AN ADAPTIVE FINITE ELEMENT EIGENVALUE SOLVER OF ASYMPTOTIC QUASI-OPTIMAL COMPUTATIONAL COMPLEXITY*

CARSTEN CARSTENSEN AND JOSCHA GEDICKE

Abstract. This paper presents a combined adaptive finite element method with an iterative algebraic eigenvalue solver for a symmetric eigenvalue problem of asymptotic quasi-optimal computational complexity. The analysis is based on a direct approach for eigenvalue problems and allows the use of higher-order conforming finite element spaces with fixed polynomial degree. The asymptotic quasi-optimal adaptive finite element eigenvalue solver (AFEMES) involves a proper termination criterion for the algebraic eigenvalue solver and does not need any coarsening. Numerical evidence illustrates the asymptotic quasi-optimal computational complexity in 2 and 3 dimensions.

1. Introduction

The eigenvalue problems for symmetric second-order elliptic boundary value problems can be discretised with some adaptive finite element method (AFEM). In practice, the resulting finite-dimensional generalised eigenvalue problems are solved iteratively. Thus, the computation involves the discretisation error of some AFEM as well as the error left from the termination of some iterative algebraic eigenvalue solver. This paper presents the first adaptive finite element eigenvalue solver (AFEMES) of overall asymptotic quasi-optimal complexity, i.e., for sufficiently small mesh-sizes the error is optimal up to a generic multiplicative constant. AFEMES is shown in the pseudocode below.

The algorithm computes one fixed simple eigenvalue. The adaptive mesh refinement via subroutines `Mark` and `Refine` is well established in the finite element community [BDD04, CKNS08, Dör96, Ste07] while `LAES` represents any state-of-the-art iterative eigenvalue solver well est-

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AFEMES
Input: Coarse triangulation T_0, initial guess (λ̃_0, ũ_0),
       parameters 0 < θ ≤ 1, 0 < ω.
δ_0 := 2√ω η_0(λ̃_0, ũ_0);
for ℓ = 0,1,... (until termination)
  while ( δ_ℓ > √ω η_ℓ(λ̃_ℓ, ũ_ℓ) )
    δ_ℓ := δ_ℓ/2; [λ̃_ℓ, ũ_ℓ] := LAES(T_ℓ, λ̃_ℓ, ũ_ℓ, δ_ℓ); end
  T_{ℓ+1} := Refine(T_ℓ, Mark(T_ℓ, θ, η_ℓ(λ̃_ℓ, ũ_ℓ)));
  δ_{ℓ+1} := 2√ω η_ℓ(λ̃_ℓ, ũ_ℓ); λ̃_{ℓ+1} := λ̃_ℓ; ũ_{ℓ+1} := P_ℓ^{ℓ+1} ũ_ℓ; end
Output: Sequence of triangulations T_ℓ and
        approximations (λ̃_ℓ, ũ_ℓ).
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ablished in the numerical linear algebra community that satisfies the convergence and complexity assumptions of Section 2. The parameters θ and ω depend on the regularity of the solution and η_ℓ denotes the error estimator from Section 4. The prolongation operator from triangulation $\mathcal{T}_\ell$ onto $\mathcal{T}_{\ell+1}$ is denoted by $P_\ell^{\ell+1}$. The pseudocode gives one possible error balance of the two error sources of asymptotic quasi-optimal complexity.

The works on asymptotic convergence [CG11, GMZ09, GG09, Sau10] as well as on asymptotic quasi-optimal convergence [DXZ08, GM11] of adaptive mesh refinement for the eigenvalue problem do assume unrealistically the exact knowledge of algebraic eigenpairs. Another optimality result for linear symmetric operator eigenvalue problems [DRSZ08] is based on coarsening. Assuming a saturation assumption, [MM11, Ney02] present combined adaptive finite element and linear algebra algorithms.

As a simple model problem for a symmetric, elliptic eigenvalue problem consider the following eigenvalue problem of the Laplace operator: Seek a nontrivial eigenpair $(\lambda, u) \in \mathbb{R} \times H^1_0(\Omega;\mathbb{R}) \cap H^2_{loc}(\Omega;\mathbb{R})$ such that

$$-\Delta u = \lambda u \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. It is well known, that problem (1.1) has countable infinite many solutions with positive eigenvalues that can be ordered increasingly. For simplicity, this paper is restricted to the case that the eigenvalue of interest λ is a simple eigenvalue; hence its algebraic and geometric multiplicity equals one. Throughout this paper, standard notations on Sobolev and Lebesgue spaces are used.

The weak problem seeks for a nontrivial eigenpair $(\lambda, u) \in \mathbb{R} \times V := \mathbb{R} \times H^1_0(\Omega;\mathbb{R})$ with $b(u,u) = 1$ and

$$a(u,v) = \lambda b(u,v) \quad \text{for all } v \in V.$$

The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \text{and} \qquad b(u,v) := \int_\Omega uv \, dx$$

and induce the norms $|\!|\!|.|\!|\!| := |.|_{H^1(\Omega)}$ on V and $\|.\| := \|.\|_{L^2(\Omega)}$ on $L^2(\Omega)$.

The conforming finite element space of order $k \in \mathbb{N}$ for the triangulation $\mathcal{T}_\ell$ is defined by

$$\mathcal{P}_k(\mathcal{T}_\ell) := \left\{ v \in H^1(\Omega) : \forall T \in \mathcal{T}_\ell,\ v|_T \text{ is polynomial of degree} \leq k \right\}.$$

Let $V_\ell := \mathcal{P}_k(\mathcal{T}_\ell) \cap V$ denote the finite-dimensional subspace of fixed order $k > 0$ and $N_\ell := \dim(V_\ell)$. The corresponding discrete eigenvalue problem reads: Seek a nontrivial eigenpair $(\lambda_\ell, u_\ell) \in \mathbb{R} \times V_\ell$ with $b(u_\ell, u_\ell) = 1$ and

$$a(u_\ell, v_\ell) = \lambda_\ell b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell.$$

This paper proves asymptotic quasi-optimal computational complexity of the proposed AFEMES: Suppose that (λ_ℓ, u_ℓ) is a discrete eigenpair to the continuous eigenpair (λ, u). Let $(\mathcal{T}_\ell)_\ell$ be a sequence of nested regular triangulations. Suppose that the continuous eigenpair (λ, u) belongs to some approximation class $\mathcal{A}_s$, i.e., there exists some $s > 0$ and some $|u|_{\mathcal{A}_s} < \infty$ such that, for any number N there is an (unknown) optimal mesh $\mathcal{T}_N$ with $|\mathcal{T}_N| \leq |\mathcal{T}_0| + N$ element domains and discrete eigenpair (λ_N, u_N) with

$$\sup_{N \in \mathbb{N}} N^{2s} \left(|\!|\!|u - u_N|\!|\!|^2 + |\lambda - \lambda_N| \right) =: |u|^2_{\mathcal{A}_s} < \infty.$$

Then the computational complexity of the AFEMES is quasi-optimal in the sense that

$$|\!|\!|u - \tilde{u}_\ell|\!|\!|^2 + |\lambda - \tilde{\lambda}_\ell| \leq \mathcal{O}(t_\ell^{-2s}),$$

where t_ℓ denotes the computational costs, i.e., the CPU time. The point is that this quasi-optimal complexity holds for any $u \in \mathcal{A}_s$ and all $s > 0$ despite the fact that AFEMES does not require any parameter s. The analysis consists of three steps and does not need any inner node property, coarsening or saturation assumption. Since in the present analysis no oscillations occur, it is not necessary to add additional inner points to reduce some oscillations [GG09]. In [DRSZ08] a coarsening of the mesh is needed in some steps to maintain optimality. The present analysis relies only on refinement of some mesh and does not need any coarsening. For hierarchical error estimators [MM11, Ney02] reliability is equivalent to the saturation assumption, namely a strict error reduction for uniform refined meshes. For the residual estimator used here the reliability is proven directly in Section 4. First the asymptotic quasi-optimal convergence for the model problem (1.1) is shown for discrete eigenpairs without using the inner node property: Suppose that (λ_ℓ, u_ℓ) is a discrete eigenpair to the continuous eigenpair (λ, u) in some approximation class $\mathcal{A}_s$ for some

$s > 0$. Then (λ_ℓ, u_ℓ) converges quasi-optimal, i.e., optimal up to a positive generic multiplicative constant C with

$$|||u - u_\ell|||^2 + |\lambda - \lambda_\ell| \leq C|u|^2_{\mathcal{A}_s} N_\ell^{-2s}.$$

In contrast to [DXZ08] the proofs are based on the eigenvalue formulation and not on a relation to its corresponding source problem. Hence, no additional oscillations arising from the corresponding source problem have to be treated. In a second step this result is extended to the case of inexact algebraic eigenvalue solutions: Suppose (λ, u) with $u \in \mathcal{A}_s$ is an eigenpair and (λ_ℓ, u_ℓ) and $(\lambda_{\ell+1}, u_{\ell+1})$ corresponding discrete eigenpairs on levels ℓ and $\ell + 1$. Let the iterative approximations $(\tilde\lambda_\ell, \tilde u_\ell)$ on $\mathcal{T}_\ell$ and $(\tilde\lambda_{\ell+1}, \tilde u_{\ell+1})$ on $\mathcal{T}_{\ell+1}$ satisfy

$$\begin{aligned} |||u_{\ell+1} - \tilde u_{\ell+1}|||^2 + |\lambda_{\ell+1} - \tilde\lambda_{\ell+1}| &\leq \omega \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell), \\ |||u_\ell - \tilde u_\ell|||^2 + |\lambda_\ell - \tilde\lambda_\ell| &\leq \omega \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell), \end{aligned}$$

for sufficiently small $\omega > 0$. Then, the iterative solutions $\tilde\lambda_\ell$ and $\tilde u_\ell$ converge quasi-optimal,

$$|||u - \tilde u_\ell|||^2 + |\lambda - \tilde\lambda_\ell| \lesssim N_\ell^{-2s}.$$

The notation $x \lesssim y$ abbreviates the inequality $x \leq Cy$ and $x \approx y$ the inequalities $Dy \leq x \leq Cy$ with constants $C > 0$ and $D > 0$ which do not depend on the mesh-size. Finally, it is shown that the AFEMES is of linear runtime $t_\ell \approx N_\ell$ provided the linear algebra eigenvalue solver satisfies some convergence and complexity assumptions of Section 2.

The outline of this paper is as follows. Section 2 concerns the basic structure of the standard AFEM for eigenvalue problems. Section 3 presents some algebraic and analytic properties for the model problem (1.1). The discrete reliability of a residual type error estimator is shown in Section 4 together with the standard reliability and efficiency. In Section 5 a contraction property for the quasi-error up to higher-order terms leads to quasi-optimal convergence of the AFEM under the usual assumption that the mesh-size is sufficiently small and that the algebraic subproblems are solved exactly. Relaxing this last assumption in Section 6, the results for quasi-optimal convergence are extended to the case of approximated discrete eigenpairs. These relaxed results are in Section 7 combined with some iterative eigenvalue solver and thus lead to the combined AFEM and iterative algebraic eigenvalue solver AFEMES with asymptotic quasi-optimal computational complexity. The numerical experiments of Section 8 show empirical quasi-optimal computational complexity of the AFEMES for some iterative algebraic eigenvalue solvers and higher-order finite element methods in 2 and 3 dimensions.

2. Adaptive Finite Element Eigenvalue Solver

The AFEM computes a sequence of discrete subspaces

$$V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_\ell \subset V$$

using local refinement of the underlying mesh of the domain Ω. The corresponding sequence of meshes $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ consists of nested regular triangulations. The AFEM consists of the following loop:

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}.$$

Solve. Given a mesh $\mathcal{T}_\ell$ on level ℓ the step Solve computes the stiffness matrix A_ℓ and the mass matrix B_ℓ and solves the finite-dimensional generalised algebraic eigenvalue problem

$$A_\ell x_\ell = \lambda_\ell B_\ell x_\ell$$

with $N_\ell := \dim(V_\ell)$ and

$$u_\ell = \sum_{k=1}^{N_\ell} x_k \varphi_k, \qquad V_\ell = \operatorname{span}\{\varphi_1, \ldots, \varphi_{N_\ell}\}.$$

Practically, these discrete eigenvalue problems are solved inexact using iterative algebraic eigenvalue solvers. In this paper the linear algebraic eigenvalue solver (LAES), used as a "black box" iterative solver in the quasi-optimal algorithm AFEMES, is assumed to be any iterative eigenvalue solver of quasi-optimal computational complexity in the sense that for any given tolerance $\varepsilon > 0$, the LAES computes some approximation $(\tilde{\lambda}_{\ell,m}, \tilde{u}_{\ell,m})$ of the generalised algebraic eigenvalue problem from a close enough initial guess $(\tilde{\lambda}_{\ell,0}, \tilde{u}_{\ell,0})$ such that

$$|\!|\!| u_\ell - \tilde{u}_{\ell,m} |\!|\!|^2 + |\lambda_\ell - \tilde{\lambda}_{\ell,m}| \leq \varepsilon^2$$

in at most, up to a generic multiplicative constant,

$$\max\left\{1, \log(\varepsilon^{-1} |\!|\!| u_\ell - \tilde{u}_{\ell,0} |\!|\!|)\right\} \times N_\ell$$

arithmetic operations. That is, each iteration of the solver requires at most $\mathcal{O}(N_\ell)$ operations and the convergence depends only on $\tilde{u}_{\ell,0}$ and not on N_ℓ.

The eigenvalue error of the preconditioned inverse iteration converges independently of h_ℓ for preconditioners that are spectrally equivalent to A_ℓ [KN03b, Theorem 5]. The complexity depends on the sparsity of the preconditioner. The geometric multigrid V-cycle is known to converge independently of h_ℓ and the number of levels ℓ for a fixed number of smoothing steps for Richardson [Bre02a] or Jacobi smoothers [Bre02b]. The preconditioned inverse iteration (PINVIT) and the locally optimal block preconditioned conjugate gradient (LOBPCG) algorithms with the V-cycle geometric multigrid preconditioner have been shown numerically to be of quasi-optimal computational complexity for uniform meshes [KN03a]. Since in this paper the mesh is refined adaptively, global smoothing might be inefficient and local smoothing needs to be applied. However, the numerical examples of Section 8 show that empirically global smoothing is efficient for those examples. The numerical examples of Subsection 8 compare the V-cycle geometric multigrid preconditioned PINVIT and LOBPCG algorithms with a standard solve of the Arnoldi method as implemented in ARPACK [LSY98] where the linear systems are solved using a LU factorisation. The stopping criteria for PINVIT [Ney02] and LOBPCG [KN03a] are based on the scalar product of the algebraic residual and the preconditioned algebraic residual.

