Visualizing Complicated Dynamics

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Abstract

The temporal evolution of real world systems can mathematically be described by dynamical systems. Global, topological information on their long term behavior is given by corresponding invariant sets, which typically have a very complex structure. We describe a new, set oriented and hierarchical approach for the visualization of invariant sets and accompanying probability measures, which give statistical information on the dynamical system.

1 Introduction

The temporal evolution of a real world system can mathematically be described by a *dynamical system*. If time is assumed to evolve continuously then this system is frequently given by an ordinary differential equation of the form $\dot{x} = g(x)$, where $g : \mathbb{R}^n \to \mathbb{R}^n$. As an interesting first example let us consider *Chua's circuit*

$$\begin{aligned} \dot{x} &= \alpha(y - m_0 x - \tfrac{1}{3} m_1 x^3) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y, \end{aligned}$$

which is a mathematical model describing the dynamical behavior of a certain electrical circuit. (Here α, β, m_0 and m_1 are real parameters related to the different components of the circuit). *Lorenz system*

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned} \tag{1}$$

is a second well–known dynamical system (For the computations we have set $\sigma = 10, \quad \rho = 28 \quad \text{and} \quad \beta = 0.4$).

On the other hand, if we consider the evolution process just at single instances of time, then the mathematical model is a discrete dynamical system of the form

$$x_{j+1} = f(x_j), \quad j = 0, 1, \ldots, \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. Observe that this type of dynamical system naturally arises when an ordinary differential equation is integrated by an explicit numerical scheme.

Topological information on the long term behavior of (2) is given by invariant sets: a set $A \subset \mathbb{R}^n$ is *invariant* if $f(A) = A$. For instance, A could simply consist of a point, i.e. a *fixed point*, but it could also have a complicated structure. In the latter case the underlying dynamical behavior is typically *chaotic*. Computing a single trajectory only very partially visualizes the typically rich and complicated structure of these objects.

The purpose of this article is to present novel visualization techniques based on recent numerical approximation methods [4]. The *set oriented*, hierarchical numerical algorithms are sketched in Section 2. By these algorithms the invariant set is approximated by an *exterior box covering* rather than by *interior testing with single trajectories*. The objects that we are going to compute and visualize are *(almost) invariant sets*, *invariant manifolds* and *invariant measures*. The invariant manifolds of an invariant set consist of those points in state space which approach the set asymptotically in forward resp. backward time: for an invariant set A define

$$\begin{aligned} W^s(A) &= \{x \in \mathbb{R}^n : f^j(x) \to A \quad \text{for } j \to +\infty\}, \\ W^u(A) &= \{x \in \mathbb{R}^n : f^j(x) \to A \quad \text{for } j \to -\infty\}. \end{aligned} \tag{3}$$

Then, under certain additional hypotheses on the invariant set A these objects are indeed manifolds and are called the *stable* resp. the *unstable manifold* of A. Obviously these manifolds are invariant sets.

The statistical information on the dynamical behavior on an invariant set is given by a corresponding invariant measure: a probability measure μ is *invariant* if $\mu(B) = \mu(f^{-1}(B))$ for all (measurable) subsets $B \subset \mathbb{R}^n$. For instance, if A is a fixed point p then the Dirac measure δ_p is an invariant measure. We will present visualization techniques for the "physically relevant" invariant measure, that is, the measure which provides the statistics for typical solutions of the underlying system.

In order to ensure efficiency the set covering is constructed as a binary, hierarchical tree of boxes. In particular very fine approximations of the complicated set with millions of boxes can be handled in compressed pyramidal data structures as already proposed by Ghavamnia [8], Tamminen and Samet [13].

Concerning the visualization, we explore volume rendering and illuminated streamline type techniques. The volume rendering is based on hierarchical splatting [11, 9]. Approximate normal spaces for shading purposes can be defined similar to Nielson's approach [3] in scattered data interpolation. The local dynamics on the invariant set is incorporated in the visualization by a dense coverage of the object with local trajectories, similar to the method by Elber [7]. Such techniques [2, 12] have been exploited for dynamical systems in 2D, or on slices in 3D by Wegenkittel et al. [14]. In contrast to this we will directly tackle the full 3D problem.

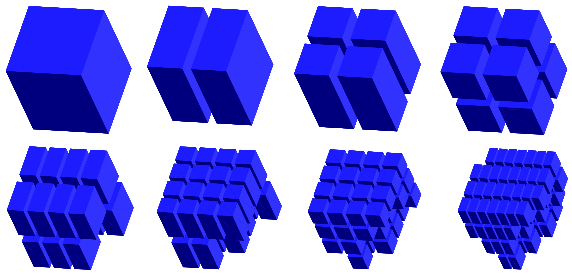

Figure 1: The box collections $\mathcal{B}_k$, $k = 0, \cdots, 7$.

