Enclosure Theorems for Extremals of Elliptic Parametric Functionals

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Abstract

In the following paper we study parametric functionals. First we introduce a generalized mean curvature (so called F*-mean curvature*). This enables us to describe extremals of parametric funcionals as surfaces of prescribed F-mean curvature. Furthermore we give a differential equation for arbitrary immersions generalizing $\Delta X = HN$ and apply this equation to surfaces of vanishing and prescribed F-mean curvature. Especially we prove non-existence results for such surfaces generalizing Theorems by Hildebrandt and Dierkes [3], [6].

Introduction

Consider an n-dimensional minimal immersion X in $\mathbb{R}^{n+1}$. If the boundary of X has parts in the two disjoint connected components of a cone K congruent to

$$K^n = \{(y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} \,|\, y_1^2 + \ldots + y_n^2 - (n-1)\, y_{n+1}^2 < 0\}$$

and if the boundary is fully contained in K, then X cannot be connected. This is a result of Dierkes [3]. The idea of the proof is due to Hildebrandt (see [6], where he obtains this result in the case $n = 2$) and is based on the maximum principle for subharmonic functions.
Indeed, using the (minimal surface-) equation $\Delta X = 0$, where Δ is the Laplace-Beltrami operator, one shows for the polynomial $t(y) = y_1^2 + \ldots + y_n^2 - (n-1)\, y_{n+1}^2$ the inequality

$$\Delta\, [t(X)] \geq 0\ .$$

The aim of this paper is to generalize the results of Dierkes to extremals of parametric functionals. Consider a C^2-immersion $X : M \to \mathbb{R}^{n+1}$ of an n-dimensional oriented manifold M into $\mathbb{R}^{n+1}$ with normal N and the induced area element dA. Then a parametric functional $\mathcal{F}$ is defined by

$$\mathcal{F}(X) := \int_M F(X,N)\,dA\ , \tag{1}$$

where $F : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is smooth and satisfies a homogeneity condition in the second entry, i.e.

$$F(y,tz) = t\,F(y,z)$$

for all $(y,z) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} - \{0\})$ and $t > 0$. This includes, for example, minimal surfaces and surfaces of prescribed mean curvature.
At first, we give an appropriate form of the Euler equation of (1). B. White considered in [9] parametric functionals (also for surfaces of higher codimension), expressing the Euler equation in terms of the non-parametric integrand associated with F. In this paper we only deal with the parametric integrand F itself. In fact, we prove

Theorem 1 *Let $X : M \to \mathbb{R}^{n+1}$ be an immersion of codimension 1 and F be an integrand of class $C^2(\mathbb{R}^{n+1} \times S^n)$[1]. If X is an extremal of $\mathcal{F}$, we have for the F-mean curvature H_F of X:*

$$H_F = \sum_{i=1}^{n+1} F_{y_i z_i}(X,N)\ .$$

The F-mean curvature is a linear combination (depending on F) of the eigenvalues of the shape operator S (see Definition 1.3). For the area integrand $F(y,z) = |z|$ one regains the classical mean curvature $H = -\mathrm{tr}\,S$.
In the special case of integrands with $F_y = 0$, this is a result of K. Räwer [7].
The next step consists of a generalization of the equation $\Delta X = HN$, where X is assumed to be an immersion. The following theorem shows that it is possible to give a similar equation adapted to the integrand F:

Theorem 2 *Let $F \in C^3(\mathbb{R}^{n+1} \times S^n)$ be a parametric integrand and $X \in C^2(M, \mathbb{R}^{n+1})$ be a C^2-immersion. Then there is a second order differential operator Θ_F such that the F-mean curvature is expressed by*

$$\Theta_F X = H_F N\ . \tag{2}$$

Furthermore we have the identity $\Theta_F = \Delta$ in the special case of the area integrand $F(y,z) = |z|$.

[1] We will use the convention that a parametric integrand F is of class $C^k(\mathbb{R}^{n+1} \times S^n)$ if $F \in C^0(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \cap C^k(\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} - \{0\}))$.

