# NUMERICAL EXPERIMENTS FOR THE ARNOLD-WINTHER MIXED FINITE ELEMENTS FOR THE STOKES PROBLEM ${ }^{\star}$ 

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#### Abstract

The stress-velocity formulation of the stationary Stokes problem allows an Arnold-Winther mixed finite element formulation with some superconvergent reconstruction of the velocity. This local postprocessing gives rise to two reliable a posteriori error estimators which recover optimal convergence order for the stress error estimates. The theoretical results are investigated in numerical benchmark examples.


## 1. Introduction

The stress-velocity-pressure formulation is the original physical model for incompressible Newtonian flows modeled by the conservation of momentum and the constitutive law. This model involves symmetric strain rates and stress tensors and is recast in a mixed form with symmetric stress tensors. The use of the deviatoric stress tensor $\mathcal{A} \boldsymbol{\sigma}$ leads to the stress-velocity formulation for the Stokes problem

$$
\operatorname{div} \boldsymbol{\sigma}=\boldsymbol{f} \quad \text { in } \Omega, \quad \mathcal{A} \boldsymbol{\sigma}-\boldsymbol{\epsilon}(\boldsymbol{u})=0 \quad \text { in } \Omega, \quad \text { and } \quad \boldsymbol{u}=\boldsymbol{g} \quad \text { on } \partial \Omega
$$

for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^{2}$ and given data $\boldsymbol{f} \in L^{2}\left(\Omega ; \mathbb{R}^{2}\right)$ and $\boldsymbol{g} \in H^{1}\left(\Omega ; \mathbb{R}^{2}\right) \cap C\left(\bar{\Omega} ; \mathbb{R}^{2}\right)$. The discretization is feasible with the symmetric Arnold-Winther mixed finite element method (MFEM) [AW02] proposed in linear elasticity. Since the Arnold-Winther MFEM has been proven to be stable for any material parameters, it is also a stable method for the Stokes problem as a limit case of linear elasticity.

[^0]Suppose $\left(\boldsymbol{\sigma}_{h}, \boldsymbol{u}_{h}\right)$ is the mixed Arnold-Winther MFEM approximation of order $k \geq 1$ to sufficiently smooth ( $\boldsymbol{\sigma}, \boldsymbol{u})$. Then it holds in terms of the standard Sobolev norms $\|\cdot\|_{s}:=\|\cdot\|_{H^{s}(\Omega)}, m \in \mathbb{N}$,

$$
\begin{aligned}
\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0} & \lesssim h^{m}\|\boldsymbol{\sigma}\|_{m}, \quad 1 \leq m \leq k+2 \\
\left\|\operatorname{div}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0} & \lesssim h^{m}\|\operatorname{div} \boldsymbol{\sigma}\|_{m}, \quad 0 \leq m \leq k+1 \\
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0} & \lesssim h^{m}\|\boldsymbol{u}\|_{m+1}, \quad 1 \leq m \leq k+1
\end{aligned}
$$

Compared to the stress errors, the last bound appears suboptimal, but is a consequence of the lower ansatz for the displacement variable reduced by two degrees when compared with the stress variable instead of only one. Here and throughout the paper, for short notation on generic constants $C$, for any two real numbers or functions or expressions $A$ and $B, A \lesssim B$ abbreviates $A \leq C B$. The point is that this multiplicative constant $C$ does not depend on the local or global mesh-sizes but may depend on the domain $\Omega$, the shape regularity of the mesh, and the polynomial degree. Similarly, $A \approx B$ abbreviates $A \lesssim B \lesssim A$. The nonstandard finite element method (FEM) for the Stokes problem started with [BW91, DDP95, GR86] and the reader is referred to [AO00, BS01, Ver96] for information on a posteriori error control. The main results of this paper concern the a priori error estimation of the superconvergence of some reconstructed velocity field $\boldsymbol{u}_{h}^{*}$ in the sense of

$$
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right\|_{0} \lesssim h^{k+3}\left(\|\boldsymbol{\sigma}\|_{k+2}+\|\operatorname{div} \boldsymbol{\sigma}\|_{k+1}\right)+h^{m+1}\|\boldsymbol{u}\|_{m+1}
$$

for the restricted class of domains $\Omega$ with sufficiently smooth boundary $\partial \Omega$. We refer the reader to [Kim07, LM08, Voh10] for some postprocessing for the Poisson problem from Stenberg [Ste88]. Based on the discontinuous postprocessed velocity field $\boldsymbol{u}_{h}^{*}$ and a smooth velocity field $\tilde{\boldsymbol{u}}_{h} \in H^{1}\left(\Omega ; \mathbb{R}^{2}\right)$ we design reliable and efficient a posteriori error estimators $\mu$ and $\eta$ such that reliability or even equivalence holds in the sense of

$$
\begin{array}{r}
\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}\left(\boldsymbol{u}-\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) \approx \mu+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right), \\
\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) \lesssim \eta+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right)
\end{array}
$$

up to oscillations

$$
\operatorname{osc}^{2}\left(\boldsymbol{f}, \mathcal{T}_{h}\right):=\sum_{T \in \mathcal{T}_{h}} h_{T}^{2}\left\|\boldsymbol{f}-\boldsymbol{f}_{h}\right\|_{0, T}^{2}
$$

of the right-hand side $\boldsymbol{f}$ and its piecewise $L^{2}$ projection $\boldsymbol{f}_{h}$ onto piecewise polynomials of degree $\leq k$. Note that for the Arnold-Winther FEM, the oscillations $\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right)$ of the right-hand side $\boldsymbol{f}$ are of the same but not of higher order compared to the stress and strain errors. In principle, the oscillations might dominate the error estimator and therefore lead to an overestimation of the stress error $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}$, which is empirically confirmed in Section 7 by an academic example
with smooth solution. However, in all practical relevant benchmark examples of this paper, oscillations vanish for the constant $\boldsymbol{f}$.

The remainder of this paper is organized as follows: Section 2 introduces necessary notation and the stress-velocity formulation. Section 3 recalls the discrete problem and its mixed Arnold-Winther finite element approximation. Section 4 establishes some superconvergent local postprocessing of the velocity for all fixed polynomial degrees. The a posteriori error analysis for the two reliable error estimators $\eta$ and $\mu$ together with the efficiency of $\mu$ is presented in Section 5. Section 6 describes some adaptive finite element method (AFEM), and Section 7 presents various numerical examples to verify the theoretical results and to illustrate the performance of the method. It turns out that AFEM is very important to meet optimal convergence rates by proper mesh-design to compensate for corner singularities.

## 2. Stress Velocity Formulation

For $\boldsymbol{v}=\left(v_{1}, v_{2}\right)^{t} \in \mathbb{R}^{2}, \boldsymbol{\tau}=\left(\tau_{i j}\right)_{2 \times 2}$, and $\boldsymbol{\sigma}=\left(\sigma_{i j}\right)_{2 \times 2} \in \mathbb{R}^{2 \times 2}$, we define

$$
\begin{aligned}
& \operatorname{curl} \boldsymbol{v}:=\left(\begin{array}{ll}
\frac{\partial v_{1}}{\partial y} & -\frac{\partial v_{1}}{\partial x} \\
\frac{\partial v_{2}}{\partial y} & -\frac{\partial v_{2}}{\partial x}
\end{array}\right), \quad \nabla \boldsymbol{v}:=\left(\begin{array}{ll}
\frac{\partial v_{1}}{\partial x} & \frac{\partial v_{1}}{\partial y} \\
\frac{\partial v_{2}}{\partial x} & \frac{\partial v_{2}}{\partial y}
\end{array}\right), \\
& \operatorname{curl} \boldsymbol{\tau}:=\binom{\frac{\partial \tau_{12}}{\partial x}-\frac{\partial \tau_{11}}{\partial y}}{\frac{\partial \tau_{22}}{\partial x}-\frac{\partial \tau_{21}}{\partial y}}, \quad \operatorname{div} \boldsymbol{\tau}:=\binom{\frac{\partial \tau_{11}}{\partial x}+\frac{\partial \tau_{12}}{\partial y}}{\frac{\partial \tau_{21}}{\partial x}+\frac{\partial \tau_{22}}{\partial y}}, \\
& \operatorname{tr} \boldsymbol{\tau}:=\tau_{11}+\tau_{22}, \quad \boldsymbol{\tau} \boldsymbol{v}:=\binom{\tau_{11} v_{1}+\tau_{12} v_{2}}{\tau_{21} v_{1}+\tau_{22} v_{2}}, \\
& \boldsymbol{\tau}: \boldsymbol{\sigma}:=\sum_{i, j} \tau_{i j} \sigma_{i j}, \quad \boldsymbol{\delta}:=2 \times 2 \text { unit matrix. }
\end{aligned}
$$

We employ the standard notation for the Sobolev spaces $H^{s}(\omega)$ for $s \geq 0$. The associated norm is denoted by $\|\cdot\|_{s, \omega}$. For $s=0$, we use the notation $L^{2}(\omega)$ instead of $H^{0}(\omega)$. In the case $\omega=\Omega$ we simply write $\|\cdot\|_{s, \Omega}=\|\cdot\|_{s}$. We define $H^{-s}(\omega):=\left(H_{0}^{s}(\omega)\right)^{*}$ as the dual space of $H_{0}^{s}(\omega)$. Extending the definitions to vector- and matrix-valued functions, we let $H^{s}\left(\omega ; \mathbb{R}^{2}\right)$ (simply $\left.\boldsymbol{H}^{s}(\omega)\right)$ and $\boldsymbol{H}^{s}\left(\omega ; \mathbb{R}^{2 \times 2}\right)$ denote the Sobolev spaces over the set of 2 -dimensional vector- and $2 \times 2$ matrix-valued functions, respectively. Finally, we define the space

$$
\boldsymbol{H}\left(\operatorname{div}, \Omega ; \mathbb{R}^{2 \times 2}\right):=\left\{\boldsymbol{\tau} \in \boldsymbol{L}^{2}\left(\Omega ; \mathbb{R}^{2 \times 2}\right) \mid \operatorname{div} \boldsymbol{\tau} \in \boldsymbol{L}^{2}(\Omega)\right\}
$$

with the norm

$$
\|\boldsymbol{\tau}\|_{\boldsymbol{H}\left(\operatorname{div}, \Omega ; \mathbb{R}^{2 \times 2}\right)}^{2}:=(\boldsymbol{\tau}, \boldsymbol{\tau})+(\operatorname{div} \boldsymbol{\tau}, \operatorname{div} \boldsymbol{\tau}) .
$$

Here and throughout the paper, $(\cdot, \cdot)_{\omega}$ denotes the $\boldsymbol{L}^{2}\left(\omega ; \mathbb{R}^{2 \times 2}\right)$ inner product $\int_{\omega} \boldsymbol{\tau}: \boldsymbol{\tau} d x$ as well as the $\boldsymbol{L}^{2}(\omega)$ inner product $\int_{\omega} \boldsymbol{\tau} \cdot \boldsymbol{\tau} d x$. In
the case $\omega=\Omega$ we simply write $(\cdot, \cdot)_{\Omega}=(\cdot, \cdot)$. The extended $L^{2}(\partial \Omega)$ product along the boundary $\partial \Omega$ is denoted by the duality brackets $\langle\cdot, \cdot\rangle$.