2 The Set Oriented Algorithms

The central object which is approximated by the subdivision algorithm developed in [5] is the so-called *relative global attractor*,

$$A_Q = \bigcap_{j \geq 0} f^j(Q), \tag{4}$$

where $Q \subset \mathbb{R}^n$ is a compact subset. Roughly speaking, the set A_Q should be viewed as the union of invariant sets inside Q together with their unstable manifolds. In particular, A_Q may contain subsets of Q which cannot be approximated by direct simulation.
The subdivision algorithm for the approximation of A_Q generates a sequence $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots$ of finite collections of boxes with the property that for all integers k the set

$$A_Q^k = \bigcup_{B \in \mathcal{B}_k} B$$

is a covering of the relative global attractor under consideration. In the implementation this sequence builds up a pyramid of boxes, where a large initial box is successively bisected in cycling directions (cf. Fig. 1). A detailed discussion of this method can be found in [4]. The following table lists the true number of boxes for the Lorenz system, respectively Chua's circuit and different levels k.

k	21	24	27	30
Lorenz ($\beta = 0.4$)	31074	124084	497740	1998566
Chua's circuit	62288	293992	1465152	-

Here, for instance $k = 30$ corresponds to a full grid of size 1024^3. The attractor under consideration may have a Hausdorff dimension which is not an integer (cf. the Lorenz attractor in Fig. 2, which is of Hausdorff dimension between two and three). Nevertheless, frequently attractors are contained in the closure of unstable manifolds which are locally m dimensional surfaces within $\mathbb{R}^n$. Unfortunately, this surface structure is hidden in our discrete box approach. In terms of a surface interpretation the fundamental question is how to give a suitable definition of normals. For given ε we consider the neighbourhood $U_\varepsilon(c_B)$ of the center point c_B of every box $B \in \mathcal{B}_k$. Then we calculate the first momentum matrix S_B and consider its eigenvalues. It turned out that, if there is a local surface structure of dimension m, the eigenvalues can be ordered:

$$\lambda_1 \leq \cdots \leq \lambda_{n-m} \ll \lambda_{n-m+1} \leq \cdots \leq \lambda_n,$$

where the corresponding eigenvalues $v_1, \cdots, v_{n-m}$ turn out to be approximate normals and we lateron use them for shading purposes.

Finally, once a box covering $\mathcal{B}$ of the attractor A_Q has been computed, we can approximate the statistics of the dynamics on A_Q by the computation of a corresponding natural discrete invariant measure as a piecewise constant measure density on the boxes of the pyramid. For details we refer to [6].
In what follows we will present two different visualization methods based on a compressed pyramidal storage scheme for the box collections.

3 Volume Rendering

One appropriate way to visualize the complicated structure of the invariant sets is to render them as transparent volumetric objects. In the typical case of relatively thin approximate sets A_Q^k our pyramidal data structures are well suited for fast volume rendering applying appropriate splatting techniques [11] on the boxes in the hierarchical tree. Color is used to represent the invariant measure or in case of almost invariant sets to mark the distinct regions. In a view dependent back to front traversal of the hierarchical tree of boxes we use splat to generate a transparent volume projection of A_Q^k (cf. Fig. 2, 3). Thereby, for each box $B \in \mathcal{B}_k$ the old image intensity I^- is updated to a new intensity I^+ according to the convex combination formula

$$I^+ = (1 - \alpha_{\hat{B}}(\mu(B)))\, I^- + \alpha_{\hat{B}}(\mu(B))\, C(\mu(B), n_B)\,.$$

Here $\alpha_{\hat{B}}(\mu(B))$ is a suitable monotone μ dependent scaling of a reference splat $\alpha_{\hat{B}}$ (boxes of small measure are more transparent than those of large measure) and the color

$$C(\mu(B), n_B) = C(\mu(B)) + C(n_B)$$

contains an additional component $C(n_B)$ due to a Goraud shading with respect to the normal n_B and given light sources illuminating the invariant manifold.
Finally, let us give another interesting application of our method. In 2001 the *Jet Propulsion Laboratory* plans to eject a craft into space which is supposed to stay on a periodic orbit between sun and earth for two years. One of the many challenges connected to this mission is to find the optimal trajectories for the transit between earth and the *Halo orbit* (and back). One can look for them on the *(un)stable manifold* of the periodic orbit, see Figure 4. It is therefore indispensable to compute and visualize reliable approximations to these objects.

