h}, \bar{\boldsymbol{\tau}}_h) = \underset{(y_{R,h},\boldsymbol{\tau})\in K_h}{\operatorname{argmin}} \frac{1}{2}\big[\beta a_h(y_{R,h}, y_{R,h}) + \beta\|\phi^*_{\boldsymbol{\tau},h}\|^2_{L_2(\Omega)} + \|(y_{R,h} + \phi_{\boldsymbol{\tau},h}) - y_d\|^2_{L_2(\Omega)}\big], \tag{3.6}$$

where $\boldsymbol{\tau} = (\tau_1, \dots, \tau_J) \in \mathbb{R}^J$,

$$\phi_{\boldsymbol{\tau},h} = \sum_{j=1}^{J} \tau_j \phi_{j,h}, \qquad \phi^*_{\boldsymbol{\tau},h} = \sum_{j=1}^{J} \tau_j \phi^*_{j,h}, \tag{3.7}$$

and

$$\begin{aligned} K_h &= \{(y_{R,h}, \boldsymbol{\tau}) \in V_h \oplus \mathbb{R}^J : y_{R,h}(p) + \phi_{\boldsymbol{\tau},h}(p) \le Y(p) \\ &\qquad \text{at all the vertices of } \mathcal{T}_h\} \\ &= \{(y_{R,h}, \boldsymbol{\tau}) \in V_h \oplus \mathbb{R}^J : I_h(y_{R,h} + \phi_{\boldsymbol{\tau},h}) \le I_h Y\}. \end{aligned} \tag{3.8}$$

Here I_h is the nodal interpolation operator for the conforming P_1 finite element space associated with $\mathcal{T}_h$.

Remark 3.3. Let the open subset G of Ω and the function Φ be defined as in Section 2.5, and let $(\bar{y}_R, \bar{\boldsymbol{\tau}}) \in K$ be the solution of (2.13). Then $(\Pi_h(\bar{y}_R - \delta_*\Phi), \bar{\boldsymbol{\tau}}) \in K_h$, where Π_h is the nodal interpolation operator for the P_k Lagrange finite element space and

$$\delta_* = \|\phi_{\bar{\boldsymbol{\tau}}} - \phi_{\bar{\boldsymbol{\tau}},h}\|_{L_\infty(G)}. \tag{3.9}$$

In particular K_h is nonempty.

The discrete optimal state $\bar{y}_h$ is then given by

$$\bar{y}_h = \bar{y}_{R,h} + \phi_{\bar{\boldsymbol{\tau}}_h,h} = \bar{y}_{R,h} + \sum_{j=1}^{J} \bar{\boldsymbol{\tau}}_{h,j}\phi_{j,h}. \tag{3.10}$$

Remark 3.4. If Ω is convex, then $\phi_{\boldsymbol{\tau},h} = \phi^*_{\boldsymbol{\tau},h} = 0$ and (3.6) reduces to the C^0 interior penalty method in [24].

We can also write (3.6) in a concise form:

$$\text{Find} \quad (\bar{y}_{R,h}, \bar{\boldsymbol{\tau}}_h) = \underset{(y_{R,h},\boldsymbol{\tau})\in K_h}{\operatorname{argmin}} \Big[\frac{1}{2}\mathcal{A}_h((y_{R,h}, \boldsymbol{\tau}), (y_{R,h}, \boldsymbol{\tau})) - (y_d, y_{R,h} + \phi_{\boldsymbol{\tau},h})\Big], \tag{3.11}$$

where the bilinear form $\mathcal{A}_h(\cdot,\cdot)$ is defined by

$$\mathcal{A}_h((y_{R,h},\boldsymbol{\tau}),(z_{R,h},\boldsymbol{\rho})) = \beta a_h(y_{R,h},z_{R,h}) + \beta(\phi^*_{\boldsymbol{\tau},h},\phi^*_{\boldsymbol{\rho},h}) + (y_{R,h}+\phi_{\boldsymbol{\tau},h}, z_{R,h}+\phi_{\boldsymbol{\rho},h}). \tag{3.12}$$

By the norm equivalence

$$\|\phi^*_{\boldsymbol{\tau},h}\|_{L_2(\Omega)} \approx |\boldsymbol{\tau}| \approx \|\phi_{\boldsymbol{\tau},h}\|_{L_2(\Omega)} \tag{3.13}$$

and the estimate (3.1), we have

$$\mathcal{A}_h((y_{R,h},\boldsymbol{\tau}),(y_{R,h},\boldsymbol{\tau})) \ge C_\flat\big(\|y_{R,h}\|_h^2 + |\boldsymbol{\tau}|^2\big) \qquad \forall\,(y_{R,h},\boldsymbol{\tau}) \in V_h \oplus \mathbb{R}^J. \tag{3.14}$$

Remark 3.5. In view of the fact that $(\phi^*_{h,j},\phi_{h,j}) \to (\phi^*_j,\phi_j)$ in $L_2(\Omega)\times L_2(\Omega)$ as $h\downarrow 0$ (cf. (4.2) and (4.3)), the hidden constants in (3.13) are independent of h.

A Discrete Variational Inequality. Since K_h is a nonempty closed convex subset of $V_h\oplus\mathbb{R}^J$ and the bilinear form $\mathcal{A}_h(\cdot,\cdot)$ is coercive by (3.14), the discrete problem (3.6)/(3.11) has a unique solution characterized by the discrete variational inequality

$$\mathcal{A}_h((\bar{y}_{R,h},\bar{\boldsymbol{\tau}}_h),(y_{R,h}-\bar{y}_{R,h},\boldsymbol{\tau}-\bar{\boldsymbol{\tau}}_h)) - (y_d,(y_{R,h}-\bar{y}_{R,h})+\phi_{\boldsymbol{\tau}-\bar{\boldsymbol{\tau}}_h,h}) \ge 0 \tag{3.15}$$

for all $(y_{R,h},\boldsymbol{\tau}) \in K_h$.

4. Convergence Analysis: Quasi-Uniform Meshes. In this section we consider the convergence of C^0 interior penalty methods on quasi-uniform meshes (cf. [31, 19]) and extend the convergence analyses in [24, 21] to nonconvex polygonal domains. We begin with an analysis for the discrete singular functions.

4.1. Estimates for the Singular Functions. Let ω_* be the largest angle at the reentrant corners of Ω and $\nu=\pi/\omega_*$.

Lemma 4.1. *We have the following error estimates*:

$$\|\zeta_j-\zeta_{j,h}\|_{L_2(\Omega)} \le Ch^{2\nu}, \tag{4.1}$$

$$\|\phi^*_j-\phi^*_{j,h}\|_{L_2(\Omega)} \le Ch^{2\nu}, \tag{4.2}$$

$$\|\phi_j-\phi_{j,h}\|_{L_2(\Omega)} \le Ch^{2\nu}. \tag{4.3}$$

Proof. Recall $\Pi_h : C(\bar{\Omega}) \longrightarrow V_h$ is the nodal interpolation operator. We have a standard error estimate [31, 19]

$$|\eta-\Pi_h\eta|_{H^1(\Omega)} \le Ch|\eta|_{H^2(\Omega)} \qquad \forall\,\eta\in H^2(\Omega)\cap H^1_0(\Omega),$$

and, by a direct calculation (cf. [7, Lemma 5.1]), $|\phi_j-\Pi_h\phi_j|_{H^1(\Omega)} \le Ch^\nu$ for $1\le j\le J$.

Combining these two estimates with (2.4) and Remark 2.6, we have

$$|\zeta-\Pi_h\zeta|_{H^1(\Omega)} \le Ch^\nu\|\zeta\|_{\mathring{E}(\Omega;\Delta)} \qquad \forall\,\zeta\in\mathring{E}(\Omega;\Delta). \tag{4.4}$$

It follows from (4.4) and Galerkin orthogonality that

$$|\zeta_j-\zeta_{j,h}|_{H^1(\Omega)} = |\zeta_j-R_h\zeta_j|_{H^1(\Omega)} \le |\zeta_j-\Pi_h\zeta_j|_{H^1(\Omega)} \le Ch^\nu\|\zeta_j\|_{\mathring{E}(\Omega;\Delta)}. \tag{4.5}$$

The estimate (4.1) follows from (4.5) and a standard duality argument. The estimate (4.2) then follows immediately from (2.6), (3.4) and (4.1).

Let $\tilde{\phi}_{j,h} \in H_0^1(\Omega)$ satisfy

$$(\nabla\tilde{\phi}_{j,h}, \nabla v) = (\phi^*_{j,h}, v) \qquad \forall\, v \in H_0^1(\Omega). \tag{4.6}$$

Then we have $\tilde{\phi}_{j,h} \in \mathring{E}(\Omega;\Delta)$ and

$$\|\phi_j - \tilde{\phi}_{j,h}\|_{\mathring{E}(\Omega;\Delta)} \le Ch^{2\nu} \tag{4.7}$$

by (2.7), (4.2) and (4.6). Note that (3.5) and (4.6) imply $\phi_{j,h} = R_h\tilde{\phi}_{j,h}$, and hence

$$\|\tilde{\phi}_{j,h}-\phi_{j,h}\|_{L_2(\Omega)} \le Ch^{\nu}|\tilde{\phi}_{j,h}-R_h\tilde{\phi}_{j,h}|_{H^1(\Omega)} \le Ch^{\nu}|\tilde{\phi}_{j,h}-\Pi_h\tilde{\phi}_{j,h}|_{H^1(\Omega)} \le Ch^{2\nu} \tag{4.8}$$

by (4.4) and a standard duality argument.