On the domain $\Omega \subset \mathbb{R}^{2}$ with sufficiently smooth Lipschitz boundary $\partial \Omega$ filled with a fluid of viscosity $\nu>0$ and given data $\boldsymbol{f} \in \boldsymbol{L}^{2}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{H}^{1}(\Omega) \cap \boldsymbol{C}(\bar{\Omega})$, the stationary Stokes problem reads

$$
\begin{align*}
-\nu \triangle \boldsymbol{u}+\nabla p & =-\boldsymbol{f} \quad \text { in } \Omega, \\
\operatorname{div} \boldsymbol{u} & =0 \quad \text { in } \Omega,  \tag{1}\\
\boldsymbol{u} & =\boldsymbol{g} \quad \text { on } \partial \Omega
\end{align*}
$$

for the unknown velocity $\boldsymbol{u}$ and pressure $p$. Suppose that the following two compatibility conditions hold:

$$
\int_{\partial \Omega} \boldsymbol{g} \cdot \boldsymbol{n}=0 \quad \text { and } \quad \int_{\Omega} p d x=0 .
$$

Let $\boldsymbol{\sigma}=\left(\sigma_{i j}\right)_{2 \times 2}$ be the stress tensor and

$$
\boldsymbol{\epsilon}(\boldsymbol{u}):=\frac{1}{2}\left(\nabla \boldsymbol{u}+(\nabla \boldsymbol{u})^{t}\right)
$$

be the deformation rate tensor. The aforementioned Stokes problem is derived from the stress-velocity-pressure formulation which is the set of original physical equations for incompressible Newtonian flow, i.e.,

$$
\begin{array}{rlrl}
\operatorname{div} \boldsymbol{\sigma} & =\boldsymbol{f} & \text { in } \Omega, \\
\boldsymbol{\sigma}+p \boldsymbol{\delta}-2 \nu \boldsymbol{\epsilon}(\boldsymbol{u}) & =\mathbf{0} & & \text { in } \Omega, \\
\operatorname{div} \boldsymbol{u} & =0 & & \text { in } \Omega, \\
\boldsymbol{u} & =\boldsymbol{g} & & \text { on } \partial \Omega .
\end{array}
$$

To design the stress-velocity formulation we define the deviatoric operator

$$
\mathcal{A}: \mathbb{S} \rightarrow \mathbb{S}
$$

for the symmetric tensors $\mathbb{S}:=\left\{\boldsymbol{\tau} \in \mathbb{R}^{2 \times 2} \mid \boldsymbol{\tau}=\boldsymbol{\tau}^{t}\right\}$ by

$$
\mathcal{A} \boldsymbol{\tau}:=\frac{1}{2 \nu}\left(\boldsymbol{\tau}-\frac{1}{2}(\operatorname{tr} \boldsymbol{\tau}) \boldsymbol{\delta}\right) \quad \text { for all } \boldsymbol{\tau} \in \mathbb{S} .
$$

Note that $\operatorname{Ker}(\mathcal{A})=\{q \boldsymbol{\delta} \in \mathbb{S} \mid q \in \mathbb{R}\}$ and $\mathcal{A} \boldsymbol{\tau}$ is a trace-free tensor called deviatoric. Further, we can easily show that the following properties of the operator $\mathcal{A}$ hold, for all $\boldsymbol{\tau}, \boldsymbol{\sigma} \in \mathbb{S}$ :

$$
\begin{aligned}
(\mathcal{A} \boldsymbol{\tau}, \boldsymbol{\sigma}) & =(\boldsymbol{\tau}, \mathcal{A} \boldsymbol{\sigma}) \\
(\mathcal{A} \boldsymbol{\tau}, 2 \nu \mathcal{A} \boldsymbol{\sigma}) & =(\mathcal{A} \boldsymbol{\sigma}, \boldsymbol{\tau})=\frac{1}{2 \nu}\left((\boldsymbol{\sigma}, \boldsymbol{\tau})-\frac{1}{2}(\operatorname{tr} \boldsymbol{\sigma}, \operatorname{tr} \boldsymbol{\tau})\right), \\
\|\mathcal{A} \boldsymbol{\tau}\| & \leq \frac{1}{2 \nu}\|\boldsymbol{\tau}\|
\end{aligned}
$$

Using the deviatoric operator, we arrive at the stress-velocity formulation for the Stokes problem (1):

$$
\begin{array}{rll}
\operatorname{div} \boldsymbol{\sigma} & =\boldsymbol{f} & \text { in } \Omega, \\
\mathcal{A} \boldsymbol{\sigma}-\boldsymbol{\epsilon}(\boldsymbol{u}) & =\mathbf{0} & \text { in } \Omega,  \tag{2}\\
\boldsymbol{u} & =\boldsymbol{g} & \\
\text { on } \partial \Omega .
\end{array}
$$

The second equation of (2) is obtained from

$$
\operatorname{tr} \boldsymbol{\epsilon}(\boldsymbol{u})=\operatorname{div} \boldsymbol{u}=0 \quad \text { and } \quad \operatorname{tr} \boldsymbol{\sigma}=-2 p
$$

and the compatibility condition $\int_{\Omega} p d x=0$ implies

$$
\int_{\Omega} \operatorname{tr} \boldsymbol{\sigma} d x=0 .
$$

We have the following well-known regularity results for sufficiently smooth boundary $\partial \Omega$ or a convex domain. For $\boldsymbol{f} \in \boldsymbol{L}^{2}(\Omega), \boldsymbol{g} \in \boldsymbol{H}^{2}(\Omega)$, the solutions to problems (1) and (2) satisfy $\boldsymbol{u} \in \boldsymbol{H}^{2}(\Omega) \cap \boldsymbol{H}^{1}(\bar{\Omega})$, $p \in H^{1}(\Omega) / \mathbb{R}, \boldsymbol{\sigma} \in \boldsymbol{H}^{1}(\Omega ; \mathbb{S})$, and

$$
\begin{equation*}
\|\boldsymbol{u}\|_{2}+\|p\|_{1}+\|\boldsymbol{\sigma}\|_{1} \lesssim\|\boldsymbol{f}\|_{0}+\|\boldsymbol{g}\|_{2} . \tag{3}
\end{equation*}
$$

With $\boldsymbol{V}:=\boldsymbol{L}^{2}(\Omega)$ and

$$
\boldsymbol{\Phi}:=\boldsymbol{H}(\operatorname{div}, \Omega, \mathbb{S}) / \mathbb{R} \sim\left\{\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}, \Omega, \mathbb{S}) \mid \int_{\Omega} \operatorname{tr} \boldsymbol{\tau} d x=0\right\}
$$

the weak form for the problem (2) reads as follows: Find $\boldsymbol{\sigma} \in \boldsymbol{\Phi}$ and $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$
\begin{align*}
(\mathcal{A} \boldsymbol{\sigma}, \boldsymbol{\tau})+(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{u}) & =\langle\boldsymbol{g}, \boldsymbol{\tau} \boldsymbol{n}\rangle \text { for all } \boldsymbol{\tau} \in \boldsymbol{\Phi},  \tag{4}\\
(\operatorname{div} \boldsymbol{\sigma}, \boldsymbol{v}) & =(\boldsymbol{f}, \boldsymbol{v}) \quad \text { for all } \boldsymbol{v} \in \boldsymbol{V} . \tag{5}
\end{align*}
$$

This problem has a unique solution from the well-known inf-sup condition in the mixed formulation and the following lemma [BF91].

Lemma 2.1. For all $\boldsymbol{\tau} \in \boldsymbol{\Phi}$, we have

$$
\|\boldsymbol{\tau}\|_{0}^{2} \lesssim\left\|\mathcal{A}^{1 / 2} \boldsymbol{\tau}\right\|_{0}^{2}+\|\operatorname{div} \boldsymbol{\tau}\|_{-1}^{2} .
$$

## 3. Mixed finite element method

Let $\left\{\mathcal{T}_{h}\right\}$ be a family of quasi-uniform triangulations of $\bar{\Omega}$ by triangles $T$ of diameter $h_{T}$. For each $\mathcal{T}_{h}$, let $\mathcal{E}_{h}$ denote the set of all edges of $\mathcal{T}_{h}$ and, given $T \in \mathcal{T}_{h}$, let $\mathcal{E}(T)$ be the set of its edges. Further, for an edge $E \in \mathcal{E}(T)$, let $\boldsymbol{t}_{E}=\left(-n_{2}, n_{1}\right)^{t}$ be the unit tangential vector along $E$ for the unit outward normal $\boldsymbol{n}_{E}=\left(n_{1}, n_{2}\right)^{t}$ to $E$ with the diameter $h_{E}$. Moreover, we define the jump $[\boldsymbol{w}]$ of $\boldsymbol{w}$ by

$$
\left.[\boldsymbol{w}]\right|_{E}:=\left.\left(\left.\boldsymbol{w}\right|_{T_{+}}\right)\right|_{E}-\left.\left(\left.\boldsymbol{w}\right|_{T_{-}}\right)\right|_{E} \quad \text { if } \quad E=\bar{T}_{+} \cap \bar{T}_{-},
$$

where $\boldsymbol{n}_{E}$ points from $T_{+}$into its neighboring element $T_{-}$, and $\left.[\boldsymbol{w}]\right|_{E}:=$ $\boldsymbol{w}-\boldsymbol{g}$ if $E=\bar{T}_{+} \cap \partial \Omega$.

We define the finite element spaces associated with the regular triangulation $\mathcal{T}_{h}$ of $\Omega$ into triangles,

$$
\begin{aligned}
A W_{k}(T) & :=\left\{\boldsymbol{\tau} \in P_{k+2}(T ; \mathbb{S}) \mid \operatorname{div} \boldsymbol{\tau} \in P_{k}\left(T ; \mathbb{R}^{2}\right)\right\} \\
\boldsymbol{\Phi}_{h} & :=\left\{\boldsymbol{\tau} \in \boldsymbol{\Phi}|\boldsymbol{\tau}|_{T} \in A W_{k}(T)\right\} \\
\boldsymbol{V}_{h} & :=\left\{\boldsymbol{v} \in \boldsymbol{L}^{2}(\Omega)|v|_{T} \in P_{k}\left(T ; \mathbb{R}^{2}\right)\right\},
\end{aligned}
$$

where $A W_{k}(T)$ is the Arnold-Winther element of index $k \geq 1$ of [AW02], and $P_{k}(T)$ is the set of polynomials of total degree $k$ on the domain $T$. The space $\boldsymbol{\Phi}_{h}$ consists of all symmetric polynomial matrix fields of degree at most $k+1$ together with the divergence-free matrix fields of degree $k+2$.
We notice that $\boldsymbol{\Phi}_{h} \subset \boldsymbol{\Phi}$ and hence if $\boldsymbol{\tau}_{h} \in \boldsymbol{\Phi}_{h}$, then $\boldsymbol{\tau}_{h}$ has continuous normal components and the constraint $\int_{\Omega} \operatorname{tr} \boldsymbol{\tau}_{h} d x=0$ holds.