4 Dense Covering with Streamlines

The volume rendering approach described so far enables us to visualize the complete geometry together with the invariant measure, respectively the splitting into almost invariant subsets. An impression of the local dynamics on the invariant set A_Q, i. e. the direction and velocity of the continuous flow according to the underlying ODE are still missing in this graphical representation. To overcome this drawback we generate a coverage of A_Q^k with streamlines at a prescribed density in a preprocessing step. Then, for the lateron interactive rendering we use transparent illuminated streamlines,

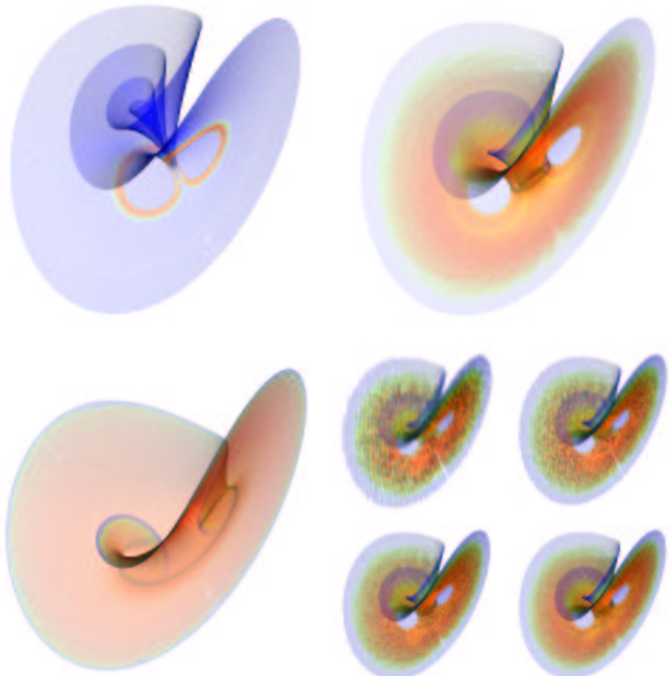

Figure 2: Three members of the Lorenz family are visualized using volume rendering for parameters $\beta = 0.4, 1.2, 2.6$ (left to right, top to bottom). In the lower right corner different levels of detail (15, 18, 21, and 24) are considered. For $\beta = 0.4$ trajectories rapidly approach a double periodic orbit, here indicated by the color concentration. The invariant set for $\beta = 2.6$ is an approximation of the famous Lorenz attractor, with an invariant measure almost equally spread all over the set. This spreading is a clear indication of the underlying chaotic behaviour. Throughout the entire sequence we realize a geometric change, but even more dominant this successive spreading of the measure.

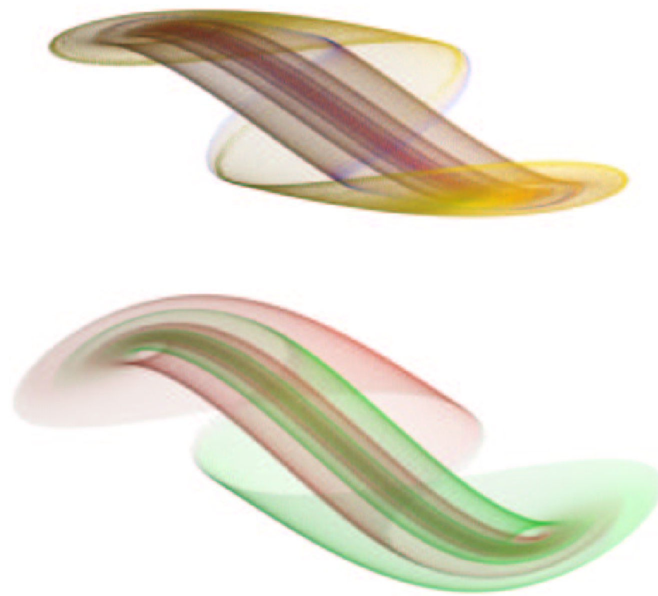

Figure 3: Volume rendering of Chua's circuit. Above color represents the invariant measure. Below, the red, respectively blue coloring indicates subsets in which trajectories stay for a very long time until they turn over to the other subset.

color them according to the invariant measure and make use of the approximate surface normals for shading purposes. We thus ensure the graphical representation of the *global geometry* and the *local dynamics* of the dynamical system at the same time while still retaining the *surface type appearance* Our coverage will be of equal density all over A_Q^k in the sense that the ratio $\gamma(B) := L(B)/m(B)$ of the sum $L(B)$ of the length of streamline segments in the boxes B of the binary tree and the local volume $m(B)$ is balanced in an iterative insertion process of streamline segments (cf. Fig. 5).

5 Conclusions and Future Work

We have presented a new approach to efficiently visualize reliable approximations of invariant sets, which characterize dynamical systems topologically. The method is based on a hierarchy of successively refined box coverings. Fundamental probability measures can be approximated on the box hierarchy. We have explained how to make use of this approach to perceive dynamical systems in a novel, considerably expanded way.