The estimate (4.3) follows from (4.7) and (4.8). ◻

From (2.10), (2.12), (3.7), (4.2) and (4.3) we immediately have the following estimate:

$$\|\phi_{\boldsymbol{\tau}} - \phi_{\boldsymbol{\tau},h}\|_{L_2(\Omega)} + \|\phi^*_{\boldsymbol{\tau}} - \phi^*_{\boldsymbol{\tau},h}\|_{L_2(\Omega)} \le Ch^{2\nu}|\boldsymbol{\tau}|. \tag{4.9}$$

Remark 4.2. It follows from (3.4) that $\phi^*_{j,h}$ belongs to $H^1_{loc}(\Omega)$ and hence $\tilde{\phi}_{j,h}$ defined by (4.6) belongs to $H^3_{loc}(\Omega)$ by interior elliptic regularity [2]. In view of (4.2), the H^3 norm of $\tilde{\phi}_{j,h}$ on any open subset of Ω away from $\partial\Omega$ is bounded by a constant independent of h.

Interior Estimates. Let G be the open neighborhood of $\mathfrak{C}$ from Section 2.5. Since $\phi_j \in C^\infty(\Omega)$, $\tilde{\phi}_{j,h} \in H^3_{loc}(\Omega)$ and $k \ge 2$, we also have the following interior error estimate (cf. [59, Theorem 9.1 and Theorem 10.1]):

$$|\phi_j - R_h\phi_j|_{H^1(G)} \le Ch^{2\nu}, \tag{4.10}$$

$$\|\tilde{\phi}_{j,h} - R_h\tilde{\phi}_{j,h}\|_{L_\infty(G)} \le Ch^{2\nu}. \tag{4.11}$$

It follows from (4.7) and (4.10) that

$$|\phi_j - \phi_{j,h}|_{H^1(G)} \le |\phi_j - R_h\phi_j|_{H^1(G)} + |R_h(\phi_j - \tilde{\phi}_{j,h})|_{H^1(G)} \le Ch^{2\nu}, \tag{4.12}$$

and from (1.3), (4.7) and (4.11) we have

$$\|\phi_j - \phi_{j,h}\|_{L_\infty(G)} \le \|\phi_j - \tilde{\phi}_{j,h}\|_{L_\infty(G)} + \|\tilde{\phi}_{j,h} - R_h\tilde{\phi}_{j,h}\|_{L_\infty(G)} \le Ch^{2\nu}. \tag{4.13}$$

In particular the number δ_* defined in (3.9) satisfies the estimate

$$\delta_* \le Ch^{2\nu}. \tag{4.14}$$

By standard interpolation error estimates (cf. [31, 19]) we also have

$$\|\phi_j - \Pi_h\phi_j\|_{H^1(G)} + \|\phi_j - I_h\phi_j\|_{L_\infty(\mathfrak{C})} \le Ch^k. \tag{4.15}$$

4.2. Connection Operators. The key ingredients for the analysis of the C^0 interior penalty methods are certain operators that connect the continuous and discrete spaces.

We can connect the continuous space to the discrete space by the standard nodal interpolation operator $\Pi_h : H^2(\Omega)\cap H^1_0(\Omega) \longrightarrow V_h$. In the other direction the discrete space is connected to the continuous space by an enriching operator $E_h : V_h \longrightarrow H^2(\Omega)\cap H^1_0(\Omega)$ constructed through averaging (cf. [24, 21]).

Recall that α is the index of elliptic regularity that appears in (2.18). The proofs of the following results can be found in [13, 24].

LEMMA 4.3. *We have*

$$\|\bar{y}_R - \Pi_h\bar{y}_R\|_{L_2(\Omega)} + h|\bar{y}_R - \Pi_h\bar{y}_R|_{H^1(\Omega)} + h^2\|\bar{y}_R - \Pi_h\bar{y}_R\|_h \le Ch^{2+\alpha}, \tag{4.16}$$

$$\sum_{k=0}^{2} h^k|\bar{y}_R - E_h\Pi_h\bar{y}_R|_{H^k(\Omega)} \le Ch^{2+\alpha}, \tag{4.17}$$

and, for all $v \in V_h$,

$$(E_h v)(p) = v(p) \qquad \text{at any vertex } p, \tag{4.18}$$

$$\sum_{k=0}^{2} h^{2k} \sum_{T\in\mathcal{T}_h} |v - E_h v|^2_{H^k(T)} \le Ch^4\|v\|_h^2, \tag{4.19}$$

$$|a(\bar{y}_R, E_h v) - a_h(\Pi_h\bar{y}_R, v)| \le Ch^{\alpha}\|v\|_h. \tag{4.20}$$

Remark 4.4. The estimate (4.17) indicates that $E_h\Pi_h$ behaves like a quasi-local interpolation operator, and the estimate (4.20) means that E_h acts like the transpose of Π_h with respect to the bilinear forms $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$.

4.3. Preliminary Estimates. The estimates below will reduce the error analysis to the continuous setting. First we observe that Remark 3.3, (3.14), (3.15) and (4.16) imply

$$\begin{aligned}
&\|\bar{y}_R - \bar{y}_{R,h}\|_h^2 + |\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h|^2\\
&\quad\le 2\|\bar{y}_R - \Pi_h\bar{y}_R\|_h^2 + 2\|\Pi_h\bar{y}_R - \bar{y}_{R,h}\|_h^2 + |\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h|^2\\
&\quad\le C_1h^{2\alpha} + C_2\mathcal{A}_h\big((\Pi_h\bar{y}_R - \bar{y}_{R,h}, \bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h), (\Pi_h\bar{y}_R - \bar{y}_{R,h}, \bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h)\big)\\
&\quad= C_1h^{2\alpha} + C_2\big[\mathcal{A}_h\big((\Pi_h\bar{y}_R - \bar{y}_{R,h}, \bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h), (\Pi_h(\delta_*\Phi), 0)\big)\\
&\qquad\qquad + \mathcal{A}_h\big((\Pi_h\bar{y}_R - \bar{y}_{R,h}, \bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h), (\Pi_h(\bar{y}_R - \delta_*\Phi) - \bar{y}_{R,h}, \bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h)\big)\big]\\
&\quad\le C_1h^{2\alpha} + C_3\delta_*\big(\|\Pi_h\bar{y}_R - \bar{y}_{R,h}\|_h + |\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h|\big)\\
&\qquad + C_2\big[\mathcal{A}_h\big((\Pi_h\bar{y}_R, \bar{\boldsymbol{\tau}}), (\Pi_h(\bar{y}_R - \delta_*\Phi) - \bar{y}_{R,h}, \bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h)\big)\\
&\qquad\qquad - \big(y_d, (\Pi_h(\bar{y}_R - \delta_*\Phi) - \bar{y}_{R,h}) + \phi_{\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h,h}\big)\big]\\
&\quad\le C_1h^{2\alpha} + C_3\delta_*\big(\|\Pi_h\bar{y}_R - \bar{y}_{R,h}\|_h + |\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h|\big) + C_4\delta_*\\
&\qquad + C_2\big[\mathcal{A}_h\big((\Pi_h\bar{y}_R, \bar{\boldsymbol{\tau}}), (\Pi_h\bar{y}_R - \bar{y}_{R,h}, \bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h)\big)\\
&\qquad\qquad - \big(y_d, (\Pi_h\bar{y}_R - \bar{y}_{R,h}) + \phi_{\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h,h}\big)\big].
\end{aligned} \tag{4.21}$$

Let $z_{R,h} = \Pi_h\bar{y}_R - \bar{y}_{R,h}$. According to (3.12), we have

$$\begin{aligned}
&\mathcal{A}_h\big((\Pi_h\bar{y}_R, \bar{\boldsymbol{\tau}}), (z_{R,h}, \bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h)\big)\\
&=\beta a_h(\Pi_h\bar{y}_R, z_{R,h}) + \beta(\phi^*_{\bar{\boldsymbol{\tau}},h}, \phi^*_{\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h,h}) + \big(\Pi_h\bar{y}_R + \phi_{\bar{\boldsymbol{\tau}},h}, z_{R,h} + \phi_{\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h,h}\big)
\end{aligned}$$

$$
\begin{aligned}
=&\beta a(\bar y_R, E_h z_{R,h}) + \beta\big[a_h(\Pi_h \bar y_R, z_{R,h}) - a(\bar y_R, E_h z_{R,h})\big]\\
&+ \beta(\phi^*_{\bar{\boldsymbol\tau}}, \phi^*_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h}) + \beta\big[(\phi^*_{\bar{\boldsymbol\tau},h} - \phi^*_{\bar{\boldsymbol\tau}}, \phi^*_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h,h}) + (\phi^*_{\bar{\boldsymbol\tau}}, \phi^*_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h,h} - \phi^*_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h})\big]\\
&+ (\bar y_R + \phi_{\bar{\boldsymbol\tau}}, E_h z_{R,h} + \phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h}) + \Big[\big((\Pi_h \bar y_R - \bar y_R) + (\phi_{\bar{\boldsymbol\tau},h} - \phi_{\bar{\boldsymbol\tau}}), z_{R,h} + \phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h,h}\big)\\
&+ \big(\bar y_R + \phi_{\bar{\boldsymbol\tau}}, (z_{R,h} - E_h z_{R,h}) + \big(\phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h,h} - \phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h}\big)\big)\Big],
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\mathcal{A}_h\big((\Pi_h \bar y_R, \bar{\boldsymbol\tau}), (\Pi_h \bar y_R - \bar y_{R,h}, \bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h)\big)\\
&\quad\le \mathcal{A}\big((\bar y_R, \bar{\boldsymbol\tau}), (E_h(\Pi_h \bar y_R - \bar y_{R,h}), \bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h)\big)\\
&\qquad + C\big[(h^\alpha + h^2)\|\Pi_h \bar y_R - \bar y_{R,h}\|_h + h^{2\nu}|\bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h|\\
&\qquad\quad + (h^{2+\alpha} + h^{2\nu})(\|\Pi_h \bar y_R - \bar y_{R,h}\|_{L_2(\Omega)} + \|\phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h,h}\|_{L_2(\Omega)})\big]\\
&\quad\le \mathcal{A}\big((\bar y_R, \bar{\boldsymbol\tau}), (E_h(\Pi_h \bar y_R - \bar y_{R,h}), \bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h)\big)\\
&\qquad + C\big[(h^\alpha + h^2 + h^{2\nu})\|\Pi_h \bar y_R - \bar y_{R,h}\|_h + (h^{2\nu} + h^{2+\alpha})|\bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h|\big]
\end{aligned}
\tag{4.22}
$$

by (2.15), (3.2), (3.13), (4.9), (4.16), (4.19) and (4.20).