It turns out that the operator Θ_F is elliptic if the integrand F itself is elliptic (see Definition 2.2).
Therefore equation (2) enables us in the elliptic case to prove a-priori estimates as for example the convex-hull property for surfaces of vanishing F-mean curvature (see Theorem 2.3 and Theorem 2.4).
Furthermore it is possible to apply the methods of Hildebrandt and Dierkes to prove the main result Theorem 2.5. In this Theorem we give a sufficient condition for immersed surfaces X for the validity of

$$\Theta_F\,[t(X)] \geq 0\ ,$$

where t is given by $t(y) = y_1^2 + \ldots + y_n^2 - (n-1)\,b\,y_{n+1}^2$ and b is an appropriate positive number. As a consequence of Theorem 2.5 we obtain a nonexistence result for $\mathcal{F}$-extremals (Corollary 2.7).
Corollary 2.8 treats surfaces of vanishing F-mean curvature. In the proof of this Corollary one encounters the problem that the F-mean curvature is not invariant under a rotation of the surface. Therefore, we modify the integrand by kind of a "translation" and "rotation", i.e. for $p \in \mathbb{R}^{n+1}$ and $Q \in O(n)$ we consider

$$\tilde{F}(\tilde{y}, \tilde{z}) := F(Q^T\tilde{y} + p, Q^T\tilde{z})\ .$$

The translated and rotated surface then has vanishing $\tilde{F}$-mean curvature. Thus, we can assume a normed geometrical situation. Now we only need to apply the maximum principle. Our results contain Theorems 2 and 3 of [3] as special cases.
For further remarks on the literature we refer to [6] and [3].

1 First variation and a differential equation for immersions

Let $M^n = M$ be an orientable manifold of dimension n and $X : M \to \mathbb{R}^{n+1}$ be an immersion of class C^2. We obtain the induced metric

$$g(V, W) := \langle DX(V), DX(W)\rangle \qquad \text{for } V, W \in T_pM$$

and the normal mapping

$$N : M \to S^n\ .$$

Consider now the parametric variational integral

$$\mathcal{F}(X) = \int_M F(X, N)\, dA\ , \tag{3}$$

where dA is the volume form. The integrand F is assumed to satisfy the homogeneity condition

$$F(y, tz) = t\, F(y, z), \tag{4}$$

for all $(y, z) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $t > 0$. In the following we say that F is a *parametric integrand* if (4) is valid. Furthermore, it will always be assumed that $F \in C^2(\mathbb{R}^{n+1} \times S^n)$.
At first we compute the first variation of X in the direction Ξ, i.e. consider $X_\epsilon = X + Z(\epsilon, \cdot)$, where $Z : (-\epsilon_0, \epsilon_0) \times M \to \mathbb{R}^{n+1}$, $\epsilon_0 > 0$, has compact support and

$$\frac{\partial}{\partial \epsilon} Z(0, \cdot\,) = \Xi \, .$$

By the area formula (see [8, p.47]) the first variation is given by

$$\delta \mathcal{F}(X, \Xi) = \frac{d}{d\epsilon}_{|\epsilon=0} \int_M F(X_\epsilon, N_\epsilon)\, J\Phi_\epsilon \, dA \, , \tag{5}$$

where $J\Phi_\epsilon = 1 + \epsilon \operatorname{div} \Xi + o(\epsilon)$.

Lemma 1.1 *The normal mapping N_ϵ of X_ϵ satisfies*

$$\frac{\partial}{\partial \epsilon}_{|\epsilon=0} N_\epsilon = -DX(\operatorname{grad} \varphi) + DN(V) \, ,$$

where $V \in T_pM$ and φ are defined by the decomposition of Ξ in tangential and normal part:

$$\Xi = \Xi^{\tan} + \Xi^N =: DX(V) + \varphi N \, . \tag{6}$$

Proof. Starting with the observation that $\partial_\epsilon N_\epsilon$ is a tangent vector of X_ϵ, we obtain

$$\partial_\epsilon N_\epsilon = -g_\epsilon^{kj} \langle N_\epsilon, \partial_j \partial_\epsilon X_\epsilon \rangle \, \partial_k X_\epsilon \, ,$$

leading to

$$\frac{\partial}{\partial \epsilon}_{|\epsilon=0} N_\epsilon = -g^{kj} \langle N, \partial_j \Xi \rangle \, \partial_k X \, .$$