The MFEM reads as follows: Find $\boldsymbol{\sigma}_{h} \in \boldsymbol{\Phi}_{h}$ and $\boldsymbol{u}_{h} \in \boldsymbol{V}_{h}$ such that

$$
\begin{align*}
\left(\mathcal{A} \boldsymbol{\sigma}_{h}, \boldsymbol{\tau}_{h}\right)+\left(\operatorname{div} \boldsymbol{\tau}_{h}, \boldsymbol{u}_{h}\right) & =\left\langle\boldsymbol{g}, \boldsymbol{\tau}_{h} \boldsymbol{n}\right\rangle \quad \text { for all } \boldsymbol{\tau}_{h} \in \boldsymbol{\Phi}_{h},  \tag{6}\\
\left(\operatorname{div} \boldsymbol{\sigma}_{h}, \boldsymbol{v}_{h}\right) & =\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right) \text { for all } \boldsymbol{v}_{h} \in \boldsymbol{V}_{h} . \tag{7}
\end{align*}
$$

By Lemma 2.1 and the discrete inf-sup condition of the $A W_{k}$ element space (cf. [BF91]), the discrete problem is well-posed and has a unique solution.
We consider a projection operator over the space $\boldsymbol{\Phi}$. Let $\tilde{\Pi}_{h}$ [AW02] denote the Arnold-Winther projection operator associated with the degrees of freedom onto $\boldsymbol{\Phi}_{h}+\operatorname{span}\{\boldsymbol{\delta}\}$. We define $\Pi_{h}: \boldsymbol{\Phi} \rightarrow \boldsymbol{\Phi}_{h}$ by

$$
\begin{equation*}
\Pi_{h} \boldsymbol{\tau}=\tilde{\Pi}_{h} \boldsymbol{\tau}-\frac{\int_{\Omega}\left(\operatorname{tr} \tilde{\Pi}_{h} \boldsymbol{\tau}\right) d x}{2|\Omega|} \boldsymbol{\delta} \quad \text { for all } \boldsymbol{\tau} \in \mathbf{\Phi} \tag{8}
\end{equation*}
$$

with the area $|\Omega|=\int_{\Omega} 1 d x$ of $\Omega$. We notice that $\int_{\Omega}\left(\operatorname{tr} \Pi_{h} \boldsymbol{\tau}\right) d x=0$. Let $\boldsymbol{P}_{h}$ be the $\boldsymbol{L}^{2}$ projection onto $\boldsymbol{V}_{h}$ with the well-known approximation property

$$
\begin{equation*}
\left\|\boldsymbol{P}_{h} \boldsymbol{v}-\boldsymbol{v}\right\|_{0} \lesssim h^{k+1}\|\boldsymbol{v}\|_{k+1} \quad \text { for all } \boldsymbol{v} \in \boldsymbol{H}^{k+1}(\Omega) \tag{9}
\end{equation*}
$$

Then the following two lemmas hold [AW02].
Lemma 3.1. The commutative property $\operatorname{div} \Pi_{h}=\boldsymbol{P}_{h} \operatorname{div}$ holds. Furthermore, for $\boldsymbol{\tau} \in \boldsymbol{\Phi} \cap \boldsymbol{H}^{k+2}(\Omega ; \mathbb{S}), m \in \mathbb{N}$, it holds that

$$
\begin{gather*}
\left\|\boldsymbol{\tau}-\Pi_{h} \boldsymbol{\tau}\right\|_{0} \lesssim h^{m}\|\boldsymbol{\tau}\|_{m}, \quad 1 \leq m \leq k+2  \tag{10}\\
\left\|\operatorname{div} \boldsymbol{\tau}-\operatorname{div}\left(\Pi_{h} \boldsymbol{\tau}\right)\right\|_{0} \lesssim h^{m}\|\operatorname{div} \boldsymbol{\tau}\|_{m} \quad 0 \leq m \leq k+1 . \tag{11}
\end{gather*}
$$

Lemma 3.2. For the exact solution $(\boldsymbol{\sigma}, \boldsymbol{u}) \in\left(\boldsymbol{\Phi} \cap \boldsymbol{H}^{k+2}(\Omega ; \mathbb{S})\right) \times$ $\boldsymbol{H}^{k+2}(\Omega)$ of problem (1) and the approximate solution $\left(\boldsymbol{\sigma}_{h}, \boldsymbol{u}_{h}\right)$ of problem (6)-(7), $m \in \mathbb{N}$, it holds that

$$
\begin{aligned}
\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0} & \lesssim h^{m}\|\boldsymbol{\sigma}\|_{m}, \quad 1 \leq m \leq k+2 \\
\left\|\operatorname{div}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0} & \lesssim h^{m}\|\operatorname{div} \boldsymbol{\sigma}\|_{m}, \quad 0 \leq m \leq k+1 \\
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0} & \lesssim h^{m}\|\boldsymbol{u}\|_{m+1}, \quad 1 \leq m \leq k+1
\end{aligned}
$$

Remark 3.3. We note that using the relation $p=-\operatorname{tr} \boldsymbol{\sigma} / 2$, we can define the approximation of the pressure by $p_{h}:=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$. Then the pressure error estimate holds:

$$
\left\|p-p_{h}\right\|_{0}=\frac{1}{2}\left\|\operatorname{tr} \boldsymbol{\sigma}-\operatorname{tr} \boldsymbol{\sigma}_{h}\right\|_{0} \leq\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0} \lesssim h^{m}\|\boldsymbol{\sigma}\|_{m}, 1 \leq m \leq k+2
$$

The estimate for $\left\|\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0}$ presented in the following theorem is used to derive the error estimates of the postprocessed velocity.

Theorem 3.4. With sufficiently smooth boundary $\partial \Omega, \boldsymbol{\sigma} \in \boldsymbol{H}^{k+2}(\Omega ; \mathbb{S})$ and $\boldsymbol{f}=\operatorname{div} \boldsymbol{\sigma} \in \boldsymbol{H}^{k+1}(\Omega)$, it holds that

$$
\begin{equation*}
\left\|\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0} \lesssim h^{k+3}\left(\|\boldsymbol{\sigma}\|_{k+2}+\|\operatorname{div} \boldsymbol{\sigma}\|_{k+1}\right) . \tag{12}
\end{equation*}
$$

Proof. We start with a duality argument. Let $(\boldsymbol{\eta}, \boldsymbol{z}) \in \boldsymbol{\Phi} \times \boldsymbol{V}$ be the dual solution to

$$
\begin{align*}
(\mathcal{A} \boldsymbol{\eta}, \boldsymbol{\tau})+(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{z}) & =0 \text { for all } \boldsymbol{\tau} \in \boldsymbol{\Phi},  \tag{13}\\
(\operatorname{div} \boldsymbol{\eta}, \boldsymbol{v}) & =\left(\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}, \boldsymbol{v}\right) \text { for all } \boldsymbol{v} \in \boldsymbol{V} . \tag{14}
\end{align*}
$$

The a priori estimate (3) implies

$$
\begin{equation*}
\|\boldsymbol{z}\|_{2} \lesssim\left\|\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0} \quad \text { and } \quad\|\boldsymbol{\eta}\|_{1} \lesssim\left\|\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0} . \tag{15}
\end{equation*}
$$

Since (14) and $\operatorname{div} \Pi_{h}=\boldsymbol{P}_{h} \operatorname{div}$, we deduce

$$
\begin{align*}
\left\|\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0}^{2} & =\left(\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}, \operatorname{div} \boldsymbol{\eta}\right) \\
& =\left(\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}, \boldsymbol{P}_{h} \operatorname{div} \boldsymbol{\eta}\right)  \tag{16}\\
& =\left(\boldsymbol{u}-\boldsymbol{u}_{h}, \operatorname{div} \Pi_{h} \boldsymbol{\eta}\right) .
\end{align*}
$$

The difference of (4)-(5) and (6)-(7) leads to

$$
\begin{array}{rll}
\left(\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right), \boldsymbol{\tau}_{h}\right)+\left(\operatorname{div} \boldsymbol{\tau}_{h}, \boldsymbol{u}-\boldsymbol{u}_{h}\right)=0 & \text { for all } \boldsymbol{\tau}_{h} \in \boldsymbol{\Phi}_{h}, \\
\left(\operatorname{div}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right), \boldsymbol{v}_{h}\right)=0 & \text { for all } \boldsymbol{v}_{h} \in \boldsymbol{V}_{h} . \tag{18}
\end{array}
$$

The identities (13), (16)-(18) and the estimates (9)-(10) yield

$$
\begin{align*}
& \left\|\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0}^{2}=-\left(\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right), \Pi_{h} \boldsymbol{\eta}-\boldsymbol{\eta}\right)-\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}, \mathcal{A} \boldsymbol{\eta}\right) \\
& \quad=-\left(\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right), \Pi_{h} \boldsymbol{\eta}-\boldsymbol{\eta}\right)+\left(\operatorname{div}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right), \boldsymbol{z}-\boldsymbol{P}_{h} \boldsymbol{z}\right)  \tag{19}\\
& \quad \lesssim h\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0}\|\boldsymbol{\eta}\|_{1}+h^{2}\left\|\operatorname{div}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}\|\boldsymbol{z}\|_{2} .
\end{align*}
$$

Lemma 3.2 and the inequalities (15) and (19) lead to

$$
\left\|\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0} \lesssim h^{k+3}\left(\|\boldsymbol{\sigma}\|_{k+2}+\|\operatorname{div} \boldsymbol{\sigma}\|_{k+1}\right) .
$$

## 4. Postprocessing

Since $\mathcal{A} \boldsymbol{\sigma}_{h}$ is expected to be a good approximation of $\boldsymbol{\epsilon}(\boldsymbol{u})$, we can obtain an improved approximate solution of the velocity $\boldsymbol{u}$ through local postprocessing in the spirit of Stenberg [Ste88]. For $m \geq k+2$ let

$$
\boldsymbol{W}_{h}^{*}=\left\{\boldsymbol{v} \in \boldsymbol{L}^{2}(\Omega)|\boldsymbol{v}|_{T} \in P_{m}\left(T ; \mathbb{R}^{2}\right) \text { for all } T \in \mathcal{T}_{h}\right\} .
$$