Moreover it follows from (4.9) and (4.19) that

$$
\begin{aligned}
&-\big(y_d, (\Pi_h \bar y_R - \bar y_{R,h}) + \phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h,h}\big)\\
&\quad\le -\big(y_d, E_h(\Pi_h \bar y_R - \bar y_{R,h})\\
&\qquad + \phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h}\big) + C(h^2\|\Pi_h \bar y_R - \bar y_{R,h}\|_h + h^{2\nu}|\bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h|).
\end{aligned}
\tag{4.23}
$$

Using (4.16), (4.22), (4.23) and the triangle inequality, we obtain

$$
\begin{aligned}
&\mathcal{A}_h\big((\Pi_h \bar y_R, \bar{\boldsymbol\tau}), (\Pi_h \bar y_R - \bar y_{R,h}, \bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h)\big) - \big(y_d, (\Pi_h \bar y_R - \bar y_{R,h}) + \phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h,h}\big)\\
&\le \mathcal{A}\big((\bar y_R, \bar{\boldsymbol\tau}), (E_h(\Pi_h \bar y_R - \bar y_{R,h}), \bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h)\big) - \big(y_d, E_h(\Pi_h \bar y_R - \bar y_{R,h}) + \phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h}\big)\\
&\qquad + C\big((h^\alpha + h^2 + h^{2\nu})\|\Pi_h \bar y_R - \bar y_{R,h}\|_h + (h^{2\nu} + h^{2+\alpha})|\bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h|\big)\\
&\le \mathcal{A}\big((\bar y_R, \bar{\boldsymbol\tau}), (E_h(\Pi_h \bar y_R - \bar y_{R,h}), \bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h)\big) - \big(y_d, E_h(\Pi_h \bar y_R - \bar y_{R,h}) + \phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h}\big)\\
&\qquad + C\big(h^{2\alpha} + h^{2+\alpha} + h^{2\nu+\alpha} + (h^\alpha + h^2 + h^{2\nu})\|\bar y_R - \bar y_{R,h}\|_h\\
&\qquad\quad + (h^{2\nu} + h^{2+\alpha})|\bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h|\big).
\end{aligned}
\tag{4.24}
$$

The main task now is to estimate the first two terms on the right-hand side of (4.24), which only involves the continuous bilinear form $\mathcal{A}(\cdot,\cdot)$.

4.4. An Estimate in the Continuous Setting. It follows from (2.11), (2.16), (2.17) and (4.18) that

$$
\begin{aligned}
&\mathcal{A}\big((\bar y_R, \bar{\boldsymbol\tau}), (E_h(\Pi_h \bar y_R - \bar y_{R,h}), \bar{\boldsymbol\tau} - \bar{\boldsymbol\tau}_h)\big) - \big(y_d, E_h(\Pi_h \bar y_R - \bar y_{R,h}) + \phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h}\big)\\
&\quad = \int_\Omega \big[E_h(\Pi_h \bar y_R - \bar y_{R,h}) + \phi_{\bar{\boldsymbol\tau}-\bar{\boldsymbol\tau}_h}\big]d\mu\\
&\quad = \int_\Omega (E_h \Pi_h \bar y_R - \bar y_R)d\mu + \int_\Omega (Y - I_h Y)d\mu\\
&\qquad + \int_\Omega \big[I_h Y - I_h(\bar y_{R,h} + \phi_{\bar{\boldsymbol\tau}_h,h})\big]d\mu + \int_\Omega (I_h E_h \bar y_{R,h} - E_h \bar y_{R,h})d\mu\\
&\qquad\quad + \int_\Omega \big[I_h \phi_{\bar{\boldsymbol\tau}_h,h} - \phi_{\bar{\boldsymbol\tau}_h}\big]d\mu.
\end{aligned}
\tag{4.25}
$$

The first integral on the right-hand side of (4.25) satisfies

$$\int_\Omega (E_h\Pi_h\bar{y}_R - \bar{y}_R)d\mu \le C\|E_h\Pi_h\bar{y}_R - \bar{y}_R\|_{H^1(\Omega)} \le Ch^{1+\alpha} \tag{4.26}$$

by (2.25) and (4.17). In view of (3.8), (3.10) and the fact that $\mu \le 0$, the third integral satisfies

$$\int_\Omega \big[I_hY - I_h(\bar{y}_{R,h} + \phi_{\bar{\boldsymbol{\tau}}_h,h})\big]d\mu \le 0. \tag{4.27}$$

Next we use (2.11) to rewrite the sum of the fourth and fifth integrals as

$$\begin{aligned}
&\int_\Omega (I_hE_h\bar{y}_{R,h} - E_h\bar{y}_{R,h})d\mu + \int_\Omega [I_h\phi_{\bar{\boldsymbol{\tau}}_h,h} - \phi_{\bar{\boldsymbol{\tau}}_h}]d\mu \\
&\quad = \int_\Omega [I_h(E_h\bar{y}_{R,h} - \bar{y}_R) - (E_h\bar{y}_{R,h} - \bar{y}_R)]d\mu + \int_\Omega (I_h\bar{y} - \bar{y})d\mu \\
&\qquad + \sum_{j=1}^J (\bar{\tau}_{h,j} - \bar{\tau}_j)\int_\Omega (I_h\phi_{j,h} - \phi_j)d\mu + \sum_{j=1}^J \bar{\tau}_j \int_\Omega I_h(\phi_{j,h} - \phi_j)d\mu.
\end{aligned} \tag{4.28}$$

We can apply a standard interpolation error estimate (cf. [31, 19]) together with (2.25), (4.17) and (4.19) to obtain the estimate

$$\begin{aligned}
&\int_\Omega [I_h(E_h\bar{y}_{R,h} - \bar{y}_R) - (E_h\bar{y}_{R,h} - \bar{y}_R)]d\mu \\
&\qquad \le C\|I_h(E_h\bar{y}_{R,h} - \bar{y}_R) - (E_h\bar{y}_{R,h} - \bar{y}_R)\|_{H^1(\Omega)} \\
&\qquad \le Ch|E_h\bar{y}_{R,h} - \bar{y}_R|_{H^2(\Omega)} \\
&\qquad \le Ch\big(|E_h(\bar{y}_{R,h} - \Pi_h\bar{y}_R)|_{H^2(\Omega)} + |E_h\Pi_h\bar{y}_R - \bar{y}_R|_{H^2(\Omega)}\big) \\
&\qquad \le Ch\big(\|\bar{y}_{R,h} - \Pi_h\bar{y}_R\|_h + h^\alpha\big).
\end{aligned} \tag{4.29}$$

We can also use (2.25), (4.3), (4.12) and (4.15) to obtain the estimate

$$\begin{aligned}
\int_\Omega I_h(\phi_{j,h} - \phi_j)d\mu &= \int_\Omega I_h(\phi_{j,h} - \Pi_h\phi_j)d\mu \\
&\le C\|\phi_{j,h} - \Pi_h\phi_j\|_{H^1(G)} \\
&\le C\big(\|\phi_{j,h} - \phi_j\|_{H^1(G)} + \|\phi_j - \Pi_h\phi_j\|_{H^1(G)}\big) \le C(h^{2\nu} + h^2).
\end{aligned} \tag{4.30}$$

Moreover it follows from (2.21), (4.15) and (4.30) that

$$\int_\Omega (I_h\phi_{j,h} - \phi_j)d\mu = \int_\Omega I_h(\phi_{j,h} - \phi_j)d\mu + \int_\Omega (I_h\phi_j - \phi_j)d\mu \le C(h^{2\nu} + h^2). \tag{4.31}$$

Combining (4.25)–(4.31) we find

$$\begin{aligned}
&\mathcal{A}\big((\bar{y}_R, \bar{\boldsymbol{\tau}}), (E_h(\Pi_h\bar{y}_R - \bar{y}_{R,h}), \bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h)\big) - \big(y_d, E_h(\Pi_h\bar{y}_R - \bar{y}_{R,h}) + \phi_{\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h}\big) \\
&\quad \le \int_\Omega \big[(Y - \bar{y}) - I_h(Y - \bar{y})\big]d\mu \\
&\qquad + C\big(h^{1+\alpha} + h^{2\nu} + h^2 + h\|\bar{y}_{R,h} - \Pi_h\bar{y}_R\|_h + (h^{2\nu} + h^2)|\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h|\big).
\end{aligned} \tag{4.32}$$