Estimate. The error in the eigenfunction or eigenvalue of interest is estimated based on the solution (λ_ℓ, u_ℓ) of the underlying algebraic eigenvalue problem

$$\eta_\ell^2(\lambda_\ell, u_\ell) := \sum_{T\in\mathcal{T}_\ell} \eta_\ell(\lambda_\ell, u_\ell; T)^2 + \sum_{E\in\mathcal{E}_\ell} \eta_\ell(\lambda_\ell, u_\ell; E)^2.$$

Mark. Based on the refinement indicators, edges and elements are marked for refinement in a bulk criterion [Dör96] such that $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \cup \mathcal{E}_\ell$ is an (almost) minimal set of marked edges with

$$\begin{aligned} \theta\eta_\ell^2(\lambda_\ell, u_\ell) &\leq \eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\ell), \\ \eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\ell) &:= \sum_{T\in\mathcal{M}_\ell\cap\mathcal{T}_\ell} \eta_\ell^2(\lambda_\ell, u_\ell; T) + \sum_{E\in\mathcal{M}_\ell\cap\mathcal{E}_\ell} \eta_\ell^2(\lambda_\ell, u_\ell; E) \end{aligned}$$

for a bulk parameter $0 < \theta \leq 1$. This is done in a greedy algorithm which marks edges and elements with larger contributions. In [Ste07] a quasi-optimal algorithm of complexity $\mathcal{O}(|\mathcal{T}_\ell \cup \mathcal{E}_\ell|)$ is proposed, where $|\mathcal{T}_\ell \cup \mathcal{E}_\ell|$ denotes the cardinality of all edges in $\mathcal{E}_\ell$ and all elements in $\mathcal{T}_\ell$. Since sorting the refinement indicators in $\mathcal{O}(|\mathcal{T}_\ell \cup \mathcal{E}_\ell| \log|\mathcal{T}_\ell \cup \mathcal{E}_\ell|)$ does not dominate the overall computational costs in practise, this simple approach is used in the numerical examples of Section 8.

Refine. In this step of the AFEM loop, the mesh is refined locally corresponding to the set $\mathcal{M}_\ell$ of marked edges and elements. Once an element is selected for refinement, all of its edges will be refined. In order to preserve the quality of the mesh, i.e., the maximal angle condition or its equivalents, additionally edges have to be marked by the closure algorithm before refinement. For each triangle let one edge be the uniquely defined reference edge $E(T)$. The closure algorithm computes a superset $\overline{\mathcal{M}_\ell} \supset \mathcal{M}_\ell$ such that

$$\big\{E(T) : T \in \mathcal{T}_\ell \quad \text{with} \quad \mathcal{E}_\ell(T) \cap \overline{\mathcal{M}}_\ell \neq \emptyset \quad \text{or} \quad T \cap \overline{\mathcal{M}}_\ell \neq \emptyset\big\} \subseteq \overline{\mathcal{M}}_\ell.$$

In other words, once a edge of a triangle or itself is marked for refinement, its reference edge $E(T)$ is among them. A similar refinement algorithm for $n = 3$ based on bisection and the concept of reference edges can be found in [AMP00].

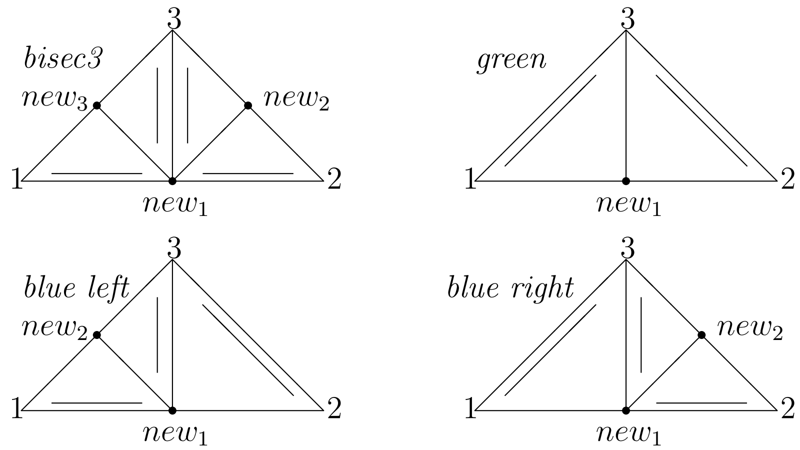

FIGURE 2.1. Refinement rules: Sub-triangles with corresponding reference edges depicted with a second edge.

Proposition 2.1 (boundedness of closure, [BDD04, Ste08]). *Let $\mathcal{T}_{\ell+1}$ be a refinement of $\mathcal{T}_\ell$, obtained using the refinement algorithm and closure. Suppose $\mathcal{T}_0$ is the initial coarse triangulation, then it holds that*

$$|\mathcal{T}_L| - |\mathcal{T}_0| \lesssim \sum_{\ell=0}^{L-1} |\mathcal{M}_\ell|,$$

where $|\mathcal{T}_\ell|$ denotes the cardinality of all triangles in $\mathcal{T}_\ell$. □

After the closure algorithm is applied one of the following refinement rules is applicable, namely no refinement, green refinement, blue left or blue right refinement and bisec3 refinement depicted in Figure 2.1.

Proposition 2.2 (overlay, [Ste07, CKNS08]). *For the smallest common refinement $\mathcal{T}_\varepsilon \oplus \mathcal{T}_\ell$ of $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\ell$ it holds that*

$$|\mathcal{T}_\varepsilon \oplus \mathcal{T}_\ell| - |\mathcal{T}_\ell| \leq |\mathcal{T}_\varepsilon| - |\mathcal{T}_0|. \quad \square$$

3. Algebraic Properties

This section summarises some known and some new algebraic properties of the model problem (1.1), such as the relation between the eigenvalue error and the error with respect to the norms $|||.|||$ and $\|.\|$ [SF73]

$$|||u - u_\ell|||^2 = \lambda \|u - u_\ell\|^2 + \lambda_\ell - \lambda. \tag{3.1}$$

Throughout this section suppose that $(\lambda_\ell, u_\ell) \in \mathbb{R}\times V_\ell$ and $(\lambda_{\ell+m}, u_{\ell+m}) \in \mathbb{R} \times V_{\ell+m}$ are discrete eigenpairs to the continuous eigenpair $(\lambda, u) \in \mathbb{R} \times V$ on the levels ℓ and $\ell + m$.

Lemma 3.1 (quasi-orthogonality). *Let $\mathcal{T}_{\ell+m}$ be a refinement of the triangulation $\mathcal{T}_\ell$ for some level ℓ such that $V_\ell \subset V_{\ell+m}$. Then, for $e_\ell :=$*

$u - u_\ell$ and $e_{\ell+m} := u - u_{\ell+m}$, the quasi-orthogonality holds,

$$\|\!|u_{\ell+m} - u_\ell|\!\|^2 = \|\!|e_\ell|\!\|^2 - \|\!|e_{\ell+m}|\!\|^2 - \lambda\|e_\ell\|^2 + \lambda\|e_{\ell+m}\|^2 + \lambda_{\ell+m}\|u_{\ell+m} - u_\ell\|^2.$$

Proof. Since $\mathcal{T}_{\ell+m}$ is a refinement of $\mathcal{T}_\ell$, (3.1) implies

$$\|\!|u_{\ell+m} - u_\ell|\!\|^2 = \lambda_{\ell+m}\|u_{\ell+m} - u_\ell\|^2 + \lambda_\ell - \lambda_{\ell+m}.$$

Hence,

$$\begin{aligned}\|\!|u_{\ell+m} - u_\ell|\!\|^2 &= \lambda_{\ell+m}\|u_{\ell+m} - u_\ell\|^2 + \lambda_\ell - \lambda - (\lambda_{\ell+m} - \lambda)\\ &= \|\!|e_\ell|\!\|^2 - \|\!|e_{\ell+m}|\!\|^2 - \lambda\|e_\ell\|^2 + \lambda\|e_{\ell+m}\|^2 + \lambda_{\ell+m}\|u_{\ell+m} - u_\ell\|^2.\end{aligned}$$

□

Let the residual $\mathrm{Res}_\ell \in V^*$ be defined by

$$\mathrm{Res}_\ell(v) := \lambda_\ell b(u_\ell, v) - a(u_\ell, v) \quad \text{for all } v \in V.$$

Notice that $V_\ell \subset \ker(\mathrm{Res}_\ell)$.

Lemma 3.2. *Let $\mathcal{T}_{\ell+m}$ be a refinement of $\mathcal{T}_\ell$ such that $V_\ell \subset V_{\ell+m} \subseteq V$, then it holds that*

$$\|\!|u_{\ell+m} - u_\ell|\!\| \le \|\!|Res_\ell|\!\|_{V^*_{\ell+m}} + \frac{(\lambda_{\ell+m} + \lambda_\ell)}{2}\frac{\|u_{\ell+m} - u_\ell\|^2}{\|\!|u_{\ell+m} - u_\ell|\!\|}.$$

Proof. Elementary algebraic manipulations, together with the assumption that $V_\ell \subset V_{\ell+m}$, show

$$\begin{aligned}\|\!|u_{\ell+m} - u_\ell|\!\|^2 &= \lambda_\ell b(u_\ell, u_{\ell+m} - u_\ell) - a(u_\ell, u_{\ell+m} - u_\ell)\\ &\qquad + a(u_{\ell+m}, u_{\ell+m} - u_\ell) - \lambda_\ell b(u_\ell, u_{\ell+m} - u_\ell)\\ &= \mathrm{Res}_\ell(u_{\ell+m} - u_\ell) + (\lambda_{\ell+m} + \lambda_\ell)(1 - b(u_{\ell+m}, u_\ell))\\ &\le \|\!|\mathrm{Res}_\ell|\!\|_{V^*_{\ell+m}}\|\!|u_{\ell+m} - u_\ell|\!\| + \frac{(\lambda_{\ell+m} + \lambda_\ell)}{2}\|u_{\ell+m} - u_\ell\|^2.\end{aligned}$$

□

The remaining part of this section is devoted to showing that the second term on the right hand side in Lemma 3.2 is of higher-order, namely

$$\|u_{\ell+m} - u_\ell\| \lesssim \|h_\ell\|^r_{L^\infty(\Omega)}\|\!|u_{\ell+m} - u_\ell|\!\|.$$

Here and throughout this paper, $h_\ell \in \mathcal{P}_0(\mathcal{T}_\ell)$ is the piecewise constant mesh-size function with $h_\ell|_T := \mathrm{diam}(T)$ for $T \in \mathcal{T}_\ell$ and $0 < r \le 1$ depends on the regularity of the solution of the corresponding boundary value problem. The first part follows the argumentation as in [SF73] for the case $u_{\ell+m} \equiv u$. The second part exploits regularity of the corresponding boundary value problem together with the Aubin–Nitsche technique. Let $G_\ell : V \to V_\ell$ denote the Galerkin projection onto V_ℓ such that for any $v \in V$ it holds that

$$a(v - G_\ell v, v_\ell) = 0 \quad \text{for all } v_\ell \in V_\ell.$$

Suppose the ith eigenvalue $\lambda = \lambda_{\infty,i}$ is simple. Let the initial mesh-size $\|h_0\|_{L^\infty(\Omega)}$ be sufficiently small such that there exist two separation bounds M and $M_{\ell+m}$, independent of h_ℓ, which satisfy for the index set $I_\ell := \{1, \dots, i-1, i+1, \dots, \dim(V_\ell)\}$

$$0 < M := \sup_{\ell\in\mathbb{N}_0} \max_{j\in I_\ell} \frac{\lambda_{\infty,i}}{|\lambda_{\ell,j} - \lambda_{\infty,i}|} < \infty;$$
$$0 < M_{\ell+m} := \max_{j\in I_\ell} \frac{\lambda_{\ell+m,i}}{|\lambda_{\ell,j} - \lambda_{\ell+m,i}|} < \infty.$$

Lemma 3.3. *Let $\mathcal{T}_{\ell+m}$ be a refinement of $\mathcal{T}_\ell$ such that $V_\ell \subset V_{\ell+m} \subseteq V$, then for the Galerkin projection $G_\ell : V \to V_\ell$ it holds that*

$$\|u_{\ell+m} - u_\ell\| \le 2(1 + M_{\ell+m})\|u_{\ell+m} - G_\ell u_{\ell+m}\|,$$
$$\|u - u_\ell\| \le 2(1 + M)\|u - G_\ell u\|.$$

Proof. Note that for the Galerkin projection it holds that

$$(\lambda_{\ell,j} - \lambda_{\ell+m,i})b(G_\ell u_{\ell+m,i}, u_{\ell,j}) = \lambda_{\ell+m,i} b(u_{\ell+m,i} - G_\ell u_{\ell+m,i}, u_{\ell,j}).$$

Since $u_{\ell,1}, \dots, u_{\ell,N_\ell}$, for $N_\ell = \dim(V_\ell)$, forms an orthogonal basis for V_ℓ, the Galerkin projection of $u_{\ell+m,i}$ can be written as

$$G_\ell u_{\ell+m,i} = \sum_{j=1}^{N_\ell} b(G_\ell u_{\ell+m,i}, u_{\ell,j})u_{\ell,j}.$$

Let $\beta := b(G_\ell u_{\ell+m,i}, u_{\ell,i})$ be the coefficient for $j = i$ in the previous formula. Because of the orthogonality of the discrete eigenfunctions $u_{\ell,1}, \dots, u_{\ell,N_\ell}$, it holds that

$$\begin{aligned}
\|G_\ell u_{\ell+m,i} - \beta u_{\ell,i}\|^2 &= \sum_{\substack{j=1\\ j\neq i}}^{N_\ell} b(G_\ell u_{\ell+m,i}, u_{\ell,j})^2 \\
&= \sum_{\substack{j=1\\ j\neq i}}^{N_\ell} \left(\frac{\lambda_{\ell+m,i}}{|\lambda_{\ell,j} - \lambda_{\ell+m,i}|}\right)^2 b(u_{\ell+m,i} - G_\ell u_{\ell+m,i}, u_{\ell,j})^2 \\
&\le M_{\ell+m}^2 \sum_{\substack{j=1\\ j\neq i}}^{N_\ell} b(u_{\ell+m,i} - G_\ell u_{\ell+m,i}, u_{\ell,j})^2 \\
&\le M_{\ell+m}^2 \|u_{\ell+m,i} - G_\ell u_{\ell+m,i}\|^2.
\end{aligned}$$