Here and in the following, the usual summation convention is used. We derive from (6)

$$\begin{aligned} \frac{\partial}{\partial \epsilon}_{|\epsilon=0} N_\epsilon &= -g^{kj} \partial_j \varphi \, \partial_k X - g^{kj} \langle N, \partial_j DX(V) \rangle \, \partial_k X \\ &= -DX(\operatorname{grad} \varphi) + g^{kj} \, g \left(S \frac{\partial}{\partial x^j}, V \right) \partial_k X \\ &= -DX(\operatorname{grad} \varphi) + DX(SV) \, , \end{aligned}$$

with the shape-operator $S : T_pM \to T_pM$ defined by $DN =: DX \circ S$.

$\square$

The following formula of integration by parts for non-tangential vector fields is proved in [2].

Lemma 1.2 *Let $\varphi \in C_0^\infty(M)$, $Z \in C^1(M, \mathbb{R}^{n+1})$, then*

$$\int_M \langle Z, DX(\operatorname{grad}\varphi)\rangle \, dA = -\int_M \varphi \left[\operatorname{div} Z + \langle Z, N\rangle \, H\right] dA \, ,$$

where $H := -\operatorname{tr} S$ denotes the mean curvature of X.

Using Lemma 1.1 and (5) we may derive a formula for the first variation:

$$\begin{aligned} \delta\mathcal{F}(X,\Xi) &= \int_M \langle F_y(X,N), \Xi\rangle + \langle F_z(X,N), DX(SV - \operatorname{grad}\varphi)\rangle \\ &\qquad + F(X,N)\operatorname{div}\Xi \, dA \\ &= \int_M \langle F_y, \Xi\rangle + \langle F_z, DX(SV - \operatorname{grad}\varphi)\rangle + F \operatorname{div} V - \varphi \, F \, H \, dA \, , \end{aligned}$$

taking into consideration the fact that $\operatorname{div} N = -H$. Partial integration (see Lemma 1.2) leads to

$$\begin{aligned} \delta\mathcal{F}(X,\Xi) &= \int_M \langle F_y, \Xi\rangle + \langle F_z, DX(SV)\rangle + \varphi \operatorname{div} F_z \\ &\qquad + \varphi \, \langle F_z, N\rangle \, H - g(\operatorname{grad} F, V) - \varphi \, F \, H \, dA \\ &= \int_M \langle F_y, \Xi\rangle + \langle F_z, DX(SV)\rangle \\ &\qquad + \varphi \operatorname{div} F_z - g(\operatorname{grad} F, V) \, dA \, . \end{aligned}$$

The last equation is valid because of the homogeneity of F, which implies $F_z(y,z)\, z = F(y,z)$. The term $g(\operatorname{grad} F, V)$ can be expressed as follows:

$$\begin{aligned} g(\operatorname{grad} F, V) &= \langle DX(\operatorname{grad} F), DX(V)\rangle \\ &= \left\langle g^{ij} \langle F_y, \partial_i X\rangle \, \partial_j X, DX(V)\right\rangle + \left\langle g^{ij} \langle F_z, \partial_i N\rangle \, \partial_j X, DX(V)\right\rangle \\ &= \langle F_y, DX(V)\rangle + \langle F_z, DX(SV)\rangle \, , \end{aligned}$$

and therefore:

$$\begin{aligned} \delta\mathcal{F}(X,\Xi) &= \int_M \langle F_y, \Xi - DX(V)\rangle + \varphi \operatorname{div} F_z \, dA \\ &= \int_M \varphi \, \left[\, \langle F_y, N\rangle + \operatorname{div} F_z \,\right] dA \, . \end{aligned}$$

Here we see that the first variation of $\mathcal{F}$ is independent of tangential variations. One obtains as Euler equation

$$\operatorname{div} F_z + \langle F_y, N\rangle = 0 \, . \tag{7}$$

Choosing an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, one has

$$\begin{aligned} \operatorname{div} F_z &= \langle e_i \, F_z, DX(e_i)\rangle \\ &= \langle F_{zy} \, DX(e_i), DX(e_i)\rangle + \langle F_{zz} \, DX(Se_i), DX(e_i)\rangle \, . \end{aligned}$$