Since $Y-\bar{y}$ belongs to $C^2(\Omega)$, we can apply (2.21) to obtain

$$\int_\Omega \big[(Y-\bar{y})-I_h(Y-\bar{y})\big]d\mu \le C\|(Y-\bar{y})-I_h(Y-\bar{y})\|_{L_\infty(\mathfrak{C})} \le Ch^2, \tag{4.33}$$

which together with (4.16), (4.32) and the triangle inequality implies

$$\begin{aligned}\mathcal{A}\big((\bar{y}_R,\bar{\boldsymbol{\tau}}),(E_h(\Pi_h\bar{y}_R-\bar{y}_{R,h}),\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h)\big)-(y_d,E_h(\Pi_h\bar{y}_R-\bar{y}_{R,h})+\phi_{\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h})\\ \le C\big(h^{1+\alpha}+h^{2\nu}+h^2+h\|\bar{y}_{R,h}-\bar{y}_R\|_h+(h^{2\nu}+h^2)|\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h|\big).\end{aligned} \tag{4.34}$$

4.5. Convergence Results. Putting (4.14), (4.21), (4.24) and (4.34) together we arrive at the estimate

$$\begin{aligned}&\|\bar{y}_R-\bar{y}_{R,h}\|_h^2+|\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h|^2\\ &\qquad\le C\big(h^{2\alpha}+h^{1+\alpha}+h^{2\nu}+h^2+(h^\alpha+h^{2\nu}+h)\|\bar{y}_R-\bar{y}_{R,h}\|_h\\ &\qquad\qquad+(h^{2\nu}+h^2)|\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h|\big).\end{aligned} \tag{4.35}$$

We can now prove our first convergence result.

THEOREM 4.5. *The following error estimate holds for C^0 interior penalty methods of order $k\ge 2$ on quasi-uniform meshes*:

$$\|\bar{y}_R-\bar{y}_{R,h}\|_h+|\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h|\le Ch^\alpha, \tag{4.36}$$

where α is the index of elliptic regularity in (2.18)*.*

Proof. It follows from (4.35) and the relation $\alpha<\nu<1$ (cf. (2.20)) that

$$\|\bar{y}_R-\bar{y}_{R,h}\|_h^2+|\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h|^2\le C(h^{2\alpha}+h^\alpha\|\bar{y}_R-\bar{y}_{R,h}\|_h+h^\alpha|\bar{\boldsymbol{\tau}}-\bar{\boldsymbol{\tau}}_h|),$$

which together with the inequality of arithmetic and geometric means implies (4.36). □

Remark 4.6. It follows from (2.11), (3.2), (3.10) and Theorem 4.5 that

$$|\bar{y}-\bar{y}_h|_{H^1(\Omega)}+\|\bar{y}-\bar{y}_h\|_{L_\infty(\Omega)}\le Ch^\alpha. \tag{4.37}$$

Since $|\cdot|_{H^1(\Omega)}$ and $\|\cdot\|_{L_\infty(\Omega)}$ are lower order norms in comparison with $\|\cdot\|_h$, the convergence of $\bar{y}_h$ in these norms should be of higher order, which is confirmed by numerical results. This remark also applies to the convergence results in Section 5.1 and Section 5.2.

Convergence of $\bar{\boldsymbol{\tau}}_h$. Numerical results in Section 6 show a better convergence for $\bar{\boldsymbol{\tau}}_h$ than that predicted by Theorem 4.5. This phenomenon can be explained as follows.

Suppose there exists a vertex x of $\mathcal{T}_h$ such that x belongs to both $\mathfrak{C}$ and the discrete active set, and $\phi_j(x)\neq 0$ for $1\le j\le J$. Then we have

$$\bar{y}_{R,h}(x)+\sum_{j=1}^J\bar{\tau}_{h,j}\phi_{j,h}(x)=Y(x)=\bar{y}_R(x)+\sum_{j=1}^J\bar{\tau}_j\phi_j(x)$$

and hence

$$\sum_{j=1}^J|\bar{\tau}_j-\bar{\tau}_{j,h}|\,|\phi_j(x)|\le|\bar{y}_R(x)-\bar{y}_{R,h}(x)|+\sum_{j=1}^J|\bar{\tau}_{j,h}||\phi_j(x)-\phi_{j,h}(x)|,$$

which together with (4.13) implies

$$\sum_{j=1}^{J} |\bar{\tau}_j - \bar{\tau}_{j,h}| \le C\big(\|\bar{y}_R - \bar{y}_{R,h}\|_{L_\infty(\mathfrak{C})} + h^{2\nu}\big). \tag{4.38}$$

Since $\|\cdot\|_{L_\infty}$ is a lower order norm, we can expect $\|\bar{y}_R - \bar{y}_{R,h}\|_{L_\infty(\mathfrak{C})}$ to be smaller than $\|\bar{y}_R - \bar{y}_{R,h}\|_h$. It then follows from (4.38) that $\bar{\boldsymbol{\tau}}_h$ converges to $\bar{\boldsymbol{\tau}}$ at a higher order.

5. Convergence Analysis: Graded Meshes. In this section we consider the convergence of C^0 interior penalty methods on graded meshes. We follow the same methodology as in Section 4 and take advantage of the improved performance afforded by graded meshes.

5.1. Grading Strategy I. The goal here is to improve the $O(h^\alpha)$ error estimate in (4.36) to $O(h)$. At a corner where the interior angle is less than $\pi/2$, there is no need for a graded mesh. Around a corner c where the interior angle ω is larger than $\pi/2$, the diameter h_T of a triangle T in the graded mesh satisfies

$$h_T \approx h|c - c_T|^{1-\tilde{\alpha}}, \tag{5.1}$$

where c_T is the center of T, $h \approx \max_{T\in\mathcal{T}_h} h_T$ is the mesh parameter,

$$0 < \tilde{\alpha} < \alpha_\dagger(\omega), \tag{5.2}$$

and $\alpha_\dagger$ is defined by (2.19).

Graded meshes defined by (5.1) and (5.2) provide $O(h)$ approximation of $\bar{y}_R$ around the corners of Ω in H^2-like norms. Since $\bar{y}_R$ belongs to H^3 away from the corners, the overall approximation is $O(h)$ for such meshes. Therefore we can replace α by 1 in Lemma 4.3. Details can be found in [24].

Remark 5.1. Shape regular graded meshes that satisfy (5.1) can be constructed by many refinement procedures (cf. [41, 5, 12, 28]). The refinement procedure in our numerical experiments is adopted from [41, 28].

Remark 5.2. Finite element methods on graded meshes were investigated in [4, 3] for optimal control problems with pointwise control constraints.

Remark 5.3. For the Poisson equation with the homogeneous Dirichlet boundary condition, the grading at a reentrant corner c with interior angle ω is defined by (5.1) with

$$0 < \tilde{\alpha} < \pi/\omega. \tag{5.3}$$

In view of (2.20), the grading condition (5.3) for the Poisson equation at a reentrant corner is implied by the grading condition (5.2).

It follows from Remark 5.3 and the improved interpolation error estimates in [6, 43] that Lemma 4.1 now holds with $\nu = 1$. Consequently the estimates (4.10) and (4.12) are now valid for $\nu = 1$. Therefore we can replace α and ν by 1 in (4.35) which then yields the following improvement of Theorem 4.5.

THEOREM 5.4. *The following error estimate holds for C^0 interior penalty methods of order $k \ge 2$ on meshes that satisfy* (5.1)–(5.2)*:*

$$\|\bar{y}_R - \bar{y}_{R,h}\|_h + |\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h| \le Ch.$$

Note that the estimate (4.11) holds for $\nu = 1-\epsilon$ for any $\epsilon > 0$ because $\tilde{\phi}_{j,h}$ only belongs to $H^3_{loc}(\Omega)$ and not $W^{2,\infty}_{loc}(\Omega)$. Consequently the estimate (4.13) also holds for $\nu = 1-\epsilon$. By the same arguments in Section 4.5, we have

$$\sum_{j=1}^{J} |\bar{\tau}_j - \bar{\tau}_{j,h}| \le C\|\bar{y}_R - \bar{y}_{R,h}\|_{L_\infty(\mathfrak{C})} + C_\epsilon h^{2-\epsilon} \qquad \text{for any } \epsilon > 0. \tag{5.4}$$

5.2. Grading Strategy II. When the desired state y_d, the constraint function Y and the free boundary $\partial\mathfrak{C}$ are sufficiently smooth, the function $\bar{y}$ (and hence $\bar{y}_R$) can have higher interior regularity and it is possible to improve the convergence of C^0 interior penalty methods of order $k \ge 3$ by using a stronger grading.

5.2.1. Additional Regularity for $\bar{y}$. We assume that

$$\partial\mathfrak{C} \text{ is smooth} \quad \text{ and } \quad \bar{y} \in H^4(G \setminus \mathfrak{C}) \cap H^4(\mathring{\mathfrak{C}}), \tag{5.5}$$

where G, as in Section 2.5, is an open neighborhood of $\mathfrak{C}$ such that $\bar{G}$ is a compact subset of Ω, and $\mathring{\mathfrak{C}}$ is the interior of $\mathfrak{C}$. It follows from (5.5) that

$$\bar{y} \in H^{3+\delta}_{loc}(\Omega), \tag{5.6}$$

where δ can be any number $< 1/2$. Therefore the approximation away from $\partial\Omega$ is $O(h^{1+\delta})$ for C^0 interior penalty methods of order $k \ge 3$ (which is assumed to be the case in the discussion below), and the grading strategy in Section 5.1, which only yields $O(h)$ approximation around the corners, is not sufficient.