We define $\boldsymbol{u}_{h}^{*} \in \boldsymbol{W}_{h}^{*}$ on each $T \in \mathcal{T}_{h}$ with $\boldsymbol{P}_{T}=\left.\boldsymbol{P}_{h}\right|_{T}$ as the solution to the system

$$
\begin{align*}
\boldsymbol{P}_{T} \boldsymbol{u}_{h}^{*} & =\boldsymbol{u}_{h},  \tag{20}\\
\left(\boldsymbol{\epsilon}\left(\boldsymbol{u}_{h}^{*}\right), \boldsymbol{\epsilon}(\boldsymbol{v})\right)_{T} & =\left(\mathcal{A} \boldsymbol{\sigma}_{h}, \boldsymbol{\epsilon}(\boldsymbol{v})\right)_{T} \quad \text { for all }\left.\boldsymbol{v} \in\left(\boldsymbol{\delta}-\boldsymbol{P}_{T}\right) \boldsymbol{W}_{h}^{*}\right|_{T} . \tag{21}
\end{align*}
$$

In other words, $\left.\boldsymbol{u}_{h}^{*}\right|_{T} \in P_{m}\left(T ; \mathbb{R}^{2}\right)$ is the Riesz representation of the linear functional $\left(\mathcal{A} \boldsymbol{\sigma}_{h}, \boldsymbol{\epsilon}(\cdot)\right)_{T}$ in the Hilbert space $\left.\left(\boldsymbol{\delta}-\boldsymbol{P}_{T}\right) \boldsymbol{W}_{h}^{*}\right|_{T} \equiv$ $\left\{\boldsymbol{v}_{m} \in P_{m}\left(T ; \mathbb{R}^{2}\right) \mid\left(\boldsymbol{v}_{m}, \boldsymbol{w}_{k}\right)_{T}=0\right.$ for all $\left.\boldsymbol{w}_{k} \in P_{k}\left(T ; \mathbb{R}^{2}\right)\right\}$ with scalar product $(\boldsymbol{\epsilon}(\cdot), \boldsymbol{\epsilon}(\cdot))_{T}$. The postpocessing on each triangle with Lagrange multiplier $\boldsymbol{\lambda}_{k} \in P_{k}\left(T ; \mathbb{R}^{2}\right)$ can be implemented as the linear system of equations

$$
\begin{array}{rlrl}
\left(\boldsymbol{\epsilon}\left(\boldsymbol{u}_{h}^{*}\right), \boldsymbol{\epsilon}\left(\boldsymbol{v}_{m}\right)\right)_{T}+\left(\boldsymbol{\lambda}_{k}, \boldsymbol{v}_{m}\right)_{T} & =\left(\mathcal{A} \boldsymbol{\sigma}_{h}, \boldsymbol{\epsilon}\left(\boldsymbol{v}_{m}\right)\right)_{T} & & \text { for all } \boldsymbol{v}_{m} \in P_{m}\left(T ; \mathbb{R}^{2}\right) \\
\left(\boldsymbol{u}_{h}^{*}, \boldsymbol{w}_{k}\right)_{T} & =\left(\boldsymbol{u}_{h}, \boldsymbol{w}_{k}\right)_{T} & \text { for all } \boldsymbol{w}_{k} \in P_{k}\left(T ; \mathbb{R}^{2}\right) .
\end{array}
$$

The Korn inequality yields positive definiteness of $(\boldsymbol{\epsilon}(\cdot), \boldsymbol{\epsilon}(\cdot))_{T}$ on $(\boldsymbol{\delta}-$ $\left.\boldsymbol{P}_{T}\right)\left.\boldsymbol{W}_{h}^{*}\right|_{T}$. Since $P_{k}\left(T ; \mathbb{R}^{2}\right) \subset P_{m}\left(T ; \mathbb{R}^{2}\right)$,

$$
\sup _{\boldsymbol{v}_{m} \in P_{m}\left(T ; \mathbb{R}^{2}\right)}\left(\boldsymbol{v}_{m}, \boldsymbol{\lambda}_{k}\right)_{T} /\left\|\boldsymbol{v}_{m}\right\|_{0, T} \geq\left\|\boldsymbol{\lambda}_{k}\right\|_{0, T} \quad \text { for all } \boldsymbol{\lambda}_{k} \in P_{k}\left(T ; \mathbb{R}^{2}\right) .
$$

Thus, the Brezzi splitting theorem [Bre74] shows that there exists a unique solution on each triangle. The identity $\mathcal{A} \boldsymbol{\sigma}=\boldsymbol{\epsilon}(\boldsymbol{u})$ and (21) imply the error identity
(22) $\left(\boldsymbol{\epsilon}\left(\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right), \boldsymbol{\epsilon}(\boldsymbol{v})\right)_{T}=\left(\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right), \boldsymbol{\epsilon}(\boldsymbol{v})\right)_{T},\left.\forall \boldsymbol{v} \in\left(\boldsymbol{\delta}-\boldsymbol{P}_{T}\right) \boldsymbol{W}_{h}^{*}\right|_{T}$.

Theorem 4.1. Let the boundary $\partial \Omega$ be sufficiently smooth, $\boldsymbol{u} \in \boldsymbol{H}^{m+1}(\Omega)$, $\boldsymbol{\sigma} \in \boldsymbol{H}^{k+2}(\Omega ; \mathbb{S})$ and $\boldsymbol{f}=\operatorname{div} \boldsymbol{\sigma} \in \boldsymbol{H}^{k+1}(\Omega)$ solve (1). Then it holds that

$$
\begin{aligned}
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right\|_{0} & \lesssim h^{k+3}\left(\|\boldsymbol{\sigma}\|_{k+2}+\|\operatorname{div} \boldsymbol{\sigma}\|_{k+1}\right)+h^{m+1}\|\boldsymbol{u}\|_{m+1}, \\
\left\|\nabla_{\mathcal{T}_{h}}\left(\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0} & \lesssim h^{k+2}\left(\|\boldsymbol{\sigma}\|_{k+2}+\|\operatorname{div} \boldsymbol{\sigma}\|_{k+1}\right)+h^{m}\|\boldsymbol{u}\|_{m+1},
\end{aligned}
$$

for the postprocessed velocity $\boldsymbol{u}_{h}^{*} \in \boldsymbol{W}_{h}^{*}$ and the piecewise gradient $\left.\left(\nabla_{\mathcal{T}_{h}}\right)\right|_{T}:=\nabla\left(\left.\cdot\right|_{T}\right)$.
Proof. Let $\hat{\boldsymbol{u}}$ be the $L^{2}$ projection of $\boldsymbol{u}$ onto $\boldsymbol{W}_{h}^{*}$. The triangle inequality shows

$$
\begin{equation*}
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right\|_{0} \leq\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|_{0}+\left\|\boldsymbol{P}_{h}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0}+\left\|\left(\boldsymbol{\delta}-\boldsymbol{P}_{h}\right)\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0} . \tag{23}
\end{equation*}
$$

Since $\hat{\boldsymbol{u}}$ is the $L^{2}$-projection of $\boldsymbol{u}$ onto $\boldsymbol{W}_{h}^{*}$, the a priori estimate (9) shows for the first term on the right-hand side of (23)

$$
\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|_{0} \lesssim h^{m+1}\|\boldsymbol{u}\|_{m+1} \quad \text { for all } \boldsymbol{u} \in \boldsymbol{H}^{m+1}(\Omega)
$$

For the second term, notice that $\boldsymbol{P}_{T} \boldsymbol{u}_{h}^{*}=\boldsymbol{u}_{h}$ on each $T$ implies $\| \boldsymbol{P}_{T}(\hat{\boldsymbol{u}}-$ $\left.\boldsymbol{u}_{h}^{*}\right)\left\|_{0, T}=\right\| \boldsymbol{P}_{T} \hat{\boldsymbol{u}}-\boldsymbol{u}_{h} \|_{0, T}$. Since $\boldsymbol{V}_{h} \subset \boldsymbol{W}_{h}^{*}$, Theorem 3.4 shows that

$$
\begin{equation*}
\left\|\boldsymbol{P}_{h} \hat{\boldsymbol{u}}-\boldsymbol{u}_{h}\right\|_{0}=\left\|\boldsymbol{P}_{h} \boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0} \lesssim h^{k+3}\left(\|\boldsymbol{\sigma}\|_{k+2}+\|\operatorname{div} \boldsymbol{\sigma}\|_{k+1}\right) . \tag{24}
\end{equation*}
$$

In order to bound the third term on the right-hand side of (23), define $\boldsymbol{v} \in \boldsymbol{W}_{h}^{*}$ by $\left.\boldsymbol{v}\right|_{T}=\left(\boldsymbol{\delta}-\boldsymbol{P}_{T}\right)\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)$ for each $T \in \mathcal{T}_{h}$. Since $\left.\boldsymbol{v}\right|_{T} \perp P_{0}\left(T ; \mathbb{R}^{2}\right)$, the Poincaré inequality yields

$$
\|\boldsymbol{v}\|_{0, T} \leq \frac{h_{T}}{\pi}\|\nabla \boldsymbol{v}\|_{0, T}
$$

Since $\left(\boldsymbol{\delta}-\boldsymbol{P}_{T}\right) \boldsymbol{w}=0$ for all $\boldsymbol{w} \in P_{1}\left(T ; \mathbb{R}^{2}\right),\left.\boldsymbol{v}\right|_{T} \perp \mathcal{R} \mathcal{M}:=\{\boldsymbol{v}=$ $\left.\boldsymbol{c}+b\left(x_{2},-x_{1}\right)^{t}, \boldsymbol{c} \in \mathbb{R}^{2}, b \in \mathbb{R}\right\}$. Thus the second Korn inequality [BS94] leads to

$$
\|\boldsymbol{v}\|_{0, T} \lesssim h_{T}\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{0, T}
$$

Then (22) and the Cauchy inequality yield

$$
\begin{aligned}
\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{0, T}^{2}= & \left(\boldsymbol{\epsilon}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right), \boldsymbol{\epsilon}(\boldsymbol{v})\right)_{T}-\left(\boldsymbol{\epsilon}\left(\boldsymbol{P}_{T}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right), \boldsymbol{\epsilon}(\boldsymbol{v})\right)_{T} \\
= & (\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}-\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v}))_{T}+\left(\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right), \boldsymbol{\epsilon}(\boldsymbol{v})\right)_{T} \\
& -\left(\boldsymbol{\epsilon}\left(\boldsymbol{P}_{T}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right), \boldsymbol{\epsilon}(\boldsymbol{v})\right)_{T} \\
\leq & \|\boldsymbol{\epsilon}(\hat{\boldsymbol{u}}-\boldsymbol{u})\|_{0, T}\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{0, T}+\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0, T}\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{0, T} \\
& +\left\|\boldsymbol{\epsilon}\left(\boldsymbol{P}_{T}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right)\right\|_{0, T}\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{0, T} .
\end{aligned}
$$

Since $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0, T} \leq\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0, T} /(2 \nu)$, we obtain

$$
\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{0, T} \leq|\hat{\boldsymbol{u}}-\boldsymbol{u}|_{1, T}+\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0, T} /(2 \nu)+\left|\boldsymbol{P}_{T}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right|_{1, T} .
$$

This inequality and the inverse estimate

$$
\left|\boldsymbol{P}_{T}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right|_{1, T} \lesssim h_{T}^{-1}\left\|\boldsymbol{P}_{T}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0, T}
$$

yield

$$
\begin{aligned}
\left\|\left(\boldsymbol{\delta}-\boldsymbol{P}_{T}\right)\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0, T} \lesssim & h_{T}\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{0, T} \lesssim\left\|\boldsymbol{P}_{T}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0, T} \\
& +h_{T}|\hat{\boldsymbol{u}}-\boldsymbol{u}|_{1, T}+h_{T}\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0, T} .
\end{aligned}
$$

Let $I_{h}$ denote the nodal interpolant $\left.I_{h}\right|_{T}:\left.\boldsymbol{H}^{m}(T) \rightarrow \boldsymbol{W}_{h}^{*}\right|_{T}$ with the interpolation estimate [BS94]
(25) $\left|\boldsymbol{u}-I_{h} \boldsymbol{u}\right|_{\mu, T} \lesssim h_{T}^{m+1-\mu}\|\boldsymbol{u}\|_{m+1, T}$ for all $\boldsymbol{u} \in \boldsymbol{H}^{m+1}(\Omega), \mu=0,1$.