The triangle inequality shows that

$$\|u_{\ell+m,i}\| - \|u_{\ell+m,i} - \beta u_{\ell,i}\| \le \|\beta u_{\ell,i}\| \le \|u_{\ell+m,i}\| + \|u_{\ell+m,i} - \beta u_{\ell,i}\|.$$

Since the eigenfunctions are normalised to one this implies

$$|\beta - 1| \le \|u_{\ell+m,i} - \beta u_{\ell,i}\|.$$

Hence,

$$\|u_{\ell+m,i}-u_{\ell,i}\| \leq \|u_{\ell+m,i}-\beta u_{\ell,i}\| + \|(\beta-1)u_{\ell,i}\| \leq 2\|u_{\ell+m,i}-\beta u_{\ell,i}\|.$$

Thus,

$$\begin{aligned}\|u_{\ell+m,i}-u_{\ell,i}\| &\leq 2\|u_{\ell+m,i}-G_\ell u_{\ell+m,i}\| + 2\|G_\ell u_{\ell+m,i}-\beta u_{\ell,i}\| \\ &\leq 2(1+M_{\ell+m})\|u_{\ell+m,i}-G_\ell u_{\ell+m,i}\|.\end{aligned}$$

The second inequality follows analogously since $V_\ell \subset V$. □

Lemma 3.4. *Let $\mathcal{T}_{\ell+m}$ be a refinement of $\mathcal{T}_\ell$ such that $V_\ell \subset V_{\ell+m} \subseteq V$. Suppose the corresponding boundary value problem to* (1.1)*, seek $z \in V$ such that*

$$a(z,v) = \int_\Omega f v\,dx \quad \textit{for all } v \in V,$$

is H^{1+r}-regular for all $f \in L^2(\Omega)$ and some $0 < r \leq 1$, i.e., $z \in H^{1+r}(\Omega) \cap V$ and $\|z\|_{H^{1+r}(\Omega)} \leq C_{reg}\|f\|_{L^2(\Omega)}$. Then it holds that

$$\begin{aligned}\|u_{\ell+m}-G_\ell u_{\ell+m}\| &\leq C_{approx}C_{reg}\|h_\ell\|^r_{L^\infty(\Omega)}|||u_{\ell+m}-u_\ell|||,\\ \|u-G_\ell u\| &\leq C_{approx}C_{reg}\|h_\ell\|^r_{L^\infty(\Omega)}|||u-u_\ell|||.\end{aligned}$$

Proof. The following convergence estimate holds for the Galerkin projection $G_\ell z \in V_\ell$ of $z \in V$

$$\|z-G_\ell z\|_{H^1(\Omega)} \leq C_{\mathrm{approx}}\|h_\ell\|^r_{L^\infty(\Omega)}\|z\|_{H^{1+r}(\Omega)}$$

for some $0 < r \leq 1$ [BS08, Theorem 14.3.3]. The Aubin-Nitzsche duality technique for the dual boundary value problem, seek $z \in V$ such that

$$a(z,v) = b(u_{\ell+m}-G_\ell u_{\ell+m}, v) \quad \text{for all } v \in V,$$

and the regularity assumption $z \in H^{1+r}(\Omega) \cap V$,

$$\|z\|_{H^{1+r}(\Omega)} \leq C_{reg}\|u_{\ell+m}-G_\ell u_{\ell+m}\|,$$

lead to

$$\begin{aligned}\|u_{\ell+m}-G_\ell u_{\ell+m}\| &\leq C_{\mathrm{approx}}C_{\mathrm{reg}}\|h_\ell\|^r_{L^\infty(\Omega)}|||u_{\ell+m}-G_\ell u_{\ell+m}|||\\ &\leq C_{\mathrm{approx}}C_{\mathrm{reg}}\|h_\ell\|^r_{L^\infty(\Omega)}|||u_{\ell+m}-u_\ell|||.\end{aligned}$$

The second inequality follows from formally $m \to \infty$. □

Lemma 3.5. *Let $\mathcal{T}_{\ell+m}$ be a refinement of $\mathcal{T}_\ell$ such that $V_\ell \subset V_{\ell+m} \subseteq V$. For sufficiently small initial mesh-size $\|h_0\|_{L^\infty(\Omega)}$ there exists a constant $C_0 > 0$ depending only on $\mathcal{T}_0$ such that $1 \leq \kappa(h_\ell) < C_0$ with*

$$|||u_{\ell+m}-u_\ell||| \leq \kappa(h_\ell)|||Res_\ell|||_{V^*_{\ell+m}}, \quad |||u-u_\ell||| \leq \kappa(h_\ell)|||Res_\ell|||_{V^*}$$

and $\lim_{\|h_\ell\|_{L^\infty(\Omega)}\to 0}\kappa(h_\ell) = 1.$

Proof. Suppose that $\|h_\ell\|_{L^\infty(\Omega)}$ is sufficiently small such that

$$\delta_\ell := 2C_{\text{approx}}^2 C_{\text{reg}}^2 (\lambda_{\ell+m} + \lambda_\ell)(1 + \max\{M, M_{\ell+m}\})^2 \|h_\ell\|_{L^\infty(\Omega)}^{2r} \ll 1.$$

Then Lemmas 3.2 and 3.3 together with Lemma 3.4 lead to

$$\begin{aligned} \|\!|u_{\ell+m} - u_\ell|\!\| &\leq (1-\delta_\ell)^{-1} \|\!|\mathrm{Res}_\ell|\!\|_{V_{\ell+m}^*}; \\ \|\!|u - u_\ell|\!\| &\leq (1-\delta_\ell)^{-1} \|\!|\mathrm{Res}_\ell|\!\|_{V^*}. \end{aligned}$$

Notice that $\kappa(h_\ell) := (1-\delta_\ell)^{-1} \to 1$ as the maximal mesh-size tends to zero and $C_0 := (1-\delta_0)^{-1}$. □

4. A Posteriori Error Estimator

This section establishes the discrete reliability and recalls the reliability and efficiency of the standard residual-based error estimator [DXZ08, DPR03, GMZ09, GG09]. Let $p_\ell := \nabla u_\ell$ denote the discrete gradient and $\mathcal{E}_\ell$ the set of inner edges ($n = 2$) or inner faces ($n = 3$) of $\mathcal{T}_\ell$. For $E \in \mathcal{E}_\ell$ let $T_+, T_- \in \mathcal{T}_\ell$ be the two neighbouring triangles such that $E = T_+ \cap T_-$. The jump of the discrete gradient p_ℓ along an inner edge $E \in \mathcal{E}_\ell$ in normal direction ν_E, pointing from T_+ to T_-, is defined by $[p_\ell]\cdot\nu_E := \big(p_\ell|_{T_+} - p_\ell|_{T_-}\big)\cdot\nu_E$. Then the residual error estimator is defined by

$$\eta_\ell^2(\lambda_\ell, u_\ell) := \sum_{T\in\mathcal{T}_\ell} \eta_\ell(\lambda_\ell, u_\ell; T)^2 + \sum_{E\in\mathcal{E}_\ell} \eta_\ell(\lambda_\ell, u_\ell; E)^2$$

with $n = 2, 3$ and

$$\begin{aligned} \eta_\ell(\lambda_\ell, u_\ell; T)^2 &:= |T|^{2/n} \|\lambda_\ell u_\ell + \mathrm{div}(p_\ell)\|_{L^2(T)}^2, \\ \eta_\ell(\lambda_\ell, u_\ell; E)^2 &:= |E|^{1/(n-1)} \|[p_\ell]\cdot\nu_E\|_{L^2(E)}^2. \end{aligned}$$

Note that the Scott-Zhang quasi-interpolation operator $J : V \to V_\ell$ is a projection $J(v_\ell) = v_\ell$ for all $v_\ell \in V_\ell$. In addition, it is locally a L^2-projection onto $(n-1)$-dimensional edges or faces. Therefore, each node is assigned any edge or face which contains it. Edge-basis functions are interpolated on their edge and element-basis functions are interpolated over the interior of their element. The element and edge patches Ω_T and Ω_E are displayed in Figure 4.1. In the following, the Scott-Zhang quasi-interpolation operator is restricted to $V_{\ell+m}$ for a refined triangulation $\mathcal{T}_{\ell+m}$ of $\mathcal{T}_\ell$. If it is possible, each nodal-basis function is assigned an edge of the boundary or an edge which is not refined. Thus, the homogeneous boundary values are preserved. Let v_ℓ denote the Scott-Zhang interpolant of $v_{\ell+m}$ in V_ℓ. Then for all elements $T \in \mathcal{T}_\ell$ and all edges $E \in \mathcal{E}_\ell$ that are not refined it holds that $v_{\ell+m}|_T = v_\ell|_T$ and $v_{\ell+m}|_E = v_\ell|_E$. The finite overlap of all the patches Ω_T and Ω_E implies the approximation property [SZ90]

$$\sum_{T\in\mathcal{T}_\ell} |T|^{-1/n} \|v_{\ell+m} - v_\ell\|_{L^2(T)} + \sum_{E\in\mathcal{E}_\ell} |E|^{-1/(2n-2)} \|v_{\ell+m} - v_\ell\|_{L^2(E)} \lesssim \|\!|v_{\ell+m}|\!\|.$$

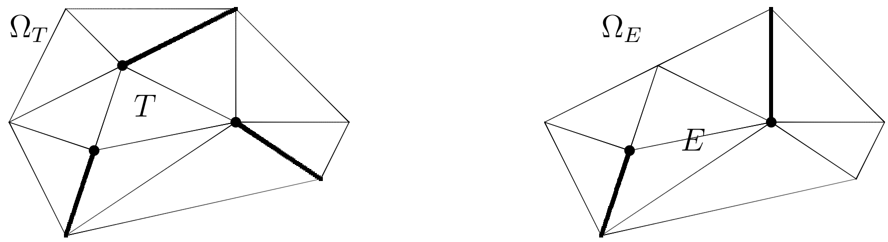

FIGURE 4.1. Patches for the Scott-Zhang interpolation operator.

Lemma 4.1 (discrete reliability). *For sufficiently small* $\|h_0\|_{L^\infty(\Omega)}$ *let* (λ_ℓ, u_ℓ) *be a discrete eigenpair on level* ℓ *and* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \cup \mathcal{E}_\ell$ *be any set of edges and elements. Suppose the refinement algorithm of Section 2 computes the refined mesh* $\mathcal{T}_{\ell+m}$*, then it holds that*

$$|||Res_\ell|||_{V^*_{\ell+m}} \lesssim \eta_\ell(\lambda_\ell, u_\ell; \mathcal{M}_\ell).$$

Proof. Let v_ℓ denote the Scott–Zhang interpolant of $v_{\ell+m} \in V_{\ell+m}$ in V_ℓ. For all common elements $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$ and all common edges $E \in \mathcal{E}_\ell \cap \mathcal{E}_{\ell+m}$ it holds that $v_\ell|_T = v_{\ell+m}|_T$ and $v_\ell|_E = v_{\ell+m}|_E$. Hence,

$$\begin{aligned}
\mathrm{Res}_\ell(v_{\ell+m}) &= \mathrm{Res}_\ell(v_{\ell+m} - v_\ell) = \lambda_\ell b(u_\ell, v_{\ell+m} - v_\ell) - a(u_\ell, v_{\ell+m} - v_\ell) \\
&\lesssim \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}} |T|^{1/n} \|\lambda_\ell u_\ell + \mathrm{div}(p_\ell)\|_{L^2(T)} \||T|^{-1/n}(v_{\ell+m} - v_\ell)\|_{L^2(T)} \\
&\quad + \sum_{E \in \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+m}} |E|^{1/(2n-2)} \|[p_\ell]\cdot\nu_E\|_{L^2(E)} \||E|^{-1/(2n-2)}(v_{\ell+m} - v_\ell)\|_{L^2(E)} \\
&\lesssim \eta_\ell(\lambda_\ell, u_\ell; \mathcal{M}_\ell) |||v_{\ell+m}|||.
\end{aligned}$$

□

Lemma 4.2. *For sufficiently small* $\|h_0\|_{L^\infty(\Omega)}$ *it holds*

$$|||Res_\ell|||_{V^*} \lesssim \eta_\ell(\lambda_\ell, u_\ell) \lesssim |||e_\ell|||.$$

Proof. The first inequality can be proven as Lemma 4.1. For the second inequality, Durán et al. [DPR03] showed the local lower bound for piecewise linear finite element functions using the bubble-function technique. In the case of higher-order finite elements the arguments of the proof remain the same as in the linear case except that $\mathrm{div}(p_\ell)$ can be nonzero. Thus the local discrete inverse inequality

$$|\omega_E|^{1/n} \|\mathrm{div}(p_\ell)\|_{L^2(\omega_E)} \lesssim \|\nabla e_\ell\|_{L^2(\omega_E)}$$

has to be applied additionally. Therefore, it holds the local lower bound

$$\begin{aligned}
|\omega_E|^{1/n} \|\lambda_\ell u_\ell + \mathrm{div}(p_\ell)\|_{L^2(\omega_E)} &+ |E|^{1/(2n-2)} \|[p_\ell]\cdot\nu_E\|_{L^2(E)} \\
&\lesssim \|\nabla e_\ell\|_{L^2(\omega_E)} + |\omega_E|^{1/n} \|\lambda u - \lambda_\ell u_\ell\|_{L^2(\omega_E)}
\end{aligned}$$

for the edge patch $\omega_E := T_+ \cup T_-$, for $T_\pm \in \mathcal{T}_\ell$ with $E = T_+ \cap T_-$. The global version reads

$$\eta_\ell^2(\lambda_\ell, u_\ell) \lesssim |||e_\ell|||^2 + \|h_\ell\|^2_{L^\infty(\Omega)} \|\lambda u - \lambda_\ell u_\ell\|^2.$$

As shown in [CG11], some elementary algebra in the spirit of Lemma 3.1 shows

$$\|\lambda u - \lambda_\ell u_\ell\|^2 = (\lambda_\ell - \lambda)^2 + \lambda\lambda_\ell \|e_\ell\|^2.$$

Equation (3.1) yields $(\lambda_\ell - \lambda)^2 \le |||e_\ell|||^4$ and $\lambda\lambda_\ell\|e_\ell\|^2 \le \lambda_\ell |||e_\ell|||^2$. Since λ_ℓ is bounded by λ_0 it holds

$$\eta_\ell(\lambda_\ell, u_\ell) \lesssim |||e_\ell|||$$

even for larger mesh-sizes $\|h_\ell\|_{L^\infty(\Omega)} \lesssim 1$. □

Remark 4.3. Lemma 3.5, Lemma 4.1 and Lemma 4.2 show that for sufficiently small $\|h_\ell\|_{L^\infty(\Omega)}$ there exist two constants $0 < C_{\mathrm{rel}}$ and $0 < C_{\mathrm{eff}}$ such that

$$\eta_\ell(\lambda_\ell, u_\ell)/C_{\mathrm{eff}} \le |||e_\ell||| \le C_{\mathrm{rel}}\eta_\ell(\lambda_\ell, u_\ell);$$
$$|||u_{\ell+m} - u_\ell||| \le C_{\mathrm{rel}}\eta_\ell(\lambda_\ell, u_\ell; \mathcal{M}_\ell).$$

Similar results as in Lemma 4.1 and 4.2 for general bilinear forms $a(\cdot,\cdot)$ with jumping coefficients include additional terms that represent data oscillations, cf. [AO00, GMZ09, Ver96].