5.2.2. Behavior of $Y - \bar{y}$. The assumption (5.5) also provides additional information on the behavior of $Y - \bar{y}$ near $\mathfrak{C}$.

LEMMA 5.5. *Let d be a sufficiently small positive number. Under assumption (5.5) we have, for any $\tau < 1$,*

$$|Y(x) - \bar{y}(x)| \le C_\tau d^{2+\tau} \tag{5.7}$$

for any $x \in \Omega$ whose distance to $\mathfrak{C}$ is less than or equal to d.

Proof. Since the C^2 function $Y - \bar{y}$ vanishes identically on $\mathfrak{C}$, we can focus on the points outside $\mathfrak{C}$. Let x be a point outside $\mathfrak{C}$ whose distance to $\mathfrak{C}$ is $\le d$. We can connect x to a point $x_* \in \partial\mathfrak{C}$ by a straight line ℓ normal to $\partial\mathfrak{C}$ because $\partial\mathfrak{C}$ is smooth and d is sufficiently small. The restriction of $Y - \bar{y}$ to the line ℓ is a C^2 function in one variable whose derivatives up to order 2 vanish at x_*, and we have $\|Y''' - \bar{y}'''\|_{L_p(\ell)} \le C_p$ for any $p < \infty$ by the Sobolev embedding theorem (cf. [1, Theorem 4.12 (B)]).

The estimate (5.7) follows from these observations, Taylor's theorem (with integral remainder) and Hölder's inequality. ∎

5.2.3. A Second Grading Strategy. We use a graded mesh that satisfies (5.1) around any corner of Ω where the interior angle is larger than $\pi/3$ and different from $\pi/2$. The grading parameter $\tilde{\alpha}$ is chosen according to

$$0 < \tilde{\alpha} < \frac{\alpha_\dagger(\omega)}{2} \tag{5.8}$$

so that we have $O(h^2)$ approximation around the corners of Ω in H^2-like norms. Since $\bar{y}_R$ belongs to $H^{3+\delta}$ away from the corners, the overall approximation in H^2-like norms is then $O(h^{1+\delta})$. Consequently we can replace α by $1+\delta$ in Lemma 4.3 and hence also in (4.21).

Observe that (4.17) (with $\alpha = 1+\delta$) and interpolation between Sobolev spaces (cf. [8, 1]) implies

$$\|\bar{y}_R - E_h\Pi_h\bar{y}_R\|_{H^{1-\delta}(\Omega)} \le C_\delta h^{2+2\delta}. \tag{5.9}$$

As in Section 5.1, we can take $\nu = 1$ in Lemma 4.1 and hence the estimate (4.24) holds with $\alpha = 1+\delta$ and $\nu = 1$, which is sufficient for the error analysis because $h^{2\nu}$ is attached to $\|\bar{y}_R - \bar{y}_{R,h}\|_h$ and $|\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h|$. However the stand-alone $h^{2\nu}$ in the estimate (4.35) is not good enough. Since it originates from (4.14) and the estimate (4.30) that depends on (4.12), we need to improve the interior estimates in Section 4.1.

5.2.4. Improved Estimates for the Singular Functions. We begin by deriving better estimates that involve the function ζ_j defined in (2.5). It is based on the following observations.

- From (2.8) and (2.5) we see that $\Delta\zeta_j$ is a C^∞ function up to the boundary except at the reentrant corner $\mathcal{C}_j$. Moreover, in a neighborhood $\mathcal{N}_j$ of $\mathcal{C}_j$ there exists a smooth function $\Phi_j(\theta)$ such that $\Delta\zeta_j + r_j^{-\pi/\omega_j}\Phi_j(\theta_j)$ belongs to $H^{1+\epsilon_j}(\mathcal{N}_j)$ for any $\epsilon_j < 1-(\pi/\omega_j)$.
- It then follows from (2.20), (5.8) and elliptic regularity for the Poisson problem on polygonal domains with the homogeneous Dirichlet boundary condition (where the right-hand side is more regular than H^1) that the approximation of ζ_j by $\Pi_h\zeta_j$ in the H^1 norm is of order $O(h^2)$ around all the corners of Ω where the mesh is graded.
- Near a corner of Ω where the interior angle is $\le \pi/3$, the mesh is quasi-uniform and the approximation of ζ_j by $\Pi_h\zeta_j$ in the H^1 norm is at least of order $O(h^3|\ln h|)$, and near a corner where the interior angles is $\pi/2$, the approximation of ζ_j by $\Pi_h\zeta_j$ in the H^1 norm is $O(h^2|\ln h|)$ for a quasi-uniform mesh.
- In view of Remark 2.4, the approximation of ζ_j by $\Pi_h\zeta_j$ in the H^1 norm is at least $O(h^3)$ away from the corners of Ω.

Putting these observations together we conclude that

$$|\zeta_j - \zeta_{j,h}|_{H^1(\Omega)} = |\zeta_j - R_h\zeta_i|_{H^1(\Omega)} \le |\zeta_j - \Pi_h\zeta_j|_{H^1(\Omega)} \le C_\tau h^{1+\tau} \tag{5.10}$$

for all $\tau < 1$, and hence, by a standard duality argument, the estimates (4.1) and (4.2) can be improved to

$$\|\zeta_j - \zeta_{j,h}\|_{L_2(\Omega)} + \|\phi_j^* - \phi_{j,h}^*\|_{L_2(\Omega)} \le C_\tau h^{2+\tau} \qquad \text{for all } \tau < 1,$$

which implies that the estimate (4.7) can be replaced by

$$\|\phi_j - \tilde{\phi}_{j,h}\|_{\mathring{E}(\Omega;\Delta)} \le C_\tau h^{2+\tau} \qquad \text{for all } \tau < 1. \tag{5.11}$$

Next we will derive better estimates that involve the function ϕ_j defined in (2.7). Observe that (i) the function $\Delta\phi_j$ belongs to H^2 around all the convex corners of Ω; (ii) at a reentrant corner $\mathcal{C}_k$ $(k \ne j)$, the function $\Delta\phi_j$ belongs to $H^{1+\epsilon_k}$ for any $\epsilon_k < \pi/\omega_k$; (iii) at the reentrant corner $\mathcal{C}_j$, the function $\Delta\phi_j + r_j^{-\pi_j/\omega_j}\sin\big((\pi_j/\omega_j)\theta_j\big)$ belongs to $H^{1+\epsilon_j}$ for any $\epsilon_j < \pi/\omega$; (iv) according to Remark 2.5, the approximation of ϕ_j by $\Pi_h\phi_j$ in the H^1 norm is $O(h^3)$ away from the corners of Ω.

Based on these observations we can conclude that, as in the case of ζ_j,

$$|\phi_j - R_h\phi_j|_{H^1(\Omega)} \le Ch^{1+\tau} \quad \text{and} \quad \|\phi_j - R_h\phi_j\|_{L_2(\Omega)} \le Ch^{2+\tau}$$

for any $\tau < 1$. It follows that the interior estimate (4.10) can be replaced by

$$|\phi_j - R_h\phi_j|_{H^1(G)} \le Ch^{2+\tau} \qquad \text{for any } \tau < 1. \tag{5.12}$$

Combining (5.11) and (5.12), we arrive at the following improvement of (4.12):

$$|\phi_j - \phi_{j,h}|_{H^1(G)} \le C_\tau h^{2+\tau} \qquad \text{for any } \tau < 1. \tag{5.13}$$

Since $k \ge 3$ and $\phi_j \in C^\infty(\Omega)$, we also have

$$|\phi_j - \Pi_h\phi_j|_{H^1(G)} \le Ch^3. \tag{5.14}$$

Lemma 5.6. *There exists a positive constant C independent of h such that*

$$\|\nabla\phi^*_{j,h}\|_{L_\infty(G)} \le C. \tag{5.15}$$

Proof. It follows from (5.10) and standard interpolation and inverse estimates that

$$\begin{aligned}
\|\nabla\zeta_{j,h}\|_{L_\infty(G)} &\le \|\nabla(\zeta_{j,h} - \Pi_h\zeta_j)\|_{L_\infty(G)} + \|\nabla(\Pi_h\zeta_j - \zeta_j))\|_{L_\infty(G)} + \|\nabla\zeta_j\|_{L_\infty(G)}\\
&\le C\big(h^{-1}|\zeta_{j,h} - \Pi_h\zeta_j|_{H^1(G)} + h|\zeta_j|_{W^{2,\infty}(G)} + |\zeta_j|_{W^{1,\infty}(G)}\big)\\
&\le C\big(h^{-1}(|\zeta_{j,h} - \zeta_j)|_{H^1(G)} + |\zeta_j - \Pi_h\zeta_j|_{H^1(G)}) + h|\zeta_j|_{W^{2,\infty}(G)} + |\zeta_j|_{W^{1,\infty}(G)}\big)\\
&\le C,
\end{aligned}$$

which, in view of (3.4), implies (5.15). ☐

It follows from (4.6), Lemma 5.6 and interior elliptic regularity estimates [47, Theorem 17.1.1] that

$$|\tilde\phi_{j,h}|_{W^{3,p}(G)} \le C_p \qquad \text{for any } p < \infty. \tag{5.16}$$