The triangle inequality and an inverse estimate show

$$
\begin{aligned}
|\hat{\boldsymbol{u}}-\boldsymbol{u}|_{1, T} & \leq\left|\hat{\boldsymbol{u}}-I_{h} \boldsymbol{u}\right|_{1, T}+\left|\boldsymbol{u}-I_{h} \boldsymbol{u}\right|_{1, T} \\
& \lesssim h_{T}^{-1}\left\|\hat{\boldsymbol{u}}-I_{h} \boldsymbol{u}\right\|_{0, T}+\left|\boldsymbol{u}-I_{h} \boldsymbol{u}\right|_{1, T} \\
& \leq h_{T}^{-1}\|\hat{\boldsymbol{u}}-\boldsymbol{u}\|_{0, T}+h_{T}^{-1}\left\|\boldsymbol{u}-I_{h} \boldsymbol{u}\right\|_{0, T}+\left|\boldsymbol{u}-I_{h} \boldsymbol{u}\right|_{1, T}
\end{aligned}
$$

The interpolation estimates (25) and the approximation property (9) yield

$$
|\hat{\boldsymbol{u}}-\boldsymbol{u}|_{1, T} \lesssim h_{T}^{m}\|\boldsymbol{u}\|_{m+1, T}
$$

After squaring and summing over all $T \in \mathcal{T}_{h}, \boldsymbol{P}_{T}=\left.\boldsymbol{P}_{h}\right|_{T}$,
$\left\|\left(\boldsymbol{\delta}-\boldsymbol{P}_{h}\right)\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0} \lesssim\left\|\boldsymbol{P}_{h}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0}+h^{m+1}\|\boldsymbol{u}\|_{m+1}+h\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0}$.

The estimate (24) and Lemma 3.2 lead to

$$
\left\|\left(\boldsymbol{\delta}-\boldsymbol{P}_{h}\right)\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0} \lesssim h^{k+3}\left(\|\boldsymbol{\sigma}\|_{k+2}+\|\operatorname{div} \boldsymbol{\sigma}\|_{k+1}\right)+h^{m+1}\|\boldsymbol{u}\|_{m+1}
$$

For the second assertion, the triangle inequality shows

$$
\left|\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right|_{1, T} \leq|\boldsymbol{u}-\hat{\boldsymbol{u}}|_{1, T}+\left|\left(\boldsymbol{\delta}-\boldsymbol{P}_{h}\right)\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right|_{1, T}+\left|\boldsymbol{P}_{h}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}_{h}^{*}\right)\right|_{1, T} .
$$

Hence, the result follows from an inverse inequality for the last two terms and the presented analysis.

## 5. A posteriori error control

This section concerns some a posteriori error estimation of the stress error. The analysis is based on the unified approach of [Car05]. Let $\boldsymbol{f}_{h}:=\boldsymbol{P}_{h} \boldsymbol{f}$ denote the piecewise $\boldsymbol{L}^{2}$ projection of $\boldsymbol{f}$ onto $\boldsymbol{V}_{h}$, i.e.,

$$
\int_{T}\left(\boldsymbol{f}-\boldsymbol{f}_{h}\right) \boldsymbol{v}_{h} d x=0 \quad \text { for all } \boldsymbol{v}_{h} \in \boldsymbol{V}_{h} .
$$

The oscillations of $\boldsymbol{f}$ are defined as

$$
\operatorname{osc}^{2}\left(\boldsymbol{f}, \mathcal{T}_{h}\right):=\sum_{T \in \mathcal{T}_{h}} h_{T}^{2}\left\|\boldsymbol{f}-\boldsymbol{f}_{h}\right\|_{0, T}^{2}
$$

Here and below, the notation $\tilde{\boldsymbol{u}}_{h} \in \boldsymbol{H}_{g}^{1}(\Omega):=\left\{\boldsymbol{v} \in \boldsymbol{H}^{1}(\Omega)|\boldsymbol{v}|_{\partial \Omega}=\boldsymbol{g}\right\}$ asserts that $\tilde{\boldsymbol{u}}_{h}$ is not necessarily a discrete function. The subscript of $\tilde{\boldsymbol{u}}_{h}$ indicates that it is closely related to the discontinuous approximation $\boldsymbol{u}_{h}$. Let $\boldsymbol{H}:=\boldsymbol{H}_{0}^{1}(\Omega)$ and

$$
\boldsymbol{L}:=\boldsymbol{L}^{2}(\Omega ; \mathbb{S}) / \mathbb{R}=\left\{\boldsymbol{\tau} \in \boldsymbol{L}^{2}(\Omega ; \mathbb{S}) \mid \int_{\Omega} \operatorname{tr} \boldsymbol{\tau} d x=0\right\}
$$

For given $\boldsymbol{f} \in \boldsymbol{L}^{2}(\Omega)$ the primal mixed formulation of the stressvelocity formulation of the Stokes problem (2) in its weak form seeks for $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{L} \times \boldsymbol{H}_{g}^{1}(\Omega)$ such that

$$
\begin{aligned}
&-(\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\boldsymbol{v}))=(\boldsymbol{f}, \boldsymbol{v}) \quad \text { for all } \boldsymbol{v} \in \boldsymbol{H}, \\
&(2 \nu \mathcal{A} \boldsymbol{\sigma}, \mathcal{A} \boldsymbol{\tau})-(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\boldsymbol{u}))=0 \quad \text { for all } \boldsymbol{\tau} \in \boldsymbol{L} .
\end{aligned}
$$

Then the following lemma from [Car05] holds.
Lemma 5.1. The operator $A: X \rightarrow X^{*}$, defined for $(\boldsymbol{\sigma}, \boldsymbol{u}) \in X:=$ $\boldsymbol{L} \times \boldsymbol{H}$ by

$$
(A(\boldsymbol{\sigma}, \boldsymbol{u}))(\boldsymbol{\tau}, \boldsymbol{v}):=(2 \nu \mathcal{A} \boldsymbol{\sigma}, \mathcal{A} \boldsymbol{\tau})-(\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\boldsymbol{v}))-(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\boldsymbol{u}))
$$

is linear, bounded and bijective.
One consequence of this lemma is that, for any approximation $\left(\boldsymbol{\sigma}_{h}, \tilde{\boldsymbol{u}}_{h}\right)$ to $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{L} \times \boldsymbol{H}_{g}^{1}(\Omega)$ such that $\left.\left(\boldsymbol{g}-\tilde{\boldsymbol{u}}_{h}\right)\right|_{\partial \Omega}=0$, it holds that

$$
\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}\left(\boldsymbol{u}-\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0} \approx\left\|\operatorname{Res}_{L}\right\|_{\boldsymbol{L}^{*}}+\left\|\operatorname{Res}_{H}\right\|_{\boldsymbol{H}^{*}}
$$

with the residuals defined by

$$
\begin{aligned}
\operatorname{Res}_{H}(\boldsymbol{v}) & :=(\boldsymbol{f}, \boldsymbol{v})+\left(\boldsymbol{\sigma}_{h}, \boldsymbol{\epsilon}(\boldsymbol{v})\right) \text { for all } \boldsymbol{v} \in \boldsymbol{H}, \\
\operatorname{Res}_{L}(\boldsymbol{\tau}) & :=\left(2 \nu \mathcal{A} \boldsymbol{\sigma}_{h}, \mathcal{A} \boldsymbol{\tau}\right)-\left(\boldsymbol{\tau}, \boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right) \text { for all } \boldsymbol{\tau} \in \boldsymbol{L} .
\end{aligned}
$$

The natural error $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}-\boldsymbol{u}_{h}\right)\right\|_{0}$ is unbalanced in the sense that the convergence rate of $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}$ is of order $k+2$, while that of $\left\|\boldsymbol{u}-\boldsymbol{u}_{h}\right\|_{0}$ is of order $k+1$. This motivates the error control of $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}\left(\boldsymbol{u}-\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}$.
Theorem 5.2. Let $(\boldsymbol{\sigma}, \boldsymbol{u}) \in L \times \boldsymbol{H}_{g}^{1}(\Omega)$ be a solution of (4)-(5). Then $\left(\boldsymbol{\sigma}_{h}, \tilde{\boldsymbol{u}}_{h}\right) \in L \times \boldsymbol{H}_{g}^{1}(\Omega)$ satisfies

$$
\begin{aligned}
& \left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}\left(\boldsymbol{u}-\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) \\
& \quad \approx \mu:=\left\|\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) .
\end{aligned}
$$

Proof. Since $\left.\boldsymbol{v}\right|_{\partial \Omega}=0$, Gauss Theorem yields

$$
\operatorname{Res}_{H}(\boldsymbol{v})=\int_{\Omega}\left(\boldsymbol{f}-\operatorname{div} \boldsymbol{\sigma}_{h}\right) \boldsymbol{v} d x=\int_{\Omega}\left(\boldsymbol{f}-\boldsymbol{f}_{h}\right) \boldsymbol{v} d x
$$

Let $\boldsymbol{v}_{h}$ denote the piecewise integral mean value of $\boldsymbol{v} \in \boldsymbol{H}$; then the Poincaré inequality leads to

$$
\operatorname{Res}_{H}(\boldsymbol{v})=\sum_{T \in \mathcal{T}} \int_{T}\left(\boldsymbol{f}-\boldsymbol{f}_{h}\right)\left(\boldsymbol{v}-\boldsymbol{v}_{h}\right) d x \lesssim \operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right)|\boldsymbol{v}|_{1}
$$

The second residual $\operatorname{Res}_{L}$ reads

$$
\operatorname{Res}_{L}(\boldsymbol{\tau})=\int_{\Omega}\left(\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right): \boldsymbol{\tau} d x \leq\left\|\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}\|\boldsymbol{\tau}\|_{0}
$$

Since $\mathcal{A} \boldsymbol{\sigma}=\boldsymbol{\epsilon}(\boldsymbol{u})$, a triangle inequality shows the efficiency

$$
\left\|\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0} \leq\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}\left(\boldsymbol{u}-\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0} .
$$