Interesting future research directions are the incooperation of adaptive multiresolutional algorithms [10] in the computation as well as the visualization, a triangular surface reconstruction from the box covering making use of the approximate normal calculus, and the appropriate animation of the long time behaviour of the dynamical system.

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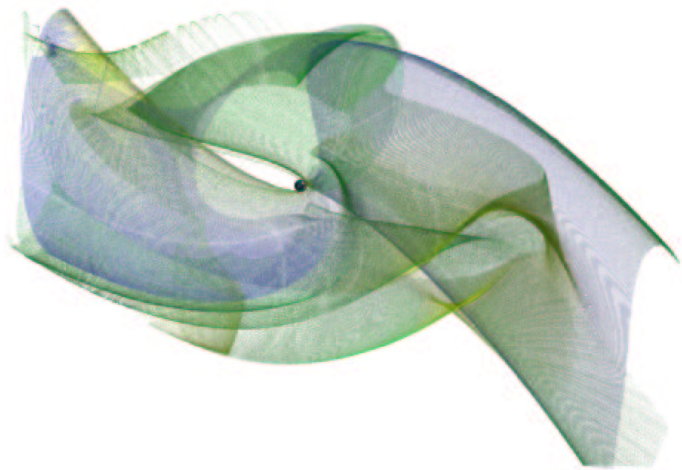

Figure 4: Part of the unstable manifold of the Halo orbit which contains the transit paths of the spacecraft from the orbit back to earth. We have clipped the complete object at a suitable intersection plane and colored the boxes with respect to their temporal distance from the Halo orbit: blue (near) $\rightarrow$ green $\rightarrow$ red $\rightarrow$ yellow (far).

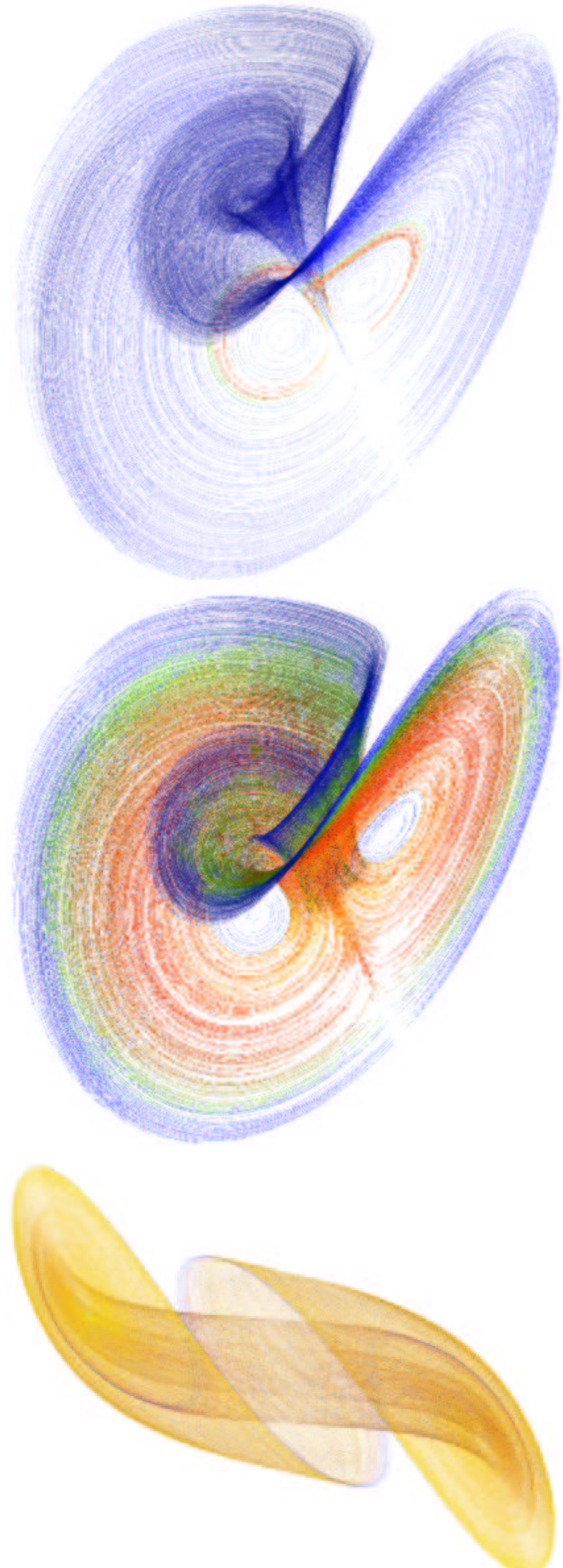

Figure 5: Invariant sets in the Lorenz family and Chua's circuit are visualized using a coverage with shaded streamlines.

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