Note that, by (5.11) and (5.12), the estimate

$$\begin{aligned}
\|R_h\tilde\phi_{j,h} - \tilde\phi_{j,h}\|_{L_2(\Omega)} &\le \|R_h(\tilde\phi_{j,h} - \phi_j) - (\tilde\phi_{j,h} - \phi_j)\|_{L_2(\Omega)} + \|R_h\phi_j - \phi_j\|_{L_2(\Omega)}\\
&\le C_\tau h^{2+\tau}
\end{aligned}$$

holds for any $\tau < 1$, which together with (5.16) implies the interior error estimate

$$\|\tilde\phi_{j,h} - R_h\tilde\phi_{j,h}\|_{L_\infty(G)} \le C_\tau h^{2+\tau}.$$

Hence we have the following improvement of (4.13):

$$\|\phi_j - \phi_{j,h}\|_{L_\infty(G)} \le C_\tau h^{2+\tau}$$

for any $\tau < 1$, which means the number δ_* defined in (3.9) satisfies the following improvement of (4.14):

$$\delta_* \le C_\tau h^{2+\tau}. \tag{5.17}$$

5.2.5. An Improved Continuous Estimate. The estimate (4.26) for the first integral on the right-hand side of (4.25) can be improved by using (2.26) and (5.9) as follows:

$$\int_\Omega (E_h\Pi_h\bar{y}_R - \bar{y}_R)d\mu \le C_\delta \|E_h\Pi_h\bar{y}_R - \bar{y}_R\|_{H^{1-\delta}(\Omega)} \le C_\delta h^{2+2\delta}. \tag{5.18}$$

Similarly we have the following improvement of (4.29):

$$\begin{aligned}\int_\Omega [I_h(E_h\bar{y}_{R,h} - \bar{y}_R) &- (E_h\bar{y}_{R,h} - \bar{y}_R)]d\mu \\ &\le C_\delta \|I_h(E_h\bar{y}_{R,h} - \bar{y}_R) - (E_h\bar{y}_{R,h} - \bar{y}_R)\|_{H^{1-\delta}(\Omega)} \\ &\le C_\delta h^{1+\delta}|E_h\bar{y}_{R,h} - \bar{y}_R|_{H^2(\Omega)} \\ &\le C_\delta h^{1+\delta}\big(|E_h(\bar{y}_{R,h} - \Pi_h\bar{y}_R)|_{H^2(\Omega)} + |E_h\Pi_h\bar{y}_R - \bar{y}_R|_{H^2(\Omega)}\big) \\ &\le C_\delta h^{1+\delta}\big(\|\bar{y}_{R,h} - \Pi_h\bar{y}_R\|_h + h^{1+\delta}\big)\end{aligned} \tag{5.19}$$

by (2.26), (4.17) (with $\alpha = 1+\delta$), (5.9) and a standard interpolation error estimate for I_h.

In view of (5.13) and (5.14), we have the estimate

$$\int_\Omega I_h(\phi_{j,h} - \phi_j)d\mu \le C_\tau h^{2+\tau} \qquad \text{for any } \tau < 1 \tag{5.20}$$

that improves (4.30).

Finally we consider the improvement of the estimate (4.33). In view of Lemma 5.5, we have, for any $\tau < 1$, $|Y(x) - \bar{y}(x)| \le C_\tau h^{2+\tau}$ if the distance between x and $\mathfrak{C}$ is comparable to h, which implies $\|I_h(Y - \bar{y})\|_{L_\infty(\mathfrak{C})} \le C_\tau h^{2+\tau}$.

It follows that

$$\int_\Omega \big[(Y - \bar{y}) - I_h(Y - \bar{y})\big]d\mu = -\int_\Omega I_h(Y - \bar{y})\, d\mu \le C_\tau h^{2+\tau}. \tag{5.21}$$

Combining (4.27), (4.31) (with $\nu = 1$), (5.18)–(5.21) and the triangle inequality, we arrive at the following improvement of (4.34):

$$\begin{aligned}\mathcal{A}\big((\bar{y}_R, \bar{\boldsymbol{\tau}}), (E_h(\Pi_h\bar{y}_R - \bar{y}_{R,h}), \bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h)\big) &- (y_d, E_h(\Pi_h\bar{y}_R - \bar{y}_{R,h}) + \phi_{\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h}) \\ &\le C_{\delta,\tau}(h^{2+2\delta} + h^{2+\tau} + h^{1+\delta}\|\bar{y}_R - \bar{y}_{R,h}\|_h + h^2|\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h|)\end{aligned} \tag{5.22}$$

for any $\delta < 1/2$ and $\tau < 1$.

5.2.6. Convergence Results. The estimates (4.21) and (4.24), where $\alpha = 1+\delta$ and $\nu = 1$, together with the estimates (5.17) and (5.22) (where $\tau = 2\delta$) yields, for any $\delta < 1/2$,

$$\|\bar{y}_R - \bar{y}_{R,h}\|_h^2 + |\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h|^2 \le C_\delta\big(h^{2+2\delta} + h^{1+\delta}\|\bar{y}_R - \bar{y}_{R,h}\|_h + h^2|\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h|\big). \tag{5.23}$$

We can then obtain the following theorem by the inequality of geometric and arithmetic means.

THEOREM 5.7. *Under the additional regularity assumption* (5.5)*, the following error estimate holds for C^0 interior penalty methods of order $k \ge 3$ on meshes that satisfy* (5.1) *and* (5.8)*:*

$$\|\bar{y}_R - \bar{y}_{R,h}\|_h + |\bar{\boldsymbol{\tau}} - \bar{\boldsymbol{\tau}}_h| \le C_\delta h^{1+\delta} \qquad \textit{for any } \delta < 1/2.$$

As in Section 5.1, we have

$$\sum_{j=1}^{J} |\bar{\tau}_j - \bar{\tau}_{j,h}| \leq C\|\bar{y}_R - \bar{y}_{R,h}\|_{L_\infty(\mathfrak{c})} + C_\epsilon h^{2-\epsilon} \qquad \forall\, \epsilon > 0. \tag{5.24}$$

6. Numerical Results. We solve the minimization problem (1.4) with $\beta = 1$ by the quadratic and cubic C^0 interior penalty methods on domains where the interior angles are right angles at all the corners except one reentrant corner, so that only the mesh around the reentrant corner needs to be graded. The initial mesh size is 1 in all the computations and the discrete variational inequalities are solved by a primal-dual active set strategy (cf. [9, 10, 45, 48]). We use a Gaussian quadrature rule with 16 points in the interior of each triangle in the computation of the discrete singular function associated with the reentrant corner (cf. (3.5)).

Example 6.1. The domain for this experiment (cf. Figure 6.1) is the L-shaped domain obtained from the square $(-1,1)^2$ by removing its lower right quadrant, and we take $y_d = 0$ and $Y = \left[\left(\frac{x_1+0.5}{0.48}\right)^2 + \left(\frac{x_2-0.5}{0.48}\right)^2\right] - 1$. For this domain the index of elliptic regularity α is $\frac{1}{3}$ (cf. Section 2.4). Therefore the order of convergence predicted by Theorem 4.5 is $\frac{1}{3}$.

The results for the quadratic (resp., cubic) C^0 interior penalty method on uniform meshes are reported in Table 6.1 (resp., Table 6.2). The value of $\bar{\tau}$ is found to be -2.218.

j	$\|\bar{y}_{R,j-1} - \bar{y}_{R,j}\|_h$	order	$|\bar{\tau}_{j-1} - \bar{\tau}_j|$	order	$\|\bar{y}_{R,j-1} - \bar{y}_{R,j}\|_{\ell_\infty(\mathfrak{C}_j)}$	order
1	5.916×10^{0}	–	8.933×10^{-1}	–	1.866×10^{-1}	–
2	3.506×10^{0}	0.75	2.060×10^{-1}	2.12	1.143×10^{-1}	0.71
3	2.064×10^{0}	0.76	6.443×10^{-2}	1.68	9.627×10^{-3}	3.57
4	1.326×10^{0}	0.64	2.554×10^{-2}	1.34	4.357×10^{-3}	1.14
5	9.255×10^{-1}	0.52	7.697×10^{-3}	1.73	8.218×10^{-4}	2.41
6	6.944×10^{-1}	0.41	5.258×10^{-3}	0.55	4.175×10^{-4}	0.98
7	5.399×10^{-1}	0.36	2.932×10^{-3}	0.84	2.344×10^{-4}	0.83
8	4.260×10^{-1}	0.34	1.720×10^{-3}	0.77	1.394×10^{-4}	0.75

Table 6.1
Results for the quadratic method on uniform meshes for Example 6.1

j	$\|\bar{y}_{R,j-1} - \bar{y}_{R,j}\|_h$	order	$|\bar{\tau}_{j-1} - \bar{\tau}_j|$	order	$\|\bar{y}_{R,j-1} - \bar{y}_{R,j}\|_{\ell_\infty(\mathfrak{C}_j)}$	order
1	2.799×10^{0}	–	4.167×10^{-1}	–	7.526×10^{-2}	–
2	2.450×10^{0}	0.19	1.829×10^{-1}	1.19	1.014×10^{-1}	-0.43
3	9.398×10^{-1}	1.38	3.386×10^{-2}	2.43	5.225×10^{-3}	4.28
4	7.319×10^{-1}	0.36	1.701×10^{-2}	0.99	3.650×10^{-3}	0.52
5	5.658×10^{-1}	0.37	8.067×10^{-3}	1.08	8.552×10^{-4}	2.09
6	4.482×10^{-1}	0.34	5.951×10^{-3}	0.44	4.899×10^{-4}	0.80
7	3.569×10^{-1}	0.33	3.633×10^{-3}	0.71	3.016×10^{-4}	0.70
8	2.840×10^{-1}	0.33	2.468×10^{-3}	0.56	2.063×10^{-4}	0.55

Table 6.2
Results for the cubic method on uniform meshes for Example 6.1

The convergence of $\bar{y}_{R,h}$ in the energy norm (cf. Table 6.1 and Table 6.2) is as predicted. The higher order convergence of $\bar{\tau}_h$ can be explained by (4.38) and the results in Table 6.1 and Table 6.2 for $|\bar{y}_{R,j-1} - \bar{y}_{R,j}|_{\ell_\infty(\mathfrak{C}_j)}$, where $\mathfrak{C}_j$ is the j-th level discrete active set and $|v|_{\ell_\infty(\mathfrak{C}_j)}$ is the maximum of the absolute values of v at the vertices of $\mathcal{T}_h$ that belong to $\mathfrak{C}_j$.