Remark 5.3. In the numerical experiments, $\tilde{\boldsymbol{u}}_{h}$ is approximated in some finite element space of order at least $k+2$ such that the boundary condition is not fulfilled. Suppose that $\boldsymbol{g}$ is sufficiently smooth, i.e., $\boldsymbol{g} \in \boldsymbol{C}(\partial \Omega)$ with $\left.\boldsymbol{g}\right|_{E} \in \boldsymbol{H}^{k+3}(E)$ for all $E \subset \partial \Omega$. Let $\boldsymbol{w} \in \boldsymbol{H}^{1}(\Omega)$ denote a harmonic extension of $\boldsymbol{g}-\boldsymbol{g}_{h}$ to the interior of $\Omega$ [BCD04] such that $\left.\boldsymbol{w}\right|_{\partial \Omega}=\boldsymbol{g}-\boldsymbol{g}_{h}$ with $\operatorname{supp}(\boldsymbol{w}) \subseteq\left\{T \in \mathcal{T}_{h}: T \cap \partial \Omega \neq \emptyset\right\}$ and the nodal interpolation $\left.\boldsymbol{g}_{h}\right|_{E} \in P_{k+2}\left(E ; \mathbb{R}^{2}\right)$ of $\boldsymbol{g}$, for all $E \subset \partial \Omega$. For $\tilde{\boldsymbol{u}}_{h} \in \boldsymbol{g}_{h}+\boldsymbol{H}_{0}^{1}(\Omega)$, Theorem 5.2 along with the triangle inequality shows

$$
\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) \lesssim\left\|\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}+|\boldsymbol{w}|_{1}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right)
$$

The proof of [BCD04, Theorem 4.2] reads

$$
\|\nabla \boldsymbol{w}\|_{0}^{2} \lesssim \sum_{E \in \mathcal{E}_{h}, E \subseteq \partial \Omega}\left(h_{E}^{-1}\left\|\boldsymbol{g}-\boldsymbol{g}_{h}\right\|_{L^{2}(E)}^{2}+h_{E}\left\|\partial\left(\boldsymbol{g}-\boldsymbol{g}_{h}\right) / \partial s\right\|_{L^{2}(E)}^{2}\right) .
$$

By interpolation it holds

$$
\begin{aligned}
\|\nabla \boldsymbol{w}\|_{0}^{2} & \lesssim h^{2(k+3)-1}\left\|\partial^{k+3} \boldsymbol{g} / \partial^{k+3} s\right\|_{L^{2}(\partial \Omega)}^{2}+h^{2(k+2)+1}\left\|\partial^{k+3} \boldsymbol{g} / \partial^{k+3} s\right\|_{L^{2}(\partial \Omega)}^{2} \\
& \lesssim h^{2 k+5}\left\|\partial^{k+3} \boldsymbol{g} / \partial^{k+3} s\right\|_{L^{2}(\partial \Omega)}^{2} .
\end{aligned}
$$

Therefore, this term is of higher order $\|\nabla \boldsymbol{w}\|_{0} \approx \mathcal{O}\left(h^{k+5 / 2}\right)$ compared to $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0} \approx \mathcal{O}\left(h^{k+2}\right)$.

The following part of this section is devoted to a second error estimator for which the continuity of $\tilde{\boldsymbol{u}}_{h}$ is not needed. Instead the estimator involves some (possibly) discontinuous function $\boldsymbol{u}_{h}^{*} \in \boldsymbol{H}^{1}\left(\mathcal{T}_{h}\right):=\{\boldsymbol{v} \in$ $\boldsymbol{L}^{2}(\Omega)|\boldsymbol{v}|_{T} \in \boldsymbol{H}^{1}(T)$ for all $\left.T \in \mathcal{T}_{h}\right\}$ from the postprocessing of Section 4. Let $\omega_{E}$ denote the edge patch $\omega_{E}:=\operatorname{int}\left(T_{+} \cup T_{-}\right), \nabla_{\mathcal{T}_{h}}$ the piecewise defined gradient and $\boldsymbol{\epsilon}_{\mathcal{T}_{h}}$ its piecewise symmetric part.
Theorem 5.4. Let $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{L} \times \boldsymbol{H}_{g}^{1}(\Omega)$ be a solution of (4)-(5). Then any $\boldsymbol{u}_{h}^{*} \in \boldsymbol{H}^{1}\left(\mathcal{T}_{h}\right)$ satisfies

$$
\begin{aligned}
& \left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) \\
& \quad \lesssim \eta:=\left\|\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}\right)\right\|_{0}+\left(\sum_{E \in \mathcal{E}_{h}} h_{E}^{-1}\left\|\left[\boldsymbol{u}_{h}^{*}\right]\right\|_{0, E}^{2}\right)^{1 / 2}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) .
\end{aligned}
$$

Proof. Let $\tilde{\boldsymbol{u}}_{h} \in \boldsymbol{H}_{g}^{1}(\Omega)$ denote the global minimizer of $\| \boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}\right)-$ $\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right) \|_{0}$. Theorem 5.2 and the triangle inequality yield, for any $\boldsymbol{u}_{h}^{*} \in$ $\boldsymbol{H}^{1}\left(\mathcal{T}_{h}\right)$, that

$$
\begin{aligned}
\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0} \\
\quad \leq\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}\left(\boldsymbol{u}-\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\tilde{\boldsymbol{u}}_{h}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0} \\
\left.\quad \lesssim\left\|\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}+\| \boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}\right)-\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right) \|_{0}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) \\
\left.\quad \lesssim\left\|\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}\right)\right\|_{0}+\| \boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}\right)-\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right) \|_{0}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) .
\end{aligned}
$$

For the set of nodes $\mathcal{N}_{h}$ let $\left(\varphi_{z}\right), z \in \mathcal{N}_{h}$, be a Lipschitz continuous partition of unity

$$
\sum_{z \in \mathcal{N}_{h}} \varphi_{z}=1 \quad \text { in } \Omega
$$

with $\varphi_{z} \in L^{2}(\Omega)$ and $\left.\varphi_{z}\right|_{T} \in P_{1}(T)$ for all $T \in \mathcal{T}_{h}$. For any edge $E \in \mathcal{E}_{h}$ let $\mathcal{N}(E)$ denote the set of all $z \in \mathcal{N}_{h}$ with $E \in \mathcal{E}_{h}(z):=\{E \in$ $\left.\mathcal{E}_{h}\left|\varphi_{z}\right|_{E} \not \equiv 0\right\}$. Theorem 3.1 of [CH07] reads

$$
\min _{\boldsymbol{v} \in \boldsymbol{H}_{g}^{1}(\Omega)}\left\|\nabla_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}-\boldsymbol{v}\right)\right\|_{0}^{2} \lesssim \sum_{E \in \mathcal{E}_{h}} \sum_{z \in \mathcal{N}(E)} h_{E}\left\|\partial\left[\varphi_{z} \boldsymbol{u}_{h}^{*}\right] / \partial s\right\|_{0, E}^{2}
$$

Thus $\left\|\epsilon_{\mathcal{T}_{h}}(\cdot)\right\|_{0} \lesssim\left\|\nabla_{\mathcal{T}_{h}}(\cdot)\right\|_{0}$ yields

$$
\begin{aligned}
\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}\right)-\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}^{2} & =\min _{\boldsymbol{v} \in \boldsymbol{H}_{g}^{1}(\Omega)}\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}-\boldsymbol{v}\right)\right\|_{0}^{2} \\
& \lesssim \sum_{E \in \mathcal{E}_{h}} \sum_{z \in \mathcal{N}(E)} h_{E}\left\|\partial\left[\varphi_{z} \boldsymbol{u}_{h}^{*}\right] / \partial s\right\|_{0, E}^{2}
\end{aligned}
$$

The local inverse inequality [BS94]

$$
h_{E}\left\|\partial\left[\varphi_{z} \boldsymbol{u}_{h}^{*}\right] / \partial s\right\|_{0, E}^{2} \lesssim h_{E}^{-1}\left\|\left[\varphi_{z} \boldsymbol{u}_{h}^{*}\right]\right\|_{0, E}^{2}
$$

leads to

$$
\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}-\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}^{2} \lesssim \sum_{E \in \mathcal{E}_{h}} \sum_{z \in \mathcal{N}(E)} h_{E}^{-1}\left\|\left[\varphi_{z} \boldsymbol{u}_{h}^{*}\right]\right\|_{0, E}^{2} .
$$

For the linear basis functions it holds that $0 \leq \varphi_{z} \leq 1$ and the cardinality of $|\mathcal{N}(E)|$ is bounded for all $E \in \mathcal{E}_{h}$. Thus, it follows that

$$
\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}-\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0}^{2} \lesssim \sum_{E \in \mathcal{E}_{h}} h_{E}^{-1}\left\|\left[\boldsymbol{u}_{h}^{*}\right]\right\|_{0, E}^{2} .
$$

Theorem 5.5. Let the boundary $\partial \Omega$ be sufficiently smooth and $(\boldsymbol{\sigma}, \boldsymbol{u}) \in$ $\boldsymbol{L} \times \boldsymbol{H}_{g}^{1}(\Omega) \cap \boldsymbol{H}^{k+3}(\Omega)$ be a solution of (4)-(5). Then the postprocessed velocity field $\boldsymbol{u}_{h}^{*} \in \boldsymbol{W}_{h}^{*}$ from Section 4 satisfies for quasi-uniform meshes

$$
\eta \lesssim\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0}+\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0}+h^{m}\|\boldsymbol{u}\|_{m+1}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) .
$$

Proof. The triangle inequality shows that

$$
\begin{aligned}
\left\|\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}\right)\right\|_{0} & \leq\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}+\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0} \\
& \lesssim\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0}+\left\|\boldsymbol{\epsilon}_{\mathcal{T}_{h}}\left(\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0} .
\end{aligned}
$$

Hence, it remains to bound the jump term. Since $\boldsymbol{u}$ is continuous,

$$
\left\|\left[\boldsymbol{u}_{h}^{*}\right]\right\|_{0, E}=\left\|\left[\boldsymbol{u}_{h}^{*}-\boldsymbol{u}\right]\right\|_{0, E}
$$

The trace inequality reads

$$
\left\|\left[\boldsymbol{u}_{h}^{*}-\boldsymbol{u}\right]\right\|_{0, E} \lesssim h_{E}^{-1 / 2}\left\|\boldsymbol{u}_{h}^{*}-\boldsymbol{u}\right\|_{0, \omega_{E}}+h_{E}^{1 / 2}\left\|\nabla_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}-\boldsymbol{u}\right)\right\|_{0, \omega_{E}}
$$

and leads to

$$
h_{E}^{-1}\left\|\left[\boldsymbol{u}_{h}^{*}\right]\right\|_{0, E}^{2} \lesssim h_{E}^{-2}\left\|\boldsymbol{u}_{h}^{*}-\boldsymbol{u}\right\|_{0, \omega_{E}}^{2}+\left\|\nabla_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}-\boldsymbol{u}\right)\right\|_{0, \omega_{E}}^{2} .
$$