5. Quasi-Optimal Convergence

This section is devoted to the asymptotic quasi-optimal convergence analysis of the adaptive eigenvalue computation based on exact solutions of the algebraic eigenvalue problems. At first the approximation class $\mathcal{A}_s$ is defined and its properties are described. Lemma 5.2 shows an estimator reduction which is used in the proof of the contraction property in Lemma 5.3. The contraction property and the bulk criterion are key arguments in the proof of the quasi-optimality in Theorem 5.4.

Definition 5.1 (approximation class)**.** *For an initial triangulation $\mathcal{T}_0$ and for $s > 0$ let the approximation class be defined by*

$$\mathcal{A}_s := \left\{ v \in V : |v|_{\mathcal{A}_s} := \sup_{\varepsilon > 0} \varepsilon \inf_{\mathcal{T}_\varepsilon : |||v - v_\varepsilon||| \le \varepsilon} (|\mathcal{T}_\varepsilon| - |\mathcal{T}_0|)^s < \infty \right\}.$$

The infimum is taken over all refinements $\mathcal{T}_\varepsilon$ of $\mathcal{T}_0$ computed by the refinement algorithm of Section 2 with $|||v - v_\varepsilon||| \le \varepsilon$ and $v_\varepsilon \in V_\varepsilon$.

Notice that $\mathcal{A}_s$ contains all functions that can be approximated within pre-described tolerance $\varepsilon > 0$ in a finite element space V_ε, $|||v - v_\varepsilon||| \le \varepsilon$ for some $v_\varepsilon \in V_\varepsilon$, based on the triangulation $\mathcal{T}_\varepsilon$ with $|\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \le \varepsilon^{-1/s}|v|_{\mathcal{A}_s}^{1/s}$. For uniform refinement classical a priori estimates show that for $0 < r \le 1$, $H^{1+r}(\Omega) \cap V \subset \mathcal{A}_{r/n}$, but the class contains many more functions which motivates the use of adaptivity.

Due to [Ste07] an equivalent formulation, similar to that of [CKNS08], reads

$$\mathcal{A}_s := \left\{ v \in V : \sup_{N \in \mathbb{N}} N^s \inf_{\mathcal{T}_\varepsilon : |\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \leq N} \|\!| v - v_\varepsilon \|\!| < \infty \right\}.$$

In the following the marking strategy of Section 2 is a key argument in the proofs.

Lemma 5.2. *Let (λ_ℓ, u_ℓ) and $(\lambda_{\ell+1}, u_{\ell+1})$ be discrete eigenpairs on the levels ℓ and $\ell+1$ to the continuous eigenpair (λ, u), then there exists some $\Lambda > 0$, such that, for all levels $\ell \geq 0$ and $0 < \theta \leq 1$, it holds that*

$$\eta_{\ell+1}(\lambda_{\ell+1}, u_{\ell+1}) \leq \sqrt{(1 - \theta(1 - 2^{-2/n}))}\eta_\ell(\lambda_\ell, u_\ell) + \Lambda \|\!| u_{\ell+1} - u_\ell \|\!|.$$

Proof. As in the proof of [CG11, Lemma 5.1], Young's inequality, some discrete inverse inequalities and the bulk criterion of Section 2 lead to

$$\begin{aligned}\eta^2_{\ell+1}(\lambda_{\ell+1}, u_{\ell+1}) \leq (1+\delta)(1 - \theta(1 - 2^{-2/n}))\eta^2_\ell(\lambda_\ell, u_\ell)\\ + \Lambda^2(1 + 1/\delta)\|\!| u_{\ell+1} - u_\ell \|\!|^2\end{aligned}$$

for any $0 < \delta$ from Young's inequality, $0 < \theta \leq 1$ bulk parameter, and $0 < \Lambda$ from application of various discrete inverse inequalities. Thereby the factor $2^{-2/n}$ results from at least one bisection of refined elements or edges. The choice

$$\delta = \frac{\Lambda \|\!| u_{\ell+1} - u_\ell \|\!|}{\sqrt{(1 - \theta(1 - 2^{-2/n}))}\eta_\ell(\lambda_\ell, u_\ell)}$$

proves the assertion. □

Lemma 5.3 (contraction property)**.** *Let (λ_ℓ, u_ℓ) and $(\lambda_{\ell+1}, u_{\ell+1})$ be discrete eigenpairs on the levels ℓ and $\ell+1$ to the same continuous eigenpair (λ, u) and let the mesh-size $\|h_\ell\|_{L^\infty(\Omega)}$ be sufficiently small, then there exist constants $0 < \varrho < 1$ and $\gamma > 0$, such that, for all $\ell = 0, 1, 2, \ldots$, it holds that*

$$\gamma \eta^2_{\ell+1}(\lambda_{\ell+1}, u_{\ell+1}) + \|\!| u - u_{\ell+1} \|\!|^2 \leq \varrho \left(\gamma \eta^2_\ell(\lambda_\ell, u_\ell) + \|\!| u - u_\ell \|\!|^2 \right). \tag{5.1}$$

Proof. Theorem 5.3 of [CG11] shows for $0 < \rho < 1$ that

$$\begin{aligned}\gamma \eta^2_{\ell+1}(\lambda_{\ell+1}, u_{\ell+1}) + \|\!| e_{\ell+1} \|\!|^2 \leq \rho \left(\gamma \eta^2_\ell(\lambda_\ell, u_\ell) + \|\!| e_\ell \|\!|^2 \right)\\ + 3\lambda_{\ell+1}\|e_{\ell+1}\|^2 + 3\lambda_\ell \|e_\ell\|^2.\end{aligned}$$

Lemmas 3.3 and 3.4 show

$$\|u - u_\ell\|^2 \leq \sigma(h_\ell)^2 \|\!| u - u_\ell \|\!|^2, \tag{5.2}$$

where $\sigma(h_\ell) := 2(1 + M)C_{approx}C_{reg}\|h_\ell\|^r_{L^\infty(\Omega)}$.

Hence, for sufficiently small mesh-size $\|h_0\|_{L^\infty(\Omega)}$, it follows (5.1) with the constant

$$0 < \varrho := \frac{\rho + 3\lambda_0 \sigma(h_\ell)^2}{1 - 3\lambda_0 \sigma(h_\ell)^2} < 1.$$

□

Theorem 5.4. *Suppose that (λ_ℓ, u_ℓ) is a discrete eigenpair to the continuous eigenpair (λ, u) with $u \in \mathcal{A}_s$ and that the initial mesh-size $\|h_0\|_{L^\infty(\Omega)}$ is sufficiently small. Then λ_ℓ and u_ℓ from the AFEM converge quasi-optimal in the sense that*

$$|\!|\!| e_\ell |\!|\!|^2 + |\lambda - \lambda_\ell| \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s} \lesssim N_\ell^{-2s}.$$

Proof. First it is shown that for a set $\mathcal{M}_\ell$ of marked edges and elements from the marking strategy of Section 2, based on the bulk criterion, $\eta_\ell(\lambda_\ell, u_\ell)$ and a bulk parameter $\theta > 0$, it holds that

$$|\mathcal{M}_\ell| \lesssim |\!|\!| e_\ell |\!|\!|^{-1/s} |u|_{\mathcal{A}_s}^{1/s}.$$

Suppose $\mathcal{T}_{\ell+\varepsilon}$ is any refinement of $\mathcal{T}_\ell$ such that

$$|\!|\!| e_{\ell+\varepsilon} |\!|\!| \le \rho |\!|\!| e_\ell |\!|\!|$$

for some $0 < \rho < 1$. Suppose that $\|h_\ell\|_{L^\infty(\Omega)}$ and θ are sufficiently small, such that

$$0 < \theta \le \frac{(1-\rho^2)}{C_{\mathrm{rel}}^2 C_{\mathrm{eff}}^2} - \lambda \sigma(h_\ell)^2,$$

where $\sigma(h_\ell)$ from Lemma 5.3 tends to zero as $\|h_\ell\|_{L^\infty(\Omega)} \to 0$. Using the efficiency estimates of Remark 4.3 together with the quasi-orthogonality of Lemma 3.1 yields

$$\begin{aligned}(1-\rho^2)\eta_\ell^2(\lambda_\ell, u_\ell)/C_{\mathrm{eff}}^2 &\le (1-\rho^2)|\!|\!| e_\ell |\!|\!|^2 \le |\!|\!| e_\ell |\!|\!|^2 - |\!|\!| e_{\ell+\varepsilon} |\!|\!|^2 \\ &= |\!|\!| u_{\ell+\varepsilon} - u_\ell |\!|\!|^2 + \lambda \|e_\ell\|^2 - \lambda \|e_{\ell+\varepsilon}\|^2 - \lambda_{\ell+\varepsilon} \|u_{\ell+\varepsilon} - u_\ell\|^2.\end{aligned}$$

Let $\mathcal{M}_\varepsilon := (\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+\varepsilon}) \cup (\mathcal{E}_\ell \backslash \mathcal{E}_{\ell+\varepsilon})$, then the reliability of Remark 4.3 and (5.2) yield

$$\begin{aligned}(1-\rho^2)\eta_\ell^2(\lambda_\ell, u_\ell)/C_{\mathrm{eff}}^2 &\le C_{\mathrm{rel}}^2 \eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\varepsilon) + \lambda \|e_\ell\|^2 \\ &\le C_{\mathrm{rel}}^2 \eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\varepsilon) + \lambda \sigma(h_\ell)^2 C_{\mathrm{rel}}^2 \eta_\ell^2(\lambda_\ell, u_\ell).\end{aligned}$$

Therefore, $\mathcal{M}_\varepsilon$ satisfies the bulk criterion. Since $\mathcal{M}_\ell$ is the set with almost minimal cardinality that fulfils the bulk criterion, it holds that

$$|\mathcal{M}_\ell| \lesssim |\mathcal{M}_\varepsilon| \lesssim |\mathcal{T}_{\ell+\varepsilon}| - |\mathcal{T}_\ell|.$$

Let $\mathcal{T}_\varepsilon$ be an optimal mesh with smallest cardinality such that

$$|\!|\!| e_\varepsilon |\!|\!| \le \rho |\!|\!| e_\ell |\!|\!|.$$

The definition of the approximation space $\mathcal{A}_s$ shows that

$$|\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \le \rho^{-1/s} |\!|\!| e_\ell |\!|\!|^{-1/s} |u|_{\mathcal{A}_s}^{1/s}.$$

Let $\mathcal{T}_{\ell+\varepsilon}$ be the smallest common refinement of $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\ell$. Then the overlay estimate yields

$$|\mathcal{M}_\ell| \lesssim |\mathcal{T}_{\ell+\varepsilon}| - |\mathcal{T}_\ell| = |\mathcal{T}_\varepsilon \oplus \mathcal{T}_\ell| - |\mathcal{T}_\ell| \le |\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \lesssim |\!|\!| e_\ell |\!|\!|^{-1/s} |u|_{\mathcal{A}_s}^{1/s}.$$

This and the boundedness of closure in Lemma 2.1 yield

$$|\mathcal{T}_L| - |\mathcal{T}_0| \lesssim \sum_{\ell=0}^{L-1} |\mathcal{M}_\ell| \lesssim |u|_{\mathcal{A}_s}^{1/s} \sum_{\ell=0}^{L-1} |||e_\ell|||^{-1/s}.$$

The efficiency estimate of Remark 4.3 yields

$$\gamma\eta_\ell^2(\lambda_\ell, u_\ell) + |||u - u_\ell|||^2 \leq \left(1 + \gamma C_{\text{eff}}^2\right) |||u - u_\ell|||^2.$$

Thus,

$$|||u - u_\ell|||^{-1/s} \leq \left(1 + \gamma C_{\text{eff}}^2\right)^{1/(2s)} \left(\gamma\eta_\ell^2(\lambda_\ell, u_\ell) + |||u - u_\ell|||^2\right)^{-1/(2s)}.$$

Lemma 5.3 leads to

$$\begin{aligned}&\left(\gamma\eta_\ell^2(\lambda_\ell, u_\ell) + |||u - u_\ell|||^2\right)^{-1/(2s)} \\ &\qquad \leq \varrho^{1/(2s)} \left(\gamma\eta_{\ell+1}^2(\lambda_{\ell+1}, u_{\ell+1}) + |||u - u_{\ell+1}|||^2\right)^{-1/(2s)}.\end{aligned}$$

Exploiting the reliability of the estimator and a geometric series argument yields that $|\mathcal{T}_L| - |\mathcal{T}_0|$ is, up to a generic multiplicative constant, bounded by

$$\begin{aligned}&|u|_{\mathcal{A}_s}^{1/s} \left(1 + \gamma C_{\text{eff}}^2\right)^{1/(2s)} \left(\gamma\eta_L^2(\lambda_L, u_L) + |||u - u_L|||^2\right)^{-1/(2s)} \sum_{\ell=1}^{L} \varrho^{\ell/(2s)} \\ &\qquad \lesssim |u|_{\mathcal{A}_s}^{1/s} \left(\frac{1 + \gamma C_{\text{eff}}^2}{1 + \gamma / C_{\text{rel}}^2}\right)^{1/(2s)} (1 - \varrho^{1/(2s)})^{-1} |||u - u_L|||^{-1/s}.\end{aligned}$$

Note that Euler's formula shows $(|\mathcal{T}_\ell| - |\mathcal{T}_0|) \approx N_\ell$. Finally equation (3.1) proves $|\lambda - \lambda_\ell| \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s}$. □

6. Quasi-Optimal Convergence for Inexact Algebraic Solutions

This section contributes to the fact that in practise the underlying algebraic eigenvalue problems are solved inexact using iterative algebraic eigenvalue solvers. A relationship between the error estimator in the discrete solution and any approximation to it is established in Lemma 6.1. As in the case of discrete solutions, the contraction property in Lemma 6.2 and the local quasi-optimality in Lemma 6.3 lead to the global asymptotic quasi-optimality in Theorem 6.4.