Because of the homogeneity of F the equality $F_{zz}(y,z)\,z = 0$ is valid, i.e.

$$F_{zz}(y,z) : z^{\perp} \to z^{\perp} \quad \text{for each } z \in \mathbb{R}^{n+1} - \{0\} \ .$$

Therefore, the linear symmetric mapping

$$\begin{array}{rcl} A_F : T_pM & \longrightarrow & T_pM \\ V & \longrightarrow & DX^{-1}\, F_{zz}(X,N)\, DX(V) \end{array}$$

is well defined. In this notation, the Euler equation (7) looks like:

$$\mathrm{tr}\,(A_F \circ S) + \langle F_{yz} DX(e_i), DX(e_i)\rangle + \langle F_y, N\rangle = 0 \ .$$

Definition 1.3 For a parametric integrand $F \in C^2(\mathbb{R}^{n+1} \times S^n)$ and an immersion $X : M \to \mathbb{R}^{n+1}$ of class $C^2(M, \mathbb{R}^{n+1})$

$$H_F := -\mathrm{tr}\,(A_F \circ S)$$

is the F-*mean curvature* of X.

Taking into consideration the relation $F_{yz}(X,N)\,N = F_y(X,N)$ that is valid on account of the homogeneity of F we arrive at:

$$H_F = \sum_{i=1}^{n+1} F_{y_i z_i}(X,N) \ ,$$

because $\{DX(e_1), \ldots, DX(e_n), N\}$ is an orthonormal basis of $\mathbb{R}^{n+1}$.
Thus, Theorem 1 is proved and we see that extremals of (3) are in a sense surfaces of prescribed F-mean cuvature.
Now we want to derive an equation for an immersion X generalizing the equation

$$\Delta X = HN \ ,$$

where $\Delta = \mathrm{div}\,\mathrm{grad}$ is the Laplace-Beltrami operator of X. To this aim, let $U, V \in T_pM$ and consider the identity:

$$\begin{aligned} & g(\nabla_U (A_F\,\mathrm{grad}\,\varphi), V) \\ & \quad = \quad U(g(A_F\,\mathrm{grad}\,\varphi, V)) - g(A_F\,\mathrm{grad}\,\varphi, \nabla_U V) \\ & \quad = \quad U(d\varphi(A_F\,V)) - d\varphi(A_F\,\nabla_U V) \ , \end{aligned} \tag{8}$$

where ∇ denotes the Levi-Civita connection. With $X = (x_i)_{i=1}^{n+1}$ and $DX = (dx_i)_{i=1}^{n+1}$, equation (8) leads to

$$\begin{aligned} & g(\nabla_U (A_F\,\mathrm{grad}\,X), V) \\ & \quad = \quad U(DX(A_F\,V)) - DX(A_F\,\nabla_U V) \\ & \quad = \quad DX(\nabla_U(A_F\,V)) + \langle U(DX(A_F\,V)), N\rangle\, N - DX(A_F\,\nabla_U V) \ , \end{aligned} \tag{9}$$

on account of the characterization of ∇ by: $DX(\nabla_U V) = [U(DX(V))]^{\tan}$. Finally we arrive at:

$$
\begin{aligned}
& g(\nabla_U(A_F \operatorname{grad}\varphi), V) \\
& \quad = DX(\nabla_U A_F\, V) + DX(A_F\, \nabla_U V) \\
& \qquad - \langle DX(A_F\, V), DN(U)\rangle\, N - DX(A_F\, \nabla_U V) \\
& \quad = DX(\nabla_U A_F\, V) - g(A_F\, V, S\, U) N \ .
\end{aligned} \tag{10}
$$