Since the exact solution is not known, we approximate the unknown error by a hierarchical error estimator, which is justified by the saturation assumption, i.e., that for uniform mesh refinements the error contracts.

We have also implemented the quadratic and cubic C^0 interior penalty methods on graded meshes. In order to apply the hierarchical error estimator, we use the grading procedure from [41, 28] that generates hierarchical meshes. The results are presented in Table 6.3 and Table 6.4.

j	$\Vert\bar{y}_{R,j-1} - \bar{y}_{R,j}\Vert_h$	order	$\vert\bar{\tau}_{j-1} - \bar{\tau}_j\vert$	order	$\Vert\bar{y}_{R,j-1} - \bar{y}_{R,j}\Vert_{\ell_\infty(\mathfrak{C}_j)}$	order
1	5.916×10^{0}	–	8.933×10^{-1}	–	1.866×10^{-1}	–
2	3.734×10^{0}	0.66	2.267×10^{-1}	1.98	1.150×10^{-1}	0.70
3	1.868×10^{0}	1.00	6.117×10^{-2}	1.89	9.812×10^{-3}	3.55
4	1.002×10^{0}	0.90	2.133×10^{-2}	1.52	4.303×10^{-3}	1.19
5	5.116×10^{-1}	0.97	4.963×10^{-3}	2.10	6.032×10^{-4}	2.83
6	2.617×10^{-1}	0.97	1.667×10^{-3}	1.57	1.600×10^{-4}	1.91
7	1.347×10^{-1}	0.96	4.326×10^{-4}	1.95	3.883×10^{-5}	2.04
8	6.843×10^{-2}	0.98	9.895×10^{-5}	2.13	9.043×10^{-6}	2.10

Table 6.3
Results for the quadratic method on graded meshes with $\tilde{\alpha} = 0.3$ for Example 6.1

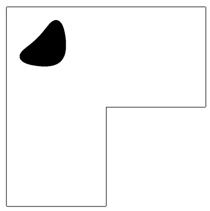
j	$\Vert\bar{y}_{R,j-1} - \bar{y}_{R,j}\Vert_h$	order	$\vert\bar{\tau}_{j-1} - \bar{\tau}_j\vert$	order	$\Vert\bar{y}_{R,j-1} - \bar{y}_{R,j}\Vert_{\ell_\infty(\mathfrak{C}_j)}$	order
1	2.799×10^{0}	–	4.167×10^{-1}	–	7.526×10^{-2}	–
2	2.525×10^{0}	0.15	2.405×10^{-1}	0.79	1.040×10^{-1}	−0.47
3	4.893×10^{-1}	2.37	1.550×10^{-2}	3.96	3.864×10^{-3}	4.75
4	2.411×10^{-1}	1.02	1.877×10^{-3}	3.05	3.217×10^{-3}	0.26
5	8.639×10^{-2}	1.48	3.525×10^{-4}	2.41	3.564×10^{-4}	3.17
6	2.802×10^{-2}	1.62	8.867×10^{-5}	1.99	6.387×10^{-5}	2.48
7	8.941×10^{-3}	1.65	2.244×10^{-5}	1.98	2.287×10^{-6}	4.80

Table 6.4
Results for the cubic method on graded meshes with $\tilde{\alpha} = 0.15$ for Example 6.1

For the quadratic method, we take the grading parameter $\tilde{\alpha}$ in (5.1) to be 0.3 so that the condition (5.2) is satisfied. The order of convergence of $\bar{y}_{R,h}$ in the energy norm is approaching 1, which agrees with Theorem 5.4. For the cubic method, we take the grading parameter $\tilde{\alpha}$ to be 0.15 so that the condition (5.8) is satisfied. The order of convergence for $\bar{y}_{R,h}$ in the energy norm is approximately 1.5, which is consistent with Theorem 5.7.

The discrete active sets computed by the quadratic C^0 interior penalty method on uniform and graded meshes are displayed in Figure 6.1. It appears that the boundary of the active set is smooth and therefore the additional regularity for $\bar{y}$ in (5.5) is justified by the smoothness of y_d and Y.

The higher order convergence of $\bar{\tau}_h$ for both methods can be justified by (5.4), (5.24) and the results for $|\bar{y}_{R,j-1} - \bar{y}_{R,j}|_{\ell_\infty(\mathfrak{C}_j)}$ in Table 6.3 and Table 6.4.


(a)

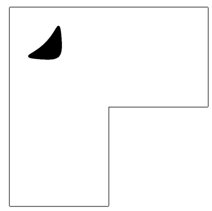

(b)

FIG. 6.1. *Discrete active sets for Example 6.1 computed by the quadratic method (6 levels of refinement): (a) uniform mesh and (b) graded mesh*

EXAMPLE 6.2. The domain Ω (cf. Figure 6.2) for this experiment is obtained from the square $(-1,1)^2$ by removing an isosceles right angled triangle. According to Section 2.4, we have $\alpha = \frac{1}{7}$. Therefore the order of convergence predicted by Theorem 4.5 is $\frac{1}{7} \approx 0.1429$.

Let (r,θ) be the polar coordinates at the reentrant corner. For this domain the leading singularity for the biharmonic equation with the boundary conditions of simply supported plates is determined by the function $\psi = -r^{8/7}\sin(8\theta/7)$, which is a negative harmonic function on Ω.

In order to check the performance of our methods on a problem with a known exact solution, we consider a modified optimal control problem defined by the cost functional (1.1) (with $\beta = 1$) and the constraint (1.2), but with $y \in \psi + H_0^1(\Omega)$ and $u \in L_2(\Omega)$. The constraint function $Y \in C^2(\Omega) \cap C(\bar{\Omega})$ is given by

$$Y(x) = \begin{cases} \psi(x) & \text{if} \quad |x - (-\frac{1}{2}, \frac{1}{2})| \le \frac{1}{4} \\ \psi(x) + (|x - (-\frac{1}{2}, \frac{1}{2})| - \frac{1}{4})^3 & \text{otherwise} \end{cases},$$

and the desired state $y_d \in L_2(\Omega)$ is given by

$$y_d(x) = \begin{cases} 0 & \text{if} \quad |x - (-\frac{1}{2}, \frac{1}{2})| \le \frac{1}{4} \\ \psi(x) & \text{otherwise} \end{cases}.$$

This optimal control problem is equivalent to the following minimization problem:

$$\text{Find} \quad \bar{y} = \operatorname*{argmin}_{y \in \mathbb{K}_\psi} \Big[\frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{1}{2}\|\Delta y\|^2_{L_2(\Omega)}\Big], \tag{6.1}$$

where $\mathbb{K}_\psi = \{y \in \psi + \mathring{E}(\Omega;\Delta) : \, y \le Y \text{ in } \Omega\}$.

Clearly $\psi \in \mathbb{K}_\psi$ and it is easy to check that it is the exact solution of (6.1). Indeed the active set for $\bar{y} = \psi$ is the disc $D = \{x : \, |x - (-\frac{1}{2}, \frac{1}{2})| \le \frac{1}{4}\}$ and we have a trivial relation

$$(\Delta\bar{y}, \Delta z) + (\bar{y} - y_d, z) = \int_D \psi z \, dx \qquad \forall \, z \in \mathring{E}(\Omega;\Delta). \tag{6.2}$$

It follows from (6.2) that $\bar{y} = \psi$ satisfies the variational inequality

$$\big(\Delta\bar{y}, \Delta(y-\bar{y})\big) + (\bar{y}-y_d, y-\bar{y}) = \int_D \psi(y-\bar{y})\,dx = \int_D \psi(y-Y)\,dx \geq 0 \qquad \forall\, y \in \mathbb{K}_\psi,$$

which characterizes the solution of (6.1).

Since $\bar{y} = \psi \in H^2(\Omega)$, we have $\bar{y}_R = \bar{y}$ and $\bar{\tau} = 0$ for this problem.

Note that (6.2) implies that the negative Borel measure μ is given by

$$\mu = \chi_D \psi\, dx, \tag{6.3}$$

where χ_D is the characteristic function for D.