Lemma 6.1. *Let $v_\ell, \tilde{v}_\ell \in V_\ell$ be arbitrary, not necessary eigenfunctions, but normalised with $\|v_\ell\| = \|\tilde{v}_\ell\| = 1$ and $\mu, \tilde{\mu} \in \mathbb{R}^+$ arbitrary positive real numbers bounded from above by $\tilde{\lambda}_0$, then it holds that*

$$|\eta_\ell(\mu, v_\ell) - \eta_\ell(\tilde{\mu}, \tilde{v}_\ell)|^2 \leq C\left(|||v_\ell - \tilde{v}_\ell|||^2 + |\mu - \tilde{\mu}|\right)$$

for a constant $0 < C$ independent of the mesh-size $\|h_\ell\|_{L^\infty(\Omega)}$.

Proof. Using twice the triangle inequality first for vectors and then for functions yields

$$
\begin{aligned}
|\eta_\ell(\mu, v_\ell) - \eta_\ell(\tilde{\mu}, \tilde{v}_\ell)|^2 \leq & \sum_{T\in\mathcal{T}_\ell} |T|^{2/n} \|\mu v_\ell - \tilde{\mu}\tilde{v}_\ell + \operatorname{div}(\nabla v_\ell - \nabla \tilde{v}_\ell)\|^2_{L^2(T)} \\
& + \sum_{E\in\mathcal{E}_\ell} |E|^{1/(n-1)} \|[\nabla v_\ell - \nabla \tilde{v}_\ell]\cdot \nu_E\|^2_{L^2(E)}.
\end{aligned}
$$

The local discrete inverse inequality

$$
|T|^{2/n} \|\operatorname{div}(\nabla v_\ell)\|^2_{L^2(T)} \lesssim \|\nabla v_\ell\|^2_{L^2(T)},
$$

together with the trace inequality

$$
\|v\|^2_{L^2(E)} \lesssim |E|^{-1/(n-1)} \|v\|^2_{L^2(\omega_E)} + |E|^{1/(n-1)} \|\nabla v\|^2_{L^2(\omega_E)},
$$

the Poincaré inequality and the finite overlay of the patches, leads to

$$
\begin{aligned}
& |\eta_\ell(\mu, v_\ell) - \eta_\ell(\tilde{\mu}, \tilde{v}_\ell)|^2 \\
& \lesssim \sum_{T\in\mathcal{T}_\ell} |T|^{2/n} \|\mu v_\ell - \tilde{\mu}\tilde{v}_\ell\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_\ell} \|\nabla v_\ell - \nabla \tilde{v}_\ell\|^2_{L^2(T)} \\
& \quad + \sum_{E\in\mathcal{E}_\ell} \|\nabla v_\ell - \nabla \tilde{v}_\ell\|^2_{L^2(\omega_E)} \\
& \lesssim \|h_\ell\|^2_{L^\infty(\Omega)} \|\mu v_\ell - \tilde{\mu}\tilde{v}_\ell\|^2 + |\!|\!| v_\ell - \tilde{v}_\ell |\!|\!|^2 \\
& \lesssim (1 + \tilde{\lambda}_0^2 \|h_0\|^2_{L^\infty(\Omega)}) |\!|\!| v_\ell - \tilde{v}_\ell |\!|\!|^2 + 2\tilde{\lambda}_0 \|h_0\|^2_{L^\infty(\Omega)} |\mu - \tilde{\mu}|. \qquad \square
\end{aligned}
$$

Lemma 6.2 (contraction property for inexact algebraic solutions)**.** *Suppose that (λ_ℓ, u_ℓ) and $(\lambda_{\ell+1}, u_{\ell+1})$ are discrete eigenpairs to the continuous eigenpair (λ, u) with $u \in \mathcal{A}_s$ on levels ℓ and $\ell+1$. Let $(\tilde{\lambda}_\ell, \tilde{u}_\ell)$ and $(\tilde{\lambda}_{\ell+1}, \tilde{u}_{\ell+1})$ be the corresponding approximations to the discrete eigenpairs, which satisfy*

$$
\begin{aligned}
|\!|\!| u_{\ell+1} - \tilde{u}_{\ell+1} |\!|\!|^2 + |\lambda_{\ell+1} - \tilde{\lambda}_{\ell+1}| &\leq \omega \eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell), \\
|\!|\!| u_\ell - \tilde{u}_\ell |\!|\!|^2 + |\lambda_\ell - \tilde{\lambda}_\ell| &\leq \omega \eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell)
\end{aligned}
$$

for sufficiently small $\omega > 0$. Then, for sufficiently small mesh-size $\|h_\ell\|_{L^\infty(\Omega)}$, there exists some $0 < \nu < 1$, such that the contraction property

$$
\gamma \eta_\ell^2(\tilde{\lambda}_{\ell+1}, \tilde{u}_{\ell+1}) + |\!|\!| u - \tilde{u}_{\ell+1} |\!|\!|^2 \leq \nu \left(\gamma \eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell) + |\!|\!| u - \tilde{u}_\ell |\!|\!|^2 \right)
$$

holds.

Proof. The assumptions, Lemma 6.1 and Young's inequality show for any $\delta > 0$

$$\begin{aligned}
&\gamma\eta_\ell^2(\tilde\lambda_{\ell+1},\tilde u_{\ell+1}) + \|\!|u-\tilde u_{\ell+1}|\!\|^2\\
&\quad\le (1+\delta)\left(\gamma\eta_\ell^2(\lambda_{\ell+1},u_{\ell+1}) + \|\!|u-u_{\ell+1}|\!\|^2\right)\\
&\qquad + (1+1/\delta)\left(\gamma|\eta_\ell(\tilde\lambda_{\ell+1},\tilde u_{\ell+1}) - \eta_\ell(\lambda_{\ell+1},u_{\ell+1})|^2 + \|\!|u_{\ell+1}-\tilde u_{\ell+1}|\!\|^2\right)\\
&\quad\le (1+\delta)\left(\gamma\eta_\ell^2(\lambda_{\ell+1},u_{\ell+1}) + \|\!|u-u_{\ell+1}|\!\|^2\right)\\
&\qquad + (1+1/\delta)\left(\gamma C|\lambda_{\ell+1}-\tilde\lambda_{\ell+1}| + (1+\gamma C)\|\!|u_{\ell+1}-\tilde u_{\ell+1}|\!\|^2\right)\\
&\quad\le (1+\delta)\left(\gamma\eta_\ell^2(\lambda_{\ell+1},u_{\ell+1}) + \|\!|u-u_{\ell+1}|\!\|^2\right)\\
&\qquad + (1+1/\delta)(1+\gamma C)\omega\eta_\ell^2(\tilde\lambda_\ell,\tilde u_\ell).
\end{aligned}$$

The contraction property Lemma 5.3 and another Young's inequality yield

$$\begin{aligned}
&\gamma\eta_\ell^2(\tilde\lambda_{\ell+1},\tilde u_{\ell+1}) + \|\!|u-\tilde u_{\ell+1}|\!\|^2\\
&\quad\le (1+\delta)\varrho\left(\gamma\eta_\ell^2(\lambda_\ell,u_\ell) + \|\!|u-u_\ell|\!\|^2\right) + (1+1/\delta)(1+\gamma C)\omega\eta_\ell^2(\tilde\lambda_\ell,\tilde u_\ell)\\
&\quad\le (1+\delta)^2\varrho\left(\gamma\eta_\ell^2(\tilde\lambda_\ell,\tilde u_\ell) + \|\!|u-\tilde u_\ell|\!\|^2\right)\\
&\qquad + (1+(1+\delta)\varrho)(1+1/\delta)(1+\gamma C)\omega\eta_\ell^2(\tilde\lambda_\ell,\tilde u_\ell).
\end{aligned}$$

Any choice of $0 < \delta < \varrho^{-1/2} - 1$ results in

$$0 < \omega < \frac{\gamma - (1+\delta)^2\varrho\gamma}{(1+(1+\delta)\varrho)(1+1/\delta)(1+\gamma C)}.$$

The choice

$$0 < \nu := (1+\delta)^2\varrho + (1+(1+\delta)\varrho)(1+1/\delta)(1+\gamma C)\omega/\gamma < 1$$

concludes the proof. □

Lemma 6.3. *Let (λ, u) with $u \in \mathcal{A}_s$ be an eigenpair and let (λ_ℓ, u_ℓ) be the corresponding discrete eigenpair with approximation $(\tilde\lambda_\ell, \tilde u_\ell)$ which satisfies*

$$\|\!|u_\ell - \tilde u_\ell|\!\|^2 + |\lambda_\ell - \tilde\lambda_\ell| \le \omega\eta_\ell^2(\tilde\lambda_\ell,\tilde u_\ell)$$

for a sufficient small $\omega > 0$. Suppose that $\mathcal{M}_{\tilde\lambda_\ell,\tilde u_\ell}$ is the set of marked edges and elements using the marking strategy of Section 2 based on the bulk criterion and $\eta_\ell(\tilde\lambda_\ell,\tilde u_\ell)$, then for sufficiently small $\|h_\ell\|_{L^\infty(\Omega)}$ and bulk parameter $\theta > 0$ it holds that

$$|\mathcal{M}_{\tilde\lambda_\ell,\tilde u_\ell}| \lesssim \|\!|u-\tilde u_\ell|\!\|^{-1/s}|u|_{\mathcal{A}_s}^{1/s}.$$

Proof. Let $\mathcal{T}_\varepsilon$ be the smallest partition of $\mathcal{T}_0$ such that

$$\|\!|u-u_\varepsilon|\!\| \le \rho\|\!|u-\tilde u_\ell|\!\|$$

for $0 < \rho < 1/2$. Thus, the definition of $|u|_{\mathcal{A}_s}$ yields

$$|\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \leq \rho^{-1/s} |||u - \tilde{u}_\ell|||^{-1/s} |u|_{\mathcal{A}_s}^{1/s}.$$

Let $\mathcal{T}_{\ell+\varepsilon} := \mathcal{T}_\ell \oplus \mathcal{T}_\varepsilon$ be the smallest common refinement of $\mathcal{T}_\ell$ and $\mathcal{T}_\varepsilon$, then it holds that

$$\begin{aligned} |||u - u_{\ell+\varepsilon}||| &\leq \rho |||u - \tilde{u}_\ell||| \leq \rho |||u - u_\ell||| + \rho |||u_\ell - \tilde{u}_\ell||| \\ &\leq \rho |||u - u_\ell||| + \rho\sqrt{\omega}\eta_\ell(\tilde{\lambda}_\ell, \tilde{u}_\ell) \\ &\leq \left(2\rho^2 |||u - u_\ell|||^2 + 2\rho^2\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell)\right)^{1/2}. \end{aligned}$$

This estimate proofs the following

$$\begin{aligned} &(1-2\rho^2)C_{\text{eff}}^{-2}\eta_\ell^2(\lambda_\ell, u_\ell) - 2\rho^2\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell) \\ &\qquad \leq (1-2\rho^2)|||u - u_\ell|||^2 - 2\rho^2\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell) \\ &\qquad \leq |||u - u_\ell|||^2 - |||u - u_{\ell+\varepsilon}|||^2. \end{aligned}$$

Let $\mathcal{M}_\varepsilon := (\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+\varepsilon}) \cup (\mathcal{E}_\ell \backslash \mathcal{E}_{\ell+\varepsilon})$, then the quasi-orthogonality from Lemma 3.1 and the discrete reliability of Lemma 4.1 yield

$$\begin{aligned} &(1-2\rho^2)C_{\text{eff}}^{-2}\eta_\ell^2(\lambda_\ell, u_\ell) - 2\rho^2\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell) \leq |||u_{\ell+\varepsilon} - u_\ell|||^2 + \lambda\|e_\ell\|^2 \\ &\quad \leq C_{\text{rel}}^2\eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\varepsilon) + \lambda\sigma(h_\ell)^2 C_{\text{rel}}^2\eta_\ell^2(\lambda_\ell, u_\ell), \end{aligned}$$

where $\sigma(h_\ell)$ from Lemma 5.3 tends to zero as $\|h_\ell\|_{L^\infty(\Omega)} \to 0$. Thus,

$$\begin{aligned} &((1-2\rho^2)C_{\text{eff}}^{-2} - \lambda\sigma(h_\ell)^2 C_{\text{rel}}^2)\eta_\ell^2(\lambda_\ell, u_\ell) \\ &\qquad \leq C_{rel}^2\eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\varepsilon) + 2\rho^2\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell). \end{aligned}$$

Lemma 6.1 together with the assumption yields

$$|\eta_\ell(\lambda_\ell, u_\ell) - \eta_\ell(\tilde{\lambda}_\ell, \tilde{u}_\ell)|^2 \leq C\left(|||u_\ell - \tilde{u}_\ell|||^2 + |\lambda_\ell - \tilde{\lambda}_\ell|\right) \leq C\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell).$$

Therefore,

$$\begin{aligned} &((1-2\rho^2)C_{\text{eff}}^{-2} - \lambda\sigma(h_\ell)^2 C_{\text{rel}}^2)2^{-1}\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell) \\ &\quad \leq ((1-2\rho^2)C_{\text{eff}}^{-2} - \lambda\sigma(h_\ell)^2 C_{\text{rel}}^2)\eta_\ell^2(\lambda_\ell, u_\ell) \\ &\qquad + ((1-2\rho^2)C_{\text{eff}}^{-2} - \lambda\sigma(h_\ell)^2 C_{\text{rel}}^2)C\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell) \\ &\quad \leq C_{\text{rel}}^2\eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\varepsilon) + 2\rho^2\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell) \\ &\qquad + ((1-2\rho^2)C_{\text{eff}}^{-2} - \lambda\sigma(h_\ell)^2 C_{\text{rel}}^2)C\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell) \\ &\quad \leq 2C_{\text{rel}}^2\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell; \mathcal{M}_\varepsilon) + 2\rho^2\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell) \\ &\qquad + (2C_{\text{rel}}^2 + (1-2\rho^2)C_{\text{eff}}^{-2} - \lambda\sigma(h_\ell)^2 C_{\text{rel}}^2)C\omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell). \end{aligned}$$