Choosing an orthonormal basis $\{e_1, \dots, e_n\} \subset T_pM$, equation (10) leads to

$$
\begin{aligned}
\operatorname{div}(A_F \operatorname{grad} X) &= g(\nabla_{e_i}(A_F \operatorname{grad} X), e_i) \\
&= DX(\nabla_{e_i} A_F\, e_i) - g(A_F\, e_i, S\, e_i) N \\
&= DX(\operatorname{div} A_F) - \operatorname{tr}(A_F\, S)\, N \ ,
\end{aligned} \tag{11}
$$

because of the symmetry of A_F or S. This motivates the following

Definition 1.4 Let $F \in C^3(\mathbb{R}^{n+1} \times S^n)$ be a parametric integrand and $X : M \to \mathbb{R}^{n+1}$ be a twice differentiable immersion. The operator

$$\Delta_F \ := \ \operatorname{div}(A_F \operatorname{grad} \cdot\,)$$

is the *F-Laplace-Beltrami operator* of X. Furthermore we define

$$\Theta_F \ := \ \Delta_F - \operatorname{div} A_F \ .$$

Using this operator, we can rewrite (11) in a shorter form. An immersed surface $X : M \to \mathbb{R}^{n+1}$ satisfies the equation

$$\Theta_F X = H_F N \ , \tag{12}$$

and so Theorem 2 is proved.

2 Enclosure properties of $\mathcal{F}$-extremals

Lemma 2.1 *Using the notation*

$$a_{ij} = g\left(A_F \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

the F-Laplace-Beltrami operator is expressed in local coordinates as

$$\Delta_F \ = \ \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} a_{jk} g^{kl} \partial_l \ ) \ .$$

The proof of the above lemma is a direct consequence of the equation

$$
\begin{aligned}
A_F \operatorname{grad} \varphi &= g^{ij}\partial_i\varphi \left(A_F \frac{\partial}{\partial x^j}\right) \\
&= g^{ij} a_{jk} g^{kl} \partial_i \varphi \, \frac{\partial}{\partial x^l}
\end{aligned}
$$

and the representation of div in coordinates (see e.g. [1, Ch.1]).

Definition 2.2 A Lagrangian $F = F(y,z)$ is called *elliptic,* if

$$F_{zz}(y,z) : z^{\perp} \to z^{\perp}$$

is a positive definite linear mapping for all $y \in \mathbb{R}^{n+1}$ and all $z \in \mathbb{R}^{n+1} - \{0\}$.

From Lemma 2.1 we obtain the ellipticity of Δ_F and Θ_F if F is a parametric, elliptic integrand.
Now we consider M as a manifold with boundary, i.e. $\overline{M} = M \cup \partial M$. Furthermore let $\mathcal{M} = X(M)$ be the image of M under X and $\partial\mathcal{M} = X(\partial M)$. From Lemma 2.1 we obtain

Theorem 2.3 (Convex-hull property)
Let $F \in C^3(\mathbb{R}^{n+1} \cap S^n)$ be an elliptic, parametric Lagrangian. If the immersion $X \in C^0(\overline{M}, \mathbb{R}^{n+1}) \cap C^2(M, \mathbb{R}^{n+1})$ has vanishing F-mean curvature then

$$\overline{\mathcal{M}} \subset \operatorname{conv}(\partial\mathcal{M}) \; ,$$

where $\operatorname{conv}\Sigma$ *denotes the convex hull of a set* Σ*.*

Proof. The operator Θ_F is elliptic by assumption. For $a \in \mathbb{R}^{n+1}$ and $b \in \mathbb{R}$ let t be the linear function

$$t(y) := \langle a, y\rangle + b \; , \quad y \in \mathbb{R}^{n+1} \; .$$

Then we have

$$\Theta_F[t(X)] = 0 \; .$$

Because of the strong maximum principle (see [5, p.32]) the following implication is true:

$$t(X)_{|\partial M} \le 0 \quad \Longrightarrow \quad t(X) \le 0 \quad \text{on } \overline{M} \; .$$

□

In the following we use the abreviation $|V|_F^2 := g(A_F V, V)$ for $V \in T_pM$. Furthermore let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ be the eigenvalues of A_F respectively $F_{zz}(X,N)$. The parametric integrand is again assumed to be elliptic, i.e. $\lambda_1 > 0$. The following remark is due to Dierkes [3]: Let $P : \mathbb{R}^{n+1} \to N(q)^{\perp}$ be the orthogonal projection on $DX(T_qM)$ and p^{ij} be the matrix representation of