The summation over all edges with finite overlap of the edge patches $\omega_{E}$ yields

$$
\sum_{E \in \mathcal{E}_{h}} h_{E}^{-1}\left\|\left[\boldsymbol{u}_{h}^{*}\right]\right\|_{0, E}^{2} \lesssim h_{\min }^{-2}\left\|\boldsymbol{u}_{h}^{*}-\boldsymbol{u}\right\|_{0, \omega_{E}}^{2}+\left\|\nabla_{\mathcal{T}_{h}}\left(\boldsymbol{u}_{h}^{*}-\boldsymbol{u}\right)\right\|_{0}^{2}
$$

with $h_{\text {min }}:=\min _{E \in \mathcal{E}_{h}} h_{E}$. The proofs of Theorems 3.4 and 4.1 show

$$
\begin{aligned}
\left\|\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right\|_{0} & \lesssim h^{m+1}\|\boldsymbol{u}\|_{m+1}+h\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0}+h \operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right), \\
\left\|\nabla_{\mathcal{T}_{h}}\left(\boldsymbol{u}-\boldsymbol{u}_{h}^{*}\right)\right\|_{0} & \lesssim h^{m}\|\boldsymbol{u}\|_{m+1}+\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|_{0}+\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right) .
\end{aligned}
$$

The quasi-uniformity of the meshes $h / h_{\min } \lesssim 1$ ends the proof.
Remark 5.6. In most practical examples, $\boldsymbol{u}$ is not as smooth as $\boldsymbol{H}^{m+1}(\Omega)$, and adaptive mesh refinement is needed. However, Theorem 5.5 gives at least a hint that $\eta$ might be not only reliable but also efficient for more general problems and meshes. This is indeed confirmed by the numerical experiments of Section 7.

## 6. The Adaptive Finite Element Method

The adaptive finite element algorithm computes a sequence of discrete subspaces

$$
\left(\boldsymbol{\Phi}_{h_{0}}, \boldsymbol{V}_{h_{0}}\right) \subsetneq\left(\boldsymbol{\Phi}_{h_{1}}, \boldsymbol{V}_{h_{1}}\right) \subsetneq \ldots \subsetneq\left(\boldsymbol{\Phi}_{h_{\ell-1}}, \boldsymbol{V}_{h_{\ell-1}}\right) \subsetneq\left(\boldsymbol{\Phi}_{h_{\ell}}, \boldsymbol{V}_{h_{\ell}}\right) \subset(\boldsymbol{\Phi}, \boldsymbol{V})
$$

throughout successive local refinement of the domain $\Omega$. The corresponding sequence of meshes $\left(\mathcal{T}_{h_{\ell}}\right)_{\ell}$ consists of nested regular triangulations. The AFEM consists of the following loop:

$$
\text { Solve } \rightarrow \text { Estimate } \rightarrow \text { Mark } \rightarrow \text { Refine. }
$$

Solve. Given a mesh $\mathcal{T}_{h}$ the step Solve calculates the solution of the finite-dimensional saddle point problem

$$
\left(\begin{array}{ccc}
A & B^{t} & c^{t} \\
B & & \\
c & &
\end{array}\right)\left(\begin{array}{c}
x \\
y \\
\lambda
\end{array}\right)=\left(\begin{array}{c}
b_{g} \\
b_{f} \\
0
\end{array}\right) .
$$

It is assumed throughout the paper that the discrete equations are solved exactly. The system matrices $A$ and $B$ and the right-hand sides $b_{g}$ and $b_{g}$ are computed for the bases $\operatorname{span}\left\{\boldsymbol{\tau}_{j}\right\}=\boldsymbol{\Phi}_{h}$ and $\operatorname{span}\left\{\boldsymbol{v}_{j}\right\}=$ $V_{h}$ by

$$
\begin{aligned}
A_{j k} & :=\int_{\Omega} \mathcal{A} \boldsymbol{\tau}_{j}: \boldsymbol{\tau}_{k} d x \quad \text { and } \quad B_{j k}:=\int_{\Omega} \boldsymbol{v}_{j} \cdot \operatorname{div} \boldsymbol{\tau}_{k} d x \\
b_{g, j} & :=\int_{\partial \Omega} \boldsymbol{g} \cdot\left(\boldsymbol{\tau}_{j} \boldsymbol{n}_{E}\right) d x \quad \text { and } \quad b_{f, j}:=\int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}_{j} d x
\end{aligned}
$$

The discrete solutions for the stress $\boldsymbol{\sigma}_{h}$ and the velocity $\boldsymbol{u}_{h}$ are given by

$$
\boldsymbol{\sigma}_{h}=\sum_{k=1}^{\operatorname{dim}\left(\boldsymbol{\Phi}_{h}\right)} x_{k} \boldsymbol{\tau}_{k} \quad \text { and } \quad \boldsymbol{u}_{h}=\sum_{k=1}^{\operatorname{dim}\left(\boldsymbol{V}_{h}\right)} y_{k} \boldsymbol{v}_{k}
$$

The condition $\int_{\Omega} \operatorname{tr} \boldsymbol{\sigma}_{h} d x=0$ is incorporated into the system using the Lagrange multiplier $\lambda$ and

$$
c_{j}:=\int_{\Omega} \operatorname{tr} \boldsymbol{\tau}_{j} d x
$$

For more details see [CGRT08].
Estimate. The error $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}$ is estimated based on the discrete solution ( $\boldsymbol{\sigma}_{h}, \boldsymbol{u}_{h}$ ) of the underlying saddle point problem

$$
\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0} \lesssim \mu_{h} \quad \text { and } \quad\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0} \lesssim \eta_{h}
$$

with

$$
\begin{aligned}
\mu_{h}^{2}= & \sum_{T \in \mathcal{T}_{h}} h_{T}^{2}\left\|\boldsymbol{f}-\boldsymbol{f}_{h}\right\|_{0, T}^{2}+\sum_{T \in \mathcal{T}_{h}}\left\|\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}\left(\tilde{\boldsymbol{u}}_{h}\right)\right\|_{0, T}^{2} \\
& \quad+\sum_{E \in \mathcal{E}_{h}, E \subseteq \partial \Omega} h_{E}\left\|\partial\left(\boldsymbol{g}-\boldsymbol{g}_{h}\right) / \partial s\right\|_{L^{2}(E)}^{2}, \\
\eta_{h}{ }^{2}= & \sum_{T \in \mathcal{T}_{h}} h_{T}^{2}\left\|\boldsymbol{f}-\boldsymbol{f}_{h}\right\|_{0, T}^{2}+\sum_{T \in \mathcal{T}_{h}}\left\|\mathcal{A} \boldsymbol{\sigma}_{h}-\boldsymbol{\epsilon}\left(\boldsymbol{u}_{h}^{*}\right)\right\|_{0, T}^{2}+\sum_{E \in \mathcal{E}_{h}} h_{E}^{-1}\left\|\left[\boldsymbol{u}_{h}^{*}\right]\right\|_{0, E}^{2} .
\end{aligned}
$$

Here $\boldsymbol{u}_{h}^{*} \in \boldsymbol{W}_{h}^{*}$ is the solution of the local postprocessing of Section 4. To obtain a conforming approximation $\tilde{\boldsymbol{u}}_{h} \in \boldsymbol{g}_{h}+\boldsymbol{H}_{0}^{1}(\Omega)$ the (possibly) discontinuous function $\boldsymbol{u}_{h}^{*}$ is smoothen by taking the arithmetic mean value

$$
\tilde{\boldsymbol{u}}_{h}(z):=\left.\frac{1}{\left|\left\{T \in \mathcal{T}_{h}: z \in T\right\}\right|} \sum_{T \in \mathcal{T}_{h}: z \in T} \boldsymbol{u}_{h}^{*}(z)\right|_{T},
$$

for each vertex and edge degree of freedom in $z \in \mathbb{R}^{2}$. The degrees of freedom on the boundary in points $z \in \mathbb{R}^{2}$ are interpolated $\tilde{\boldsymbol{u}}_{h}(z)=$ $\boldsymbol{g}(z)$.

Remark 6.1. In the academic case $\boldsymbol{f} \not \equiv \boldsymbol{f}_{h}$ the oscillations might dominate the other terms in the a posteriori error estimators and therefore lead to a high overestimation of the error $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}$. For the realistic benchmark problems of Section 7 the oscillation vanish, and Theorem 5.2 shows for $\tilde{\boldsymbol{u}}_{h} \equiv \boldsymbol{u}$ that the error estimate is sharp. Because of that and $\tilde{\boldsymbol{u}}_{h}$ being a higher order approximation of $\boldsymbol{u}$, the efficiency indices $\mu_{h} /\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}$ are expected to be close to one.

Mark. Based on the refinement indicators, edges and elements are marked for refinement in a bulk criterion such that $\mathcal{M}_{h} \subseteq \mathcal{T}_{h} \cup \mathcal{E}_{h}$ is an (almost) minimal set of marked edges with

$$
\theta \eta_{h}^{2} \leq \eta_{h}^{2}\left(\mathcal{M}_{h}\right), \quad \eta_{h}^{2}\left(\mathcal{M}_{h}\right):=\sum_{T \in \mathcal{M}_{h} \cap \mathcal{T}_{h}} \eta_{h}^{2}(T)+\sum_{E \in \mathcal{M}_{h} \cap \mathcal{E}_{h}} \eta_{h}^{2}(E)
$$

for a bulk parameter $0<\theta \leq 1$. This is done in a greedy algorithm which marks edges and elements with larger contributions.

Refine. In this step of the AFEM loop, the mesh is refined locally corresponding to the set $\mathcal{M}_{h}$ of marked edges and elements. Once an element is selected for refinement, all of its edges will be refined. In order to preserve the quality of the mesh, i.e., the maximal angle condition or its equivalents, additionally edges have to be marked by the closure algorithm before refinement. For each triangle let one edge be the uniquely defined reference edge $E(T)$. The closure algorithm computes a superset $\overline{\mathcal{M}}_{h} \supset \mathcal{M}_{h}$ such that

$$
\left\{E(T): T \in \mathcal{T}_{h} \quad \text { with } \quad \mathcal{E}(T) \cap \overline{\mathcal{M}}_{h} \neq \emptyset \quad \text { or } \quad T \cap \overline{\mathcal{M}}_{h} \neq \emptyset\right\} \subseteq \overline{\mathcal{M}}_{h} .
$$



Figure 1. Refinement rules: Subtriangles with corresponding reference edges depicted with a second edge.

In other words, once an edge of a triangle or itself is marked for refinement, its reference edge $E(T)$ is among them. After the closure algorithm is applied, one of the following refinement rules is applicable: no refinement, red refinement, green refinement, blue left refinement, or blue right refinement (see Figure 1).