The choice $\sigma(h_\ell) \ll 1$ and $0 < \omega \ll 1$ shows $0 < \theta \leq \Theta \leq 1$ with

$$\Theta := \frac{((1-2\rho^2)C_{\text{eff}}^{-2} - \lambda\sigma(h_\ell)^2 C_{\text{rel}}^2)(2^{-1} - C\omega) - 2(C_{\text{rel}}^2 C + \rho^2)\omega}{2C_{\text{rel}}^2}$$

and hence the bulk criterion for the set $\mathcal{M}_\varepsilon$ based on $\eta_\ell(\tilde\lambda_\ell, \tilde u_\ell)$ is satisfied. Since the set $\mathcal{M}_{\tilde\lambda_\ell,\tilde u_\ell}$ has been chosen with almost minimal cardinality, the overlay estimate leads to

$$|\mathcal{M}_{\tilde\lambda_\ell,\tilde u_\ell}| \lesssim |\mathcal{M}_\varepsilon| \lesssim |\mathcal{T}_{\ell+\varepsilon}| - |\mathcal{T}_\ell| \le |\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \lesssim \|\!|u - \tilde u_\ell|\!\|^{-1/s} |u|_{\mathcal{A}_s}^{1/s}. \square$$

Theorem 6.4. *Suppose that (λ, u) with $u \in \mathcal{A}_s$ is an eigenpair and let (λ_ℓ, u_ℓ) and $(\lambda_{\ell+1}, u_{\ell+1})$ be the corresponding discrete eigenpairs on levels ℓ and $\ell + 1$. Let the iterative approximations $(\tilde\lambda_\ell, \tilde u_\ell)$ on $\mathcal{T}_\ell$ and $(\tilde\lambda_{\ell+1}, \tilde u_{\ell+1})$ on $\mathcal{T}_{\ell+1}$ satisfy*

$$\begin{aligned} \|\!|u_{\ell+1} - \tilde u_{\ell+1}|\!\|^2 + |\lambda_{\ell+1} - \tilde\lambda_{\ell+1}| &\le \omega \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell), \\ \|\!|u_\ell - \tilde u_\ell|\!\|^2 + |\lambda_\ell - \tilde\lambda_\ell| &\le \omega \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell) \end{aligned}$$

for sufficiently small $\omega > 0$. Then, for sufficiently small initial mesh-size $\|h_0\|_{L^\infty(\Omega)}$, the iterative solutions $\tilde\lambda_\ell$ and $\tilde u_\ell$ converge quasi-optimal

$$\|\!|u - \tilde u_\ell|\!\|^2 + |\lambda - \tilde\lambda_\ell| \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s} \lesssim N_\ell^{-2s}.$$

Proof. Lemma 6.3 and Proposition 2.1 yield

$$|\mathcal{T}_L| - |\mathcal{T}_0| \lesssim \sum_{\ell=0}^{L-1} |\mathcal{M}_{\tilde\lambda_\ell,\tilde u_\ell}| \lesssim |u|_{\mathcal{A}_s}^{1/s} \sum_{\ell=0}^{L-1} \|\!|u - \tilde u_\ell|\!\|^{-1/s}.$$

The efficiency estimate of Remark 4.3 and Lemma 6.1 show

$$\begin{aligned} \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell) &\le 2\eta_\ell^2(\lambda_\ell, u_\ell) + 2C \left(\|\!|u_\ell - \tilde u_\ell|\!\|^2 + |\lambda_\ell - \tilde\lambda_\ell| \right) \\ &\le 4C_{\text{eff}}^2 \|\!|u - \tilde u_\ell|\!\|^2 + (2C + 4C_{\text{eff}}^2) \left(\|\!|u_\ell - \tilde u_\ell|\!\|^2 + |\lambda_\ell - \tilde\lambda_\ell| \right) \\ &\le 4C_{\text{eff}}^2 \|\!|u - \tilde u_\ell|\!\|^2 + (2C + 4C_{\text{eff}}^2) \omega \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell). \end{aligned}$$

Hence, for $0 < \omega < (2C + 4C_{\text{eff}}^2)^{-1}$, it holds that

$$\eta_\ell(\tilde\lambda_\ell, \tilde u_\ell) \lesssim \|\!|u - \tilde u_\ell|\!\|.$$

For the other direction, notice that

$$\|\!|u - \tilde u_\ell|\!\| \le \|\!|u - u_\ell|\!\| + \|\!|u_\ell - \tilde u_\ell|\!\| \le C_{\text{rel}} \eta_\ell(\lambda_\ell, u_\ell) + \sqrt{\omega}\, \eta_\ell(\tilde\lambda_\ell, \tilde u_\ell),$$

implies

$$\begin{aligned} \|\!|u - \tilde u_\ell|\!\|^2 &\le 2C_{\text{rel}}^2 \eta_\ell^2(\lambda_\ell, u_\ell) + 2\omega \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell) \\ &\le \left(4C_{\text{rel}}^2 + 4C_{\text{rel}}^2 C\omega + 2\omega\right) \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell). \end{aligned}$$

Thus,

$$\|\!|u - \tilde u_\ell|\!\|^{-1/s} \lesssim \left(\gamma \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell) + \|\!|u - \tilde u_\ell|\!\|^2 \right)^{-1/(2s)}.$$

Lemma 6.2 leads to

$$\begin{aligned} &\left(\gamma \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell) + \|\!|u - \tilde u_\ell|\!\|^2 \right)^{-1/(2s)} \\ &\qquad \le \nu^{1/(2s)} \left(\gamma \eta_{\ell+1}^2(\tilde\lambda_{\ell+1}, \tilde u_{\ell+1}) + \|\!|u - \tilde u_{\ell+1}|\!\|^2 \right)^{-1/(2s)}. \end{aligned}$$

A geometric series argument yields

$$|\mathcal{T}_L| - |\mathcal{T}_0| \lesssim |u|_{\mathcal{A}_s}^{1/s} \left(\gamma\eta_L^2(\tilde{\lambda}_L, \tilde{u}_L) + \|u - \tilde{u}_L\|^2\right)^{-1/(2s)} \sum_{\ell=1}^{L} \nu^{\ell/(2s)}$$
$$\lesssim |u|_{\mathcal{A}_s}^{1/s}(1-\nu^{1/(2s)})^{-1}\|u - \tilde{u}_L\|^{-1/s}.$$

Since

$$|\lambda - \tilde{\lambda}_\ell| \le |\lambda - \lambda_\ell| + |\lambda_\ell - \tilde{\lambda}_\ell| \le |\lambda - \lambda_\ell| + \omega\eta_\ell^2(\tilde{\lambda}_\ell, \tilde{u}_\ell)$$
$$\le |\lambda - \lambda_\ell| + 2\omega C_{\mathrm{eff}}^2 \|u - u_\ell\|^2 + 2\omega C\left(\|u_\ell - \tilde{u}_\ell\|^2 + |\lambda_\ell - \tilde{\lambda}_\ell|\right),$$

it holds that

$$|\lambda - \tilde{\lambda}_\ell| \lesssim |\lambda - \lambda_\ell| + \|u - u_\ell\|^2 + \|u - \tilde{u}_\ell\|^2$$

for sufficiently small $\omega > 0$. Thus, Theorem 5.4 proves $|\lambda - \tilde{\lambda}_\ell| \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s}$ and Euler's formula shows $(|\mathcal{T}_\ell| - |\mathcal{T}_0|) \approx N_\ell$. $\square$

The choice of the bulk parameter θ is asymptotically independent of λ and depends on the reliability and efficiency constants as well as on ω. The choice of the parameter ω in particular depends on the constant of Lemma 6.1 and therefore on the initial mesh-size $\|h_0\|_{L^\infty(\Omega)}$ and the initial guess $\tilde{\lambda}_0$. Empirical choices of these parameters for some numerical examples are discussed in Section 8.

7. Quasi-Optimal Complexity

In this section the proof of the quasi-optimal computational complexity of the AFEMES is presented. The proposed algorithm combines the AFEM with some iterative algebraic eigenvalue solver. In order to prove overall asymptotic quasi-optimal complexity, the iterative solver needs to have a constant contraction factor independent of the size of the discrete problem and to be of linear complexity. In other words for any $\varepsilon > 0$ the algorithm LAES has to compute an iterative solution of the algebraic eigenvalue problem $(\tilde{\lambda}_{\ell,m}, \tilde{u}_{\ell,m})$ from an initial guess $(\tilde{\lambda}_{\ell,0}, \tilde{u}_{\ell,0})$ such that

$$\|u_\ell - \tilde{u}_{\ell,m}\|^2 + |\lambda_\ell - \tilde{\lambda}_{\ell,m}| \le \varepsilon^2$$

in at most, up to a generic multiplicative constant,

$$\max\left\{1, \log(\varepsilon^{-1}\|u_\ell - \tilde{u}_{\ell,0}\|)\right\} \times N_\ell$$

arithmetic operations.

Theorem 7.1. *Let (λ, u) with $u \in \mathcal{A}_s$ be an eigenpair. Then for sufficiently small $\|h_0\|_{L^\infty(\Omega)}$, $0 < \theta \ll 1$ and $0 < \omega \ll 1$, the algorithm AFEMES computes from a coarse triangulation $\mathcal{T}_0$ and an initial guess*

$(\tilde\lambda_0, \tilde u_0)$ *sufficiently close to* (λ, u) *a sequence of triangulations* $(\mathcal{T}_\ell)_\ell$ *and corresponding approximated eigenpairs* $(\tilde\lambda_\ell, \tilde u_\ell)$ *such that*

$$|\!|\!| u - \tilde u_\ell |\!|\!|^2 + |\lambda - \tilde\lambda_\ell| \lesssim \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell) \lesssim t_\ell^{-2s}$$

where t_ℓ *denotes the computational costs, i.e., the CPU-time.*

Proof. First it is shown that the while-loop is terminating after a finite number of iterations on each level. Remark that the while-loop is executed at least once and that in further runs it holds that

$$|\!|\!| u_\ell - \tilde u_\ell |\!|\!|^2 + |\lambda_\ell - \tilde\lambda_\ell| \le \delta_\ell^2$$

because of the previous calls of LAES. Using Lemma 6.1 yields

$$\begin{aligned}\sqrt{\omega}\eta_\ell(\tilde\lambda_\ell, \tilde u_\ell) &\ge \sqrt{\omega}\eta_\ell(\lambda_\ell, u_\ell) - \sqrt{\omega}|\eta_\ell(\tilde\lambda_\ell, \tilde u_\ell) - \eta_\ell(\lambda_\ell, u_\ell)| \\ &\ge \sqrt{\omega}\eta_\ell(\lambda_\ell, u_\ell) - \sqrt{\omega C}\left(|\!|\!| u_\ell - \tilde u_\ell |\!|\!|^2 + |\lambda_\ell - \tilde\lambda_\ell|\right)^{1/2} \\ &\ge \sqrt{\omega}\eta_\ell(\lambda_\ell, u_\ell) - \delta_\ell\sqrt{\omega C}.\end{aligned}$$

Therefore, the while-loop is at least terminated on the level ℓ if

$$\delta_\ell \le \frac{\sqrt{\omega}\eta_\ell(\lambda_\ell, u_\ell)}{1 + \sqrt{\omega C}}.$$

Due to the geometric decrease of δ_ℓ this is achieved in a bounded constant number of steps for all levels ℓ. The choice of the initial value for δ_ℓ on each level ℓ and the fact that after the while-loop terminates $\delta_\ell \le \sqrt{\omega}\eta_\ell(\tilde\lambda_\ell, \tilde u_\ell)$ shows that the conditions of Theorem 6.4 are satisfied. Thus, the convergence of

$$|\!|\!| u - \tilde u_\ell |\!|\!| \lesssim N_\ell^{-s}$$

is quasi-optimal. Moreover the proof of Theorem 6.4 shows

$$|\!|\!| u - \tilde u_\ell |\!|\!| \lesssim \eta_\ell(\tilde\lambda_\ell, \tilde u_\ell) \lesssim |\!|\!| u - \tilde u_\ell |\!|\!| \tag{7.1}$$

for sufficiently small $\omega > 0$. For the eigenvalue error it holds that

$$\begin{aligned}|\lambda - \tilde\lambda_\ell| &\le |\lambda - \lambda_\ell| + |\lambda_\ell - \tilde\lambda_\ell| \le C_{\mathrm{rel}}^2\eta_\ell^2(\lambda_\ell, u_\ell) + \delta_\ell^2 \\ &\le 2C_{\mathrm{rel}}^2\eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell) + (2C_{\mathrm{rel}}^2 C + 1)\delta_\ell^2 \\ &\le (2C_{\mathrm{rel}}^2 + (2C_{\mathrm{rel}}^2 C + 1)\omega)\eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell).\end{aligned}$$

Hence,

$$|\!|\!| u - \tilde u_\ell |\!|\!|^2 + |\lambda - \tilde\lambda_\ell| \lesssim \eta_\ell^2(\tilde\lambda_\ell, \tilde u_\ell) \lesssim N_\ell^{-2s}.$$

Because of the quasi-optimal convergence and the finitely many number of iterations of the while-loop, it remains to show that Mark, Refine and LAES are of linear computational complexity. An quasi-optimal algorithm for Mark and Refine can be found in [Ste07]. In the first execution of the while-loop, except for the first level for which the costs can be bounded by a constant separately, before LAES is executed, it holds that

$$|||u_\ell - \tilde{u}_\ell||| = |||u_\ell - \tilde{u}_{\ell-1}||| \leq |||u - u_\ell||| + |||u - \tilde{u}_{\ell-1}|||.$$

Lemma 5.3 reads

$$|||u - u_\ell|||^2 \leq 2\varrho \left(\gamma C_{\mathrm{eff}}^2 + 1\right) \left(|||u - \tilde{u}_{\ell-1}|||^2 + |||u_{\ell-1} - \tilde{u}_{\ell-1}|||^2\right).$$