Since the boundary condition is now nonhomogeneous, we modify the discrete problem (3.6) by replacing $a_h(\cdot,\cdot)$ and K_h by $a_{h,\psi}(\cdot,\cdot)$ and $K_{h,\psi}$ respectively, where

$$a_{h,\psi}(w,v) = a_h(w,v) - \sum_{e\in\mathcal{E}_h^b} \int_e \Big(\frac{\partial^2\psi}{\partial n^2}\Big)\Big(\frac{\partial v}{\partial n}\Big)ds,$$

$\mathcal{E}_h^b$ is the set of the edges of $\mathcal{T}_h$ that are subsets of $\partial\Omega$, n is the outward pointing unit normal,

$$K_{h,\psi} = \{(y_{R,h},\tau) \in \big(\Pi_h\psi + V_h\big)\times\mathbb{R} : y_{R,h}(p) + \bar{\tau}\phi_h(p) \leq Y(p) \text{ at all the vertices of } \mathcal{T}_h\},$$

and ϕ_h is the discrete singular function associated with the reentrant corner (cf. Section 3.1). The error analysis in Section 4 and Section 5 can be extended to this discretization of (6.1).

The numerical results for the quadratic (resp., cubic) method on uniform meshes are reported in Table 6.5 (resp., Table 6.6). The order of convergence of $\bar{y}_{R,h}$ in the energy norm is as predicted and τ_h converges to 0.

j	$\|\bar{y}_R - \bar{y}_{R,j}\|_h$	order	$\lvert\bar{\tau}_j\rvert$
0	7.738×10^{-1}	–	2.925×10^{-3}
1	7.187×10^{-1}	0.11	2.749×10^{-3}
2	6.600×10^{-1}	0.12	2.430×10^{-3}
3	6.036×10^{-1}	0.13	7.723×10^{-4}
4	5.509×10^{-1}	0.13	5.936×10^{-4}
5	5.020×10^{-1}	0.13	5.262×10^{-4}
6	4.569×10^{-1}	0.14	5.508×10^{-4}
7	4.156×10^{-1}	0.14	5.468×10^{-4}
8	3.776×10^{-1}	0.14	5.344×10^{-4}

Table 6.5
Results for the quadratic method on uniform meshes for Example 6.2

The results for the quadratic and cubic C^0 interior penalty methods on graded meshes are reported in Table 6.7 and Table 6.8, respectively. The grading parameter $\tilde{\alpha}$ for the mesh around the reentrant corner is 0.1 (resp., 0.07) for the quadratic (resp., cubic) method so that the condition (5.2) (resp., (5.8)) is satisfied. We have also tested the quadratic method on graded meshes with $\tilde{\alpha} = 0.14$. The performance of the method is better when the smaller grading parameter $\tilde{\alpha} = 0.1$ is used.

For the quadratic method the order of convergence for $\bar{y}_{R,h}$ is approaching 1 as predicted by Theorem 5.4. For the cubic method the order of convergence for $\bar{y}_{R,h}$ is

j	$\|\bar{y}_R - \bar{y}_{R,j}\|_h$	order	$\lvert\bar{\tau}_j\rvert$
0	6.553×10^{-1}	–	2.857×10^{-3}
1	5.977×10^{-1}	0.13	2.730×10^{-3}
2	5.458×10^{-1}	0.13	1.290×10^{-3}
3	4.976×10^{-1}	0.13	7.653×10^{-4}
4	4.532×10^{-1}	0.13	5.984×10^{-4}
5	4.123×10^{-1}	0.14	5.440×10^{-4}
6	3.749×10^{-1}	0.14	5.314×10^{-4}
7	3.406×10^{-1}	0.14	5.173×10^{-4}

TABLE 6.6
Results for the cubic method on uniform meshes for Example 6.2

j	$\|\bar{y}_R - \bar{y}_{R,j}\|_h$	order	$\lvert\bar{\tau} - \bar{\tau}_j\rvert = \lvert\bar{\tau}_j\rvert$	order	$\|\bar{y}_R - \bar{y}_{R,j}\|_{\ell_\infty(\mathfrak{C})}$	order
0	7.738×10^{-1}	–	2.925×10^{-3}	–	–	–
1	7.187×10^{-1}	0.11	2.749×10^{-3}	0.09	–	–
2	4.120×10^{-1}	0.80	5.774×10^{-5}	5.57	6.193×10^{-6}	–
3	2.198×10^{-1}	0.91	6.200×10^{-4}	−3.42	5.953×10^{-5}	−3.26
4	1.164×10^{-1}	0.92	2.540×10^{-4}	1.29	2.721×10^{-5}	1.13
5	6.046×10^{-2}	0.94	7.339×10^{-5}	1.79	8.280×10^{-6}	1.72
6	3.102×10^{-2}	0.96	1.887×10^{-5}	1.96	2.162×10^{-6}	1.94
7	1.579×10^{-2}	0.97	9.707×10^{-6}	0.96	1.123×10^{-6}	0.94

TABLE 6.7
Results for the quadratic method on graded meshes with $\tilde{\alpha} = 0.1$ for Example 6.2

approaching 2, which is better than the one predicted by Theorem 5.7. This can be explained as follows.

First we observe that $\bar{y} = \psi \in C^\infty(\Omega)$ and hence we can take $\delta = 1$ in (5.6). Secondly we can take $\tau = 1$ in (5.7) because $Y - \bar{y}$ is piecewise C^∞, and we can combine this observation with (6.3) to replace (5.21) by

$$\int_\Omega \big[(Y - \bar{y}) - I_h(Y - \bar{y})\big] d\mu = -\int_D \psi I_h(Y - \bar{y}) dx \leq Ch^4,$$

where we have used the fact that the restriction of $I_h(Y - \bar{y})$ to D vanishes outside a strip neighboring $\partial\Omega$ whose width is $\approx h$. Putting these together we can replace δ by 1 in Theorem 5.7.

Since $\bar{\tau} = 0$, we can replace the estimate (5.24) by

$$|\tau_h| \leq C\|\bar{y}_R - \bar{y}_{R,h}\|_{L_\infty(\mathfrak{C})},$$

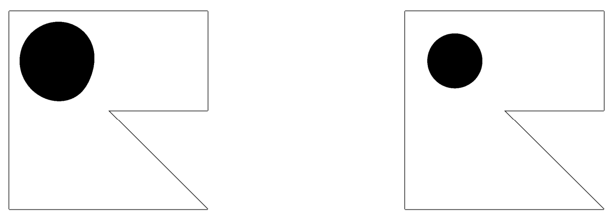
and the higher order convergence of $\bar{\tau}_h$ for both quadratic and cubic methods can be justified by the results for $\|\bar{y}_R - \bar{y}_{R,j}\|_{\ell_\infty(\mathfrak{C})}$ in Table 6.7 and Table 6.8.

The discrete active sets generated by the quadratic C^0 interior method on uniform and graded meshes are displayed in Figure 6.2. The one generated on graded meshes is almost identical with the exact active set D.

7. Concluding Remarks. We have designed C^0 interior penalty methods for an elliptic distributed optimal control problem with pointwise state constraint on general polygonal domains. From our experience with this problem on convex polygonal

j	$\|\bar{y}_R - \bar{y}_{R,j}\|_h$	order	$|\bar{\tau} - \bar{\tau}_j| = |\bar{\tau}_j|$	order	$\|\bar{y}_R - \bar{y}_{R,j}\|_{\ell_\infty(\mathfrak{C})}$	order
0	6.553×10^{-1}	–	2.857×10^{-3}	–	–	–
1	5.977×10^{-1}	0.13	2.730×10^{-3}	0.07	–	–
2	1.769×10^{-1}	1.76	1.950×10^{-4}	3.81	2.032×10^{-5}	–
3	4.723×10^{-2}	1.91	1.377×10^{-5}	3.82	1.477×10^{-6}	3.78
4	1.282×10^{-2}	1.88	3.862×10^{-7}	5.16	4.403×10^{-8}	5.07
5	3.379×10^{-3}	1.92	8.166×10^{-9}	5.56	9.510×10^{-10}	5.53

TABLE 6.8
Results for the cubic method on graded meshes with $\tilde{\alpha} = 0.07$ for Example 6.2



(a) (b)

FIG. 6.2. *Discrete active sets for Example 6.2 computed by the quadratic method (6 levels of refinement): (a) uniform mesh and (b) graded mesh*

domains, we expect that the C^0 interior penalty methods can be replaced by any convergent finite element method for the biharmonic equation with the boundary conditions of simply supported plates. Therefore all such methods are relevant for the optimal control problem studied in this paper.

We have only treated the Dirichlet boundary condition in the elliptic constraint, but the results can be extended to other boundary conditions. They can also be extended to problems with both upper and lower constraints.

We can take

$$\bar{u}_h = -\Delta_h \bar{y}_h + \sum_{j=1}^{J} \bar{\boldsymbol{\tau}}_{h,j} \phi^*_{j,h}$$

to be an approximation of the optimal control $\bar{u}$, where Δ_h is the piecewise defined Laplace operator. It is trivial to show that $\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)}$ satisfies the same error estimates in Theorems 4.5, 5.4 and 5.7. We can also use more sophisticated post-processing techniques [25] to generate $\bar{u}_h$ from $\bar{y}_h$.

Note that the solution for the obstacle problem for simply supported plates belongs to $H^2(\Omega) \cap H^1_0(\Omega)$ for any polygonal domain Ω. Therefore we can solve this problem using the approach in this paper by simply removing all the singular functions. The theory developed in this paper can also be applied to this simpler problem.

Since the singularities (and hence the graded meshes) for three dimensional domains are more complicated (cf. [34]), it would be useful to extend the results in [15] for adaptive C^0 interior penalty methods for the obstacle problem of clamped Kirchhoff plates to the problem (1.4) on nonconvex domains. This will be investigated in the future.

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