## 7. Numerical Experiments

This section is devoted to three numerical experiments with known exact solution and two well-known benchmark problems. The experiments compare the convergence behavior for uniform and adaptive meshes using the error estimators of Section 6 for the stress error. The experiments restrict themselves to the lowest order Arnold-Winther finite element $(k=1)$ [CGRT08] and the parameter $\nu=1$. Since the experiments show some superconvergence phenomenon of fourth order for the stress error if the right hand side $\boldsymbol{f} \equiv 0$, the polynomial order for the postprocessing is accordingly increased by one to $\boldsymbol{u}_{h}^{*} \in P_{4}\left(\mathcal{T}_{h} ; \mathbb{R}^{2}\right)$. In the absence of superconvergence phenomena this is only a minor computational overhead compared to the postprocessing in $P_{3}\left(\mathcal{T}_{h} ; \mathbb{R}^{2}\right)$.

Smooth example. The first example concerns the model problem (2) in $\Omega=(0,1) \times(0,1)$, with source $\boldsymbol{f}=\left(4 \pi^{2} \sin (\pi(x-y)), 0\right)^{t}$ and the Dirichlet data $\boldsymbol{g}$ chosen in such a way that the smooth solution reads $\boldsymbol{u}=(\sin (\pi x) \cos (\pi y)-\cos (\pi x) \sin (\pi y), \sin (\pi x) \cos (\pi y)-\cos (\pi x) \sin (\pi y))^{t}$, $\boldsymbol{\sigma}=2 \boldsymbol{\epsilon}(\boldsymbol{u})-p \boldsymbol{\delta}$, with $p=-2 \pi(\cos (\pi x) \cos (\pi y)+\sin (\pi x) \sin (\pi y))+8 / \pi$.
The computed solution is displayed in Figure 2 as streamline plot for a uniform mesh, and the discrete pressure $p_{h}:=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$ is visualized on an adaptive mesh. The convergence history in Figure 3 shows empirically optimal convergence rates of $\mathcal{O}\left(N^{-3 / 2}\right), N:=\operatorname{dim}\left(\boldsymbol{\Phi}_{h}\right)+\operatorname{dim}\left(\boldsymbol{V}_{h}\right)$, for both uniform and adaptive meshes. Note that for uniform meshes


Figure 2. Streamline plot of the discrete solution on a uniform mesh (left) and discrete pressure $p_{h}=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$ on an adaptive mesh with 409 nodes (right) for the smooth example.


Figure 3. Convergence history of $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}, \| \boldsymbol{u}-$ $\boldsymbol{u}_{h}\left\|_{0},\right\| \boldsymbol{u}-\boldsymbol{u}_{h}^{*} \|_{0}, \mu_{h}$ and $\eta_{h}$ on adaptively and uniformly refined meshes for the smooth example.
it holds that $\mathcal{O}\left(N^{-3 / 2}\right) \approx \mathcal{O}\left(h^{3}\right)$. Due to the $H^{2}$-regularity of the solution, the numerical results for uniform and adaptive refinements do not differ much. Both error estimators are empirically reliable and efficient. In this academically smooth example the efficiency index is not close to one since the oscillations $\operatorname{osc}\left(\boldsymbol{f}, \mathcal{T}_{h}\right)$ dominate both error estimators. Since the solution is smooth, the postprocessed velocity $\boldsymbol{u}_{h}^{*}$ is of higher order $\mathcal{O}\left(N^{-2}\right)$, which confirms the theoretical result.

Colliding flow. As second example consider the model problem (2) in $\Omega=(-1,1) \times(-1,1)$ with source $\boldsymbol{f} \equiv 0$ and the Dirichlet condition


Figure 4. Streamline plot of the discrete solution on a uniform mesh (left) and discrete pressure $p_{h}=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$ on an adaptive mesh with 429 nodes (right) for the colliding flow example.


Figure 5. Convergence history of $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}, \| \boldsymbol{u}-$ $\boldsymbol{u}_{h}\left\|_{0},\right\| \boldsymbol{u}-\boldsymbol{u}_{h}^{*} \|_{0}, \mu_{h}$ and $\eta_{h}$ on adaptively and uniformly refined meshes for the colliding flow example.
$\boldsymbol{g}$ chosen in such a way that

$$
\boldsymbol{u}=\left(20 x y^{4}-4 x^{5}, 20 x^{4} y-4 y^{5}\right)^{t} \quad \text { and } \quad \boldsymbol{\sigma}=2 \boldsymbol{\epsilon}(\boldsymbol{u})-p \boldsymbol{\delta}
$$

with $p=120 x^{2} y^{2}-20 x^{4}-20 y^{4}-32 / 6$ is the solution. Figure 4 shows the streamlines of the approximated velocity field on a uniform refined mesh and the discrete pressure $p_{h}=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$ on an adaptive refined mesh. Both error estimators are numerically reliable and efficient as shown in Figure 5. Since there are no oscillations in this example, the efficiency indices of both estimators are much closer to one. It is remarkable that the order of convergence approaches experimentally


Figure 6. Streamline plot of the discrete solution on a uniform mesh (left) and discrete pressure $p_{h}=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$ on an adaptive mesh with 223 nodes (right) for the slit example.


Figure 7. Convergence history of $\left\|\mathcal{A}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right)\right\|_{0}, \| \boldsymbol{u}-$ $\boldsymbol{u}_{h}\left\|_{0},\right\| \boldsymbol{u}-\boldsymbol{u}_{h}^{*} \|_{0}, \mu_{h}$ and $\eta_{h}$ on adaptively and uniformly refined meshes for the slit example.
$\mathcal{O}\left(N^{-2}\right)$. This superconvergence was previously observed in [CGRT08] where it is conjectured that this effect takes place due to $f \equiv 0$. Consequently, the postprocessed velocity $\boldsymbol{u}_{h}^{*} \in P_{4}\left(\mathcal{T}_{h} ; \mathbb{R}^{2}\right)$ shows also an increased empirical convergence rate of $\mathcal{O}\left(N^{-5 / 2}\right)$. The fact that the adaptive mesh refinement algorithm destroys the symmetry of the mesh might be a reason that the error for adaptively refined meshes is larger than that for uniform meshes.

Slit example. As an example with a nonconvex domain, consider the model problem (2) in $\Omega=(-1,1)^{2} \backslash[0,1) \times\{0\}$. The right hand side


Figure 8. Adaptive meshes obtained using $\mu_{h}$ (left) with 672 nodes and $\eta_{h}$ (right) with 761 nodes for the slit example.
$\boldsymbol{f}$ and the Dirichlet data $\boldsymbol{g}$ are chosen in such a way that the exact solution in polar coordinates reads

$$
\begin{aligned}
\boldsymbol{u} & =\frac{3 \sqrt{r}}{2}\left(\cos \left(\frac{\theta}{2}\right)-\cos \left(\frac{3 \theta}{2}\right), 3 \sin \left(\frac{\theta}{2}\right)-\sin \left(\frac{3 \theta}{2}\right)\right)^{t}, \\
\boldsymbol{\sigma} & =2 \boldsymbol{\epsilon}(\boldsymbol{u})-p \boldsymbol{\delta} \quad \text { with } \quad p=-\frac{6}{\sqrt{r}} \cos \left(\frac{\theta}{2}\right) .
\end{aligned}
$$

Due to the re-entrant corner at the origin of the domain, this example allows a singular solution. A discrete approximation of the velocity and the pressure $p_{h}=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$ on uniformly and adaptively refined meshes is shown in Figure 6. The convergence history in Figure 7 shows poor convergence for the error in the case of uniform meshes. On the other hand, adaptive refinement results in optimal convergence of the error and in reliable and efficient a posteriori error control, which underlines the importance of adaptivity. It can be observed that $\eta_{h}$ is underestimating the error and that the efficiency index is much smaller than that from $\mu_{h}$, which is close to one. Additionally the error for the adaptive meshes generated with $\mu_{h}$ is significant smaller than that generated with $\eta_{h}$. Figure 8 shows pictures of adaptively refined meshes for $\mu_{h}$ and $\eta_{h}$, which show strong refinement towards the singularity at the origin. The postprocessed velocity $\boldsymbol{u}_{h}^{*}$ shows empirical superconvergence with convergence rates of $\mathcal{O}\left(N^{-2}\right)$ for adaptive meshes for both estimators.

Backward facing step. This example is a well-known benchmark problem for flow problems in the domain $\Omega$ of Figure 9. Consider the model problem (2) with $\boldsymbol{f} \equiv 0, \boldsymbol{g}(x, y)=(0,0)^{t}$ for $-2<x<8$, $\boldsymbol{g}(x, y)=(-y(y-1) / 10,0)^{t}$ for $x=-2$ and $\boldsymbol{g}(x, y)=(-(y+1)(y-$ 1) $/ 80,0)^{t}$ for $x=8$. The numerical solution of the velocity field and the pressure $p_{h}=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$ on uniformly refined meshes are shown in Figure 9. Note that the pressure is high on the left and low on


Figure 9. Approximated velocity field (top) and discrete pressure $p_{h}=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$ (bottom) for the backward facing step example on uniform meshes.


Figure 10. Adaptive meshes obtained using $\mu_{h}$ (top) with 760 nodes and $\eta_{h}$ (bottom) with 791 nodes for the backward facing step example.


Figure 11. Approximated velocity field near the bottom corner for the backward facing step example on uniform meshes.


Figure 12. Approximated velocity field (left) and discrete pressure $p_{h}=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$ (right) for the driven cavity example on uniform meshes.


Figure 13. Approximated velocity field near the left and right lower corner for the driven cavity example on uniform meshes.
the right. Figure 10 shows two adaptively refined meshes for both error estimators which look quite similar and show strong refinement towards the singularity at the origin. A zoom of the lower left corner in Figure 11 shows not only one eddy, but two streamlines at the left corner indicate a second one. This indicates a high stability of the numerical scheme.

Lid-driven cavity flow. As last example consider the lid-driven cavity flow benchmark problem. Consider the model problem (2) in $\Omega=$ $(-1,1) \times(1,1)$ with $\boldsymbol{f} \equiv 0, \boldsymbol{g}(x, y)=(0,0)^{t}$ for $y<1$ and $\boldsymbol{g}(x, y)=$ $(1,0)^{t}$ for $y=1$. Figure 12 displays the numerical solution of the velocity on a uniform mesh with two Moffat eddies at the bottom corners and an approximation of the pressure $p_{h}=-\operatorname{tr} \boldsymbol{\sigma}_{h} / 2$ on an adaptive mesh. The absolute largest values for the pressure occur in the top corners; in the other areas there seems to be almost no pressure. Figure 13 shows a zoom towards the Moffat eddies in the left and right lower corners.


Figure 14. Adaptive meshes obtained using $\mu_{h}$ (left) with 422 nodes and $\eta_{h}$ (right) with 355 nodes for the driven cavity example.

Near the bottom corners one or two lines indicate a more detailed resolution of the Moffat eddies, which again illustrates the high stability of the numerical scheme. Both adaptively refined meshes show strong refinement towards the two left and right top corners in Figure 14. It seems that the area away from the corners is refined only to prevent hanging nodes and not due to a high refinement indicator.

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