Thus, (7.1), the termination of the while-loop on the previous level $\ell - 1$ and the initialisation of δ_ℓ, yield

$$|||u_\ell - \tilde{u}_\ell||| \lesssim \eta_{\ell-1}(\tilde{\lambda}_{\ell-1}, \tilde{u}_{\ell-1}) + \delta_{\ell-1} \lesssim \eta_{\ell-1}(\tilde{\lambda}_{\ell-1}, \tilde{u}_{\ell-1}) \lesssim \delta_\ell.$$

If it is not the first evaluation of the while-loop, then $|||u_\ell - \tilde{u}_\ell||| \leq 2\delta_\ell$ because of the previous call of LAES. Thus, before any call of LAES for $\ell > 0$ it holds that $|||u_\ell - \tilde{u}_\ell||| \lesssim \delta_\ell$ which shows that LAES can be executed in linear time $t_\ell \approx N_\ell$. □

8. Numerical Experiments

The numerical experiments for $n = 2, 3$ show asymptotic quasi-optimal computational complexity of the AFEMES for linear P_1 up to fourth order P_4 finite elements. The AFEMES is implemented in MATLAB for $n = 2, 3$. The aim of the implementation is not to be the fastest one but to verify the asymptotic quasi-optimal complexity of the AFEMES in numerical experiments. The implementation of the AFEM follows the ideas of [ACF99] and in an enhanced way of [FPW11]. The mesh refinement for $n = 3$ is based on a bisection type strategy [AMP00]. The quasi-optimal complexity is measured by plotting the number of seconds a computation needs to finish on a single CPU-core of a AMD-Opteron processor 8378 at 2.4 GHz and with 128GB ram versus the eigenvalue error or the a posteriori error estimator. The numerical experiments compare the computational performance of different algebraic eigenvalue solvers in combination with the asymptotic quasi-optimal AFEMES. These are the ARPACK solver as implemented in the MATLAB function "eigs", the PINVIT with one multigrid V-cycle as preconditioner, and the LOBPCG implementation in MATLAB [Kny10] using also one multigrid V-cycle as preconditioner. The reference algorithm to solve the eigenvalue problem only once on an arbitrary uniform refined mesh with ARPACK (eigs) will be denoted by "ARPACK uniform" and the measured time involves the assembly of the matrices, the time to solve the algebraic eigenvalue problem, and the calculation of the a posteriori error estimator. The standard AFEM algorithm with the ARPACK solver for default tolerance in the range of the machine precision is denoted by "ARPACK AFEM". For the V-cycle geometric multigrid preconditioner global Richardson smoothing (n=2) and Jacobi smoothing (n=3)

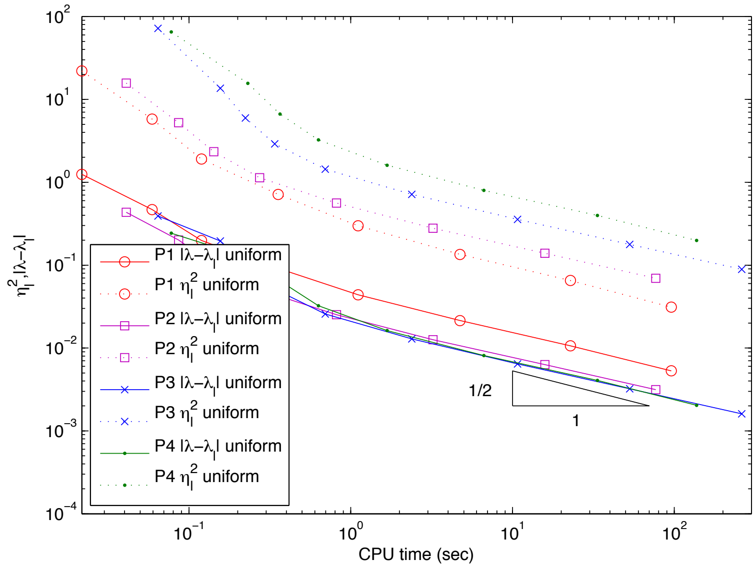

FIGURE 8.1. Eigenvalue errors and estimated errors on the slit domain for uniform meshes for $\theta = 1$ and $\omega = 10^{-3}$.

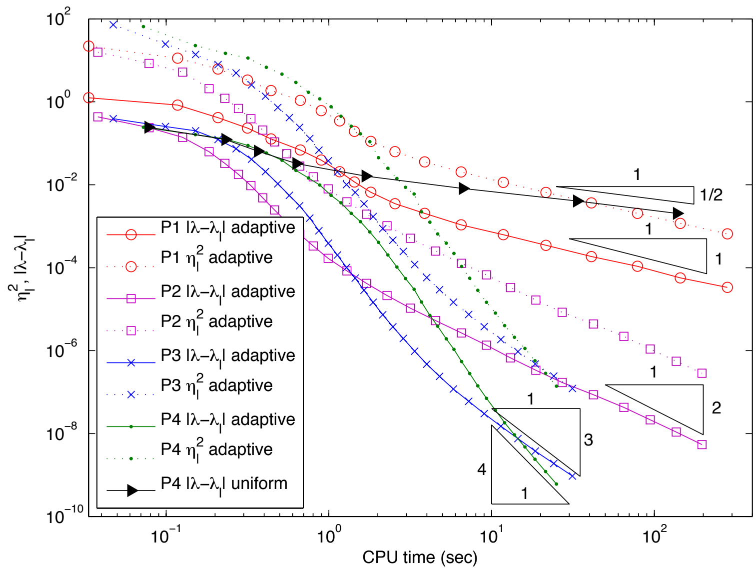

FIGURE 8.2. Eigenvalue errors and estimated errors on the slit domain for adaptive meshes for $\theta = 0.5$ and $\omega = 10^{-3}$.

with empirical optimal scaling factors independently of h_ℓ are used. All eigensolvers start from the same initial guess $x_0 = (1, \ldots, 1)^t$ on $\mathcal{T}_0$.

Example 8.1. Consider the two-dimensional model eigenvalue problem (1.1) on the slit domain $\Omega = ((-1,1) \times (-1,1)) \backslash ([0,1] \times \{0\})$ with

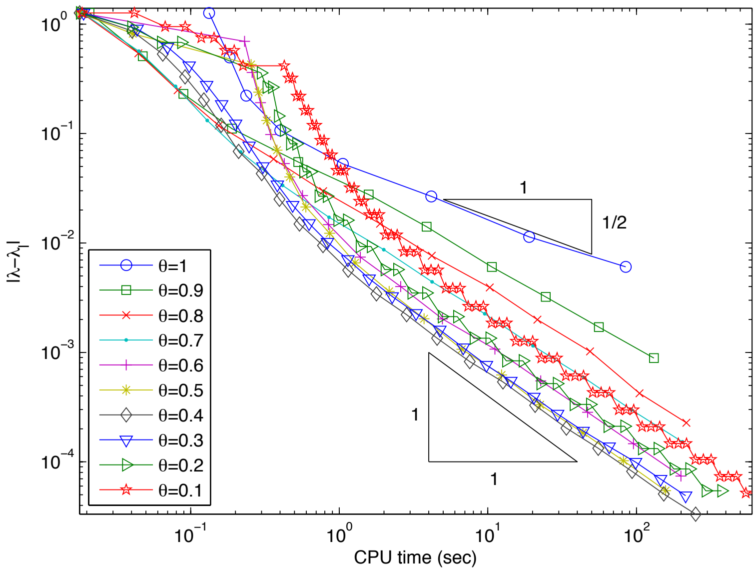

FIGURE 8.3. Eigenvalue errors for different values of $\theta = 0.1, \ldots, 1$ on the slit domain for P_1 and $\omega = 10^{-1}$.

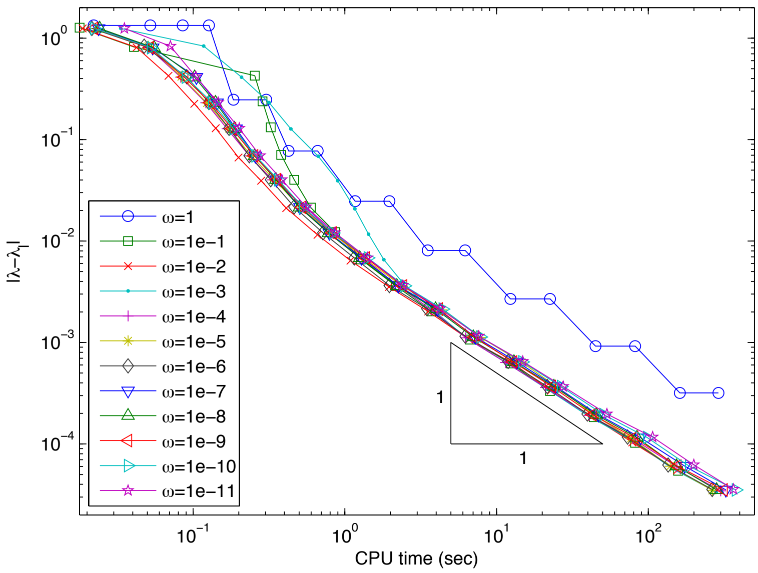

FIGURE 8.4. Eigenvalue errors for different orders of magnitude of ω on the slit domain for P_1 and $\theta = 0.5$.

tip at the origin. An approximation of the smallest eigenvalue with high accuracy is computed with higher-order finite elements on fine meshes

$$\lambda = 8.3713297112,$$

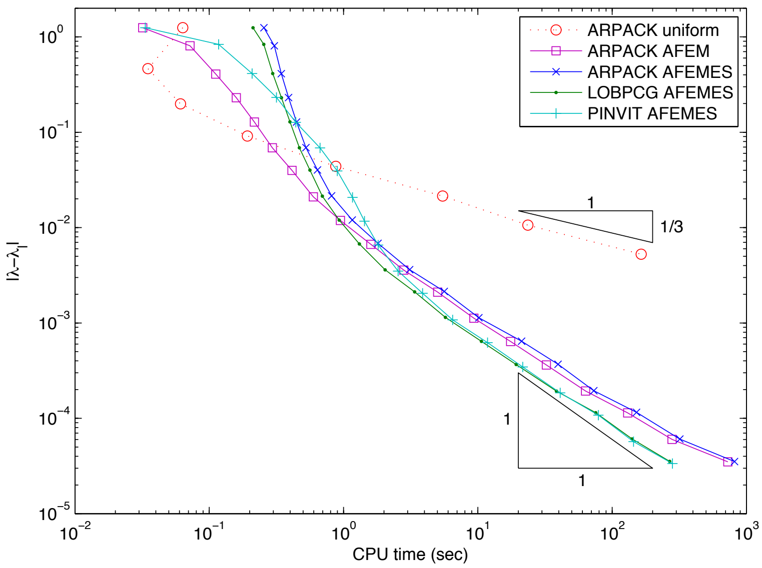

FIGURE 8.5. Eigenvalue errors for different algebraic solvers on the slit domain for P_1, $\theta = 0.5$ and $\omega = 10^{-3}$.

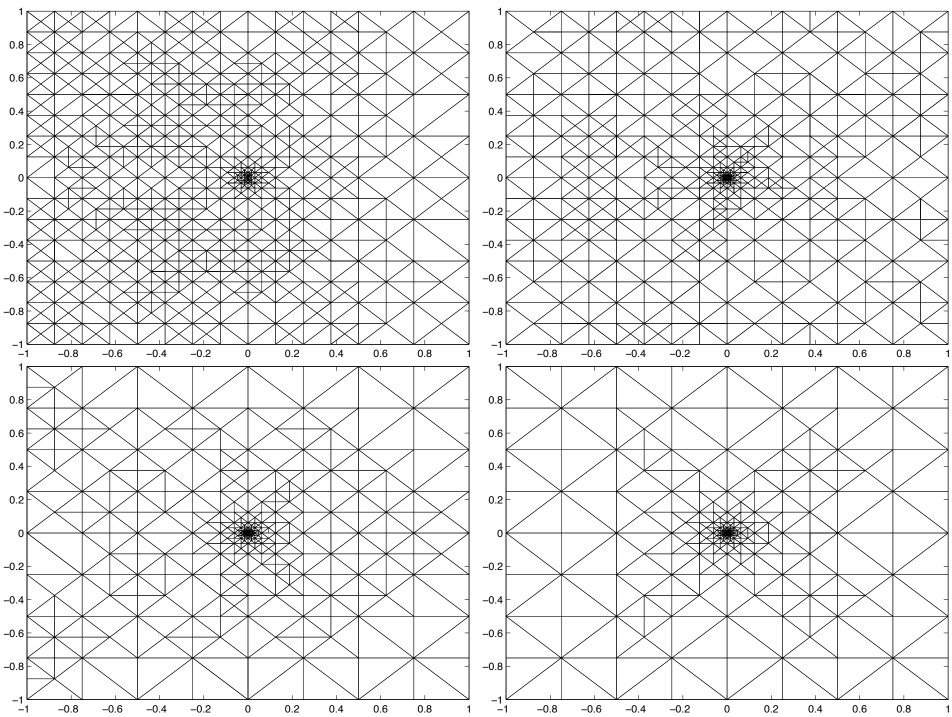

FIGURE 8.6. Adaptive refined meshes for P_k, $k = 1, 2, 3, 4$ (top left to bottom right), with about 500 nodes.

where the authors believe that all digits except the last one are exact. Note that for uniform meshes and $n = 2$ it holds that $N_\ell^{-1/2} \approx h_\ell$. Thus, for P_k, $k = 1, \ldots, 4$, convergence rates of $\mathcal{O}(t_\ell^{-k})$ are optimal for the eigenvalue error of the AFEMES. For the following experiments the PINVIT algebraic eigenvalue solver is used and the parameters are $\theta = 0.5$ and $\omega = 10^{-3}$. The algorithm stops when a tolerance of 10^{-9} in the eigenvalue error is reached due to the accuracy of the reference eigenvalue or the number of degrees of freedom exceeds 10^6. In Figure 8.1 it is shown that the error estimator is numerically reliable and efficient for uniform meshes but these meshes result in suboptimal convergence rates of about $\mathcal{O}(t_\ell^{-1/2})$ due to the singularity at the origin. Note that the same rates are obtained for N_ℓ instead of t_ℓ. Thus the computational costs are quasi-optimal for uniform meshes. In contrast using adaptive refinement results in experimental optimal convergence rates of $\mathcal{O}(t_\ell^{-k})$, $k = 1, \ldots, 4$, as shown in Figure 8.2 and the error estimator shows to be numerically reliable and efficient.

The asymptotic quasi-optimal AFEMES involves two parameters $\omega > 0$ and $0 < \theta \leq 1$ which have to be sufficiently small. Figure 8.3 shows a numerical strong dependency of the size of the eigenvalue error on θ for $\omega = 0.1$. For $\theta = 1$ uniform refinement results in suboptimal convergence rates. Smaller values lead to optimal convergence rates and down to $\theta = 0.4$ the error decreases. Then for even smaller values for θ, the convergence rates are numerically optimal, but $\theta \ll 1$ leads to more iterations of the algebraic eigenvalue solver and thus to more computational work. Note that for values $\theta \leq 0.2$ the algorithm marks too few elements such that the algorithm accepts the value of the previous level as approximation for the next one from time to time. This results in the effect that those convergence plots look like a stair. Different values for ω lead almost all (asymptotically) to optimal convergence rates as depicted in Figure 8.4. Only the value $\omega = 1$ is not small enough. The computational costs for smaller values only moderately increases.

The asymptotic quasi-optimal complexity of AFEMES depends on the choice of the algebraic eigenvalue solver. Figure 8.5 shows that the AFEMES is in the long term faster than one solve of ARPACK on an uniform mesh for linear P_1 finite elements (“ARPACK uniform”). The results obtained with the multigrid preconditioned PINVIT and LOBPCG solver show asymptotic quasi-optimal computational complexity. The AFEMES shows larger computational time for ARPACK than for PINVIT and LOBPCG due to the use of matrix factorisations instead of multigrid and the convergence rate deteriorates for larger number of unknowns because the time for the matrix factorisations dominates the computational costs. PINVIT and LOBPCG with matrix factorisations would lead to similar large computational costs.

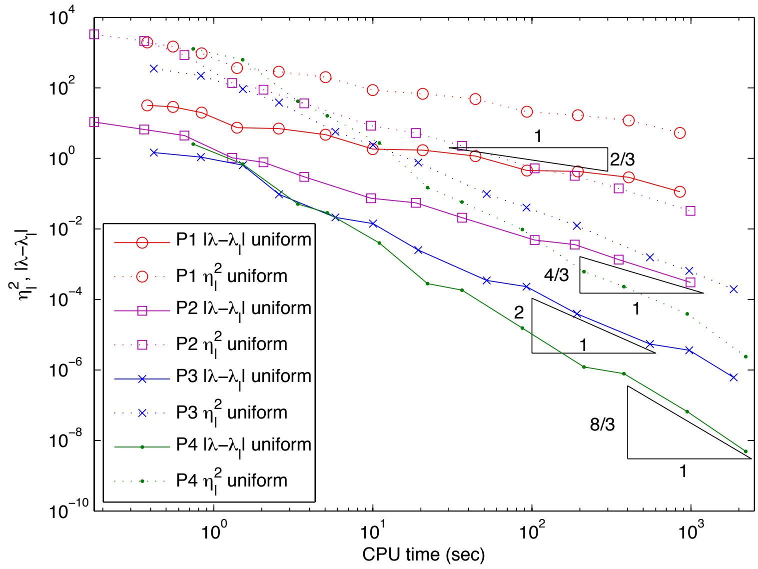

FIGURE 8.7. Eigenvalue errors and estimated errors for the 11th eigenvalue on the cube for uniform meshes with $\theta = 1$ and $\omega = 10^{-4}$.

Different adaptive refined meshes for P_k, $k = 1, 2, 3, 4$, with about 500 nodes are displayed in Figure 8.6. Note that the meshes are strongly refined towards the corner singularity at the origin.

Example 8.2. Consider the three-dimensional model eigenvalue problem (1.1) on the cube $\Omega = (0, 1) \times (0, 1) \times (0, 1)$ for the 11th eigenvalue $\lambda_{11} = 12\pi^2$ which is simple. Note that for uniform meshes and $n = 3$ it holds that $N_\ell^{-1/3} \approx h_\ell$. Thus, for P_k, $k = 1, \dots, 4$, convergence rates of $\mathcal{O}(t_\ell^{-2k/3})$ for the eigenvalue error are optimal. The asymptotic quasi-optimal AFEMES is stopped when 10^6 degrees of freedom are reached because of hardware limitations. Figure 8.7 shows optimal convergence rates for uniform meshes of $\mathcal{O}(t_\ell^{-2k/3})$, $k = 1, \dots, 4$, computing the 11th eigenvalue with the AFEMES using the LOBPCG solver. The 11th eigenvalue is computed without any shift but from a subspace iteration.

Example 8.3. Consider the three-dimensional model eigenvalue problem (1.1) on the L-shaped domain $\Omega = ((-1, 1)^3)\backslash([0, 1]^2 \times [-1, 1])$. The first eigenvalue is the sum of π^2 and the first eigenvalue of the two-dimensional L-shaped domain with approximation 9.6397238440219 [BT05],

$$\lambda = 19.509328245111$$

(all displayed digits are correct). The asymptotic quasi-optimal AFEMES is stopped when 10^6 degrees of freedom are reached. In this non-convex

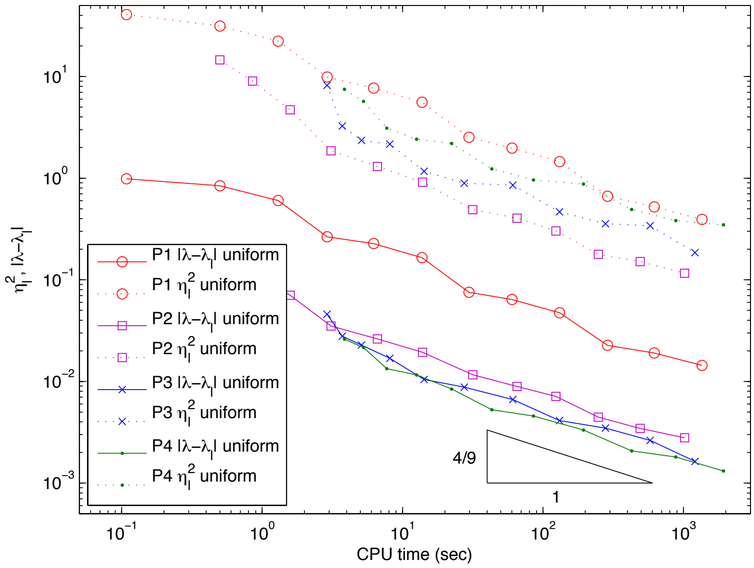

FIGURE 8.8. Eigenvalue errors and estimated errors on the three-dimensional L-shaped domain for uniform meshes with $\theta = 1$ and $\omega = 10^{-3}$.

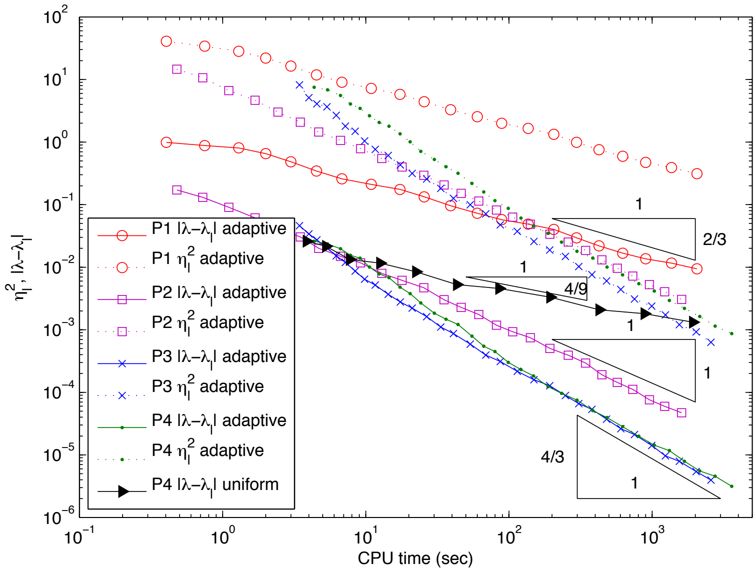

FIGURE 8.9. Eigenvalue errors and estimated errors on the three-dimensional L-shaped domain for adaptive meshes with $\theta = 0.5$ and $\omega = 10^{-3}$.

three-dimensional example uniform refinement results in suboptimal convergence rates $\mathcal{O}(t_\ell^{-4/9})$ as shown in Figure 8.8. Note that the same rates are obtained for N_ℓ. Note that the AFEMES is based on isotropic refinement and therefore cannot create anisotropic meshes. Thus, we

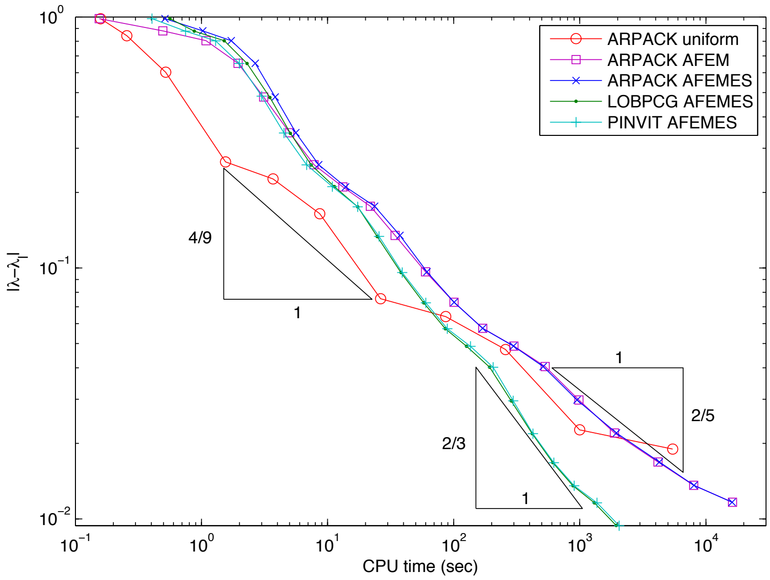

FIGURE 8.10. Eigenvalue errors for the first eigenvalue and different algebraic solvers on the L-shaped domain for P_1, $\theta = 0.5$ and $\omega = 10^{-3}$.

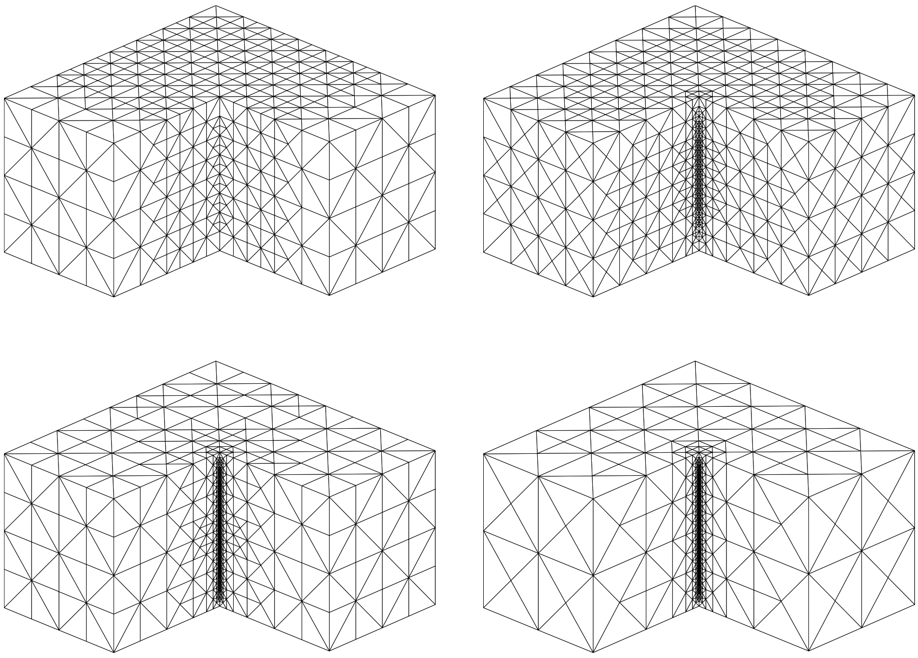

FIGURE 8.11. Adaptive refined meshes for P_k, $k = 1, 2, 3, 4$ (top left to bottom right), with about 3000 nodes.

do not expect similar optimal rates for adaptively refined meshes as for the two-dimensional case due to the edge singularity. This is no contradiction to the theory because the definition of the approximation spaces involves only all possible isotropic and no anisotropic refinements. For isotropic refinement for domains with edges [Ape99, Section 4.2] states the optimal relation $N_\ell \approx h_\ell^{-3}$ for linear P_1 and the suboptimal relations $N_\ell \approx h_\ell^{-3}|\ln h_\ell|$ for P_2, $N_\ell \approx h_\ell^{-2/9}$ for P_3 and $N_\ell \approx h_\ell^{-1/6}$ for P_4 finite elements. Therefore, isotropic meshes are not optimal for P_k, $k \geq 2$ and convergence rates of $\mathcal{O}(t_\ell^{-2/3})$ for P_1, rates slightly less that than $\mathcal{O}(t_\ell^{-4/3})$ for P_2 and rates of $\mathcal{O}(t_\ell^{-4/3})$ for P_3 and P_4 are the best possible for isotropic refinements. Figure 8.9 shows that the asymptotic quasi-optimal algorithm AFEMES with the PINVIT solver, $\theta = 0.5$ and $\omega = 10^{-3}$ leads to these rates and that the error estimator is reliable and efficient for P_k, $k = 1, \ldots, 4$.

The computational time for the complete AFEMES with linear finite elements is faster compared to one uniform solve with ARPACK as shown in Figure 8.10 for larger degrees of freedom. For smaller numbers of unknowns the computational costs for the assembly of the matrices and the calculation of the error estimator dominates and the convergence rate of ARPACK uniform is the best possible for uniform meshes but deteriorates for larger systems because of the computation of the matrix factorisations. Since the computational costs for the matrix factorisations get more severe for $n = 3$ and larger number of degrees of freedom, this example shows that ARPACK with matrix factorisations leads to suboptimal computational complexity even for adaptively refined meshes. The PINVIT and the LOBPCG solver with multigrid preconditioner lead to almost the same quasi-optimal complexity. Note that both graphs almost cover each other.

Different adaptive refined meshes for P_k, $k = 1, 2, 3, 4$, with about 3000 nodes are displayed in Figure 8.11. The meshes are strongly refined towards the edge singularity for the higher-order methods.

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(C. Carstensen) Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany; Department of Computational Science and Engineering, Yonsei University, 120–749 Seoul, Korea.
E-mail address: cc@mathematik.hu-berlin.de

(J. Gedicke) Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany.
E-mail address: gedicke@mathematik.hu-berlin.de