P with respect to the canonical basis of $\mathbb{R}^{n+1}$. Then we have $p^{ii} = |\mathrm{grad}\, x_i|^2$ leading to

$$\sum_{i=1}^{n+1} |\mathrm{grad}\, x_i|^2 = n \; . \tag{13}$$

To derive further enclosure theorems, we consider instead of a linear function quadratic polynomials. Our first result belonging to quadratic polynomials is:

Theorem 2.4 *Let $X : M \to \mathbb{R}^{n+1}$ be an $\mathcal{F}$-extremal of class $X \in C^0(\overline{M}, \mathbb{R}^{n+1}) \cap C^2(M, \mathbb{R}^{n+1})$. If*

$$\sup_{(y,z)\in\mathbb{R}^{n+1}\times S^n} |y| \left| \sum_{i=1}^{n+1} F_{y_i z_i}(y,z) \right| \le n \inf_{(y,z)\in\mathbb{R}^{n+1}\times S^n} \lambda_1 \tag{14}$$

then $\partial\mathcal{M} \subset \overline{B_r(0)}$ implies $\overline{\mathcal{M}} \subset \overline{B_r(0)}$, where $\overline{B_r(0)}$ is the closed ball of radius r with center 0.

Proof. We consider for $y = (y_1, \ldots, y_{n+1})$ the polynomial $t(y) = y_1^2 + \ldots y_{n+1}^2$ and find for $\Theta_F[t(X)]$ using (13):

$$\begin{aligned} \Theta_F[t(X)] &= 2 \sum_{i=1}^{n+1} |\mathrm{grad}\, x_i|_F^2 \;+\; 2\,\langle X, \Theta_F X \rangle \\ &\ge 2\lambda_1 \sum_{i=1}^{n+1} |\mathrm{grad}\, x_i|^2 \;-\; 2|H_F||X| \\ &= 2\,n\,\lambda_1 \;-\; 2 \left| \sum_{i=1}^{n+1} F_{y_i z_i}(X,N) \right| \, |X| \; . \end{aligned}$$

Now, condition (14) implies $\Theta_F[t(X)] \ge 0$. The maximum principle gives the desired result.

□

For a proof of an enclosure theorem implying a nonexistence result we set

$$t(y) = y_1^2 + \ldots + y_n^2 - c y_{n+1}^2 \; ; \;\; c > 0 \; .$$

Then any immersion $X \in C^2(M, \mathbb{R}^{n+1})$ of F-mean curvature H_F satisfies:

$$\begin{aligned} \frac{1}{2}\,\Theta_F[t(X)] &= |\mathrm{grad}\, x_1|_F^2 + \ldots + |\mathrm{grad}\, x_n|_F^2 - c|\mathrm{grad}\, x_{n+1}|_F^2 \\ &\quad + H_F\, x_1 N_1 + \ldots + H_F\, x_n N_n - c\, H_F\, x_{n+1} N_{n+1} \\ &\ge \lambda_1 \left[|\mathrm{grad}\, x_1|^2 + \ldots + |\mathrm{grad}\, x_n|^2 \right] - c\,\lambda_n |\mathrm{grad}\, x_{n+1}|^2 \\ &\quad - |H_F| \sqrt{x_1^2 + \ldots + x_n^2 + c^2 x_{n+1}^2} \; . \end{aligned}$$

Using equation (13) we obtain

$$\begin{aligned}
\frac{1}{2}\,\Theta_F[t(X)] &\geq \lambda_1(p^{11}+\ldots+p^{nn}) - c\,\lambda_n\,p^{n+1\,n+1} \\
&\quad - |H_F|\sqrt{x_1^2+\ldots+x_n^2+c^2x_{n+1}^2} \\
&= \lambda_1\,(p^{11}+\ldots+p^{nn}+p^{n+1\,n+1}) - (\lambda_1+c\lambda_n)p^{n+1\,n+1} \\
&\quad - |H_F|\sqrt{x_1^2+\ldots+x_n^2+c^2x_{n+1}^2} \\
&\geq (n-1)\lambda_1 - c\lambda_n - \,|H_F|\sqrt{x_1^2+\ldots+x_n^2+c^2x_{n+1}^2}\,. \qquad (15)
\end{aligned}$$

Setting $c =: (n-1)\,b$, $b \in (0,1]$, one can rewrite (15):

$$\frac{1}{2}\,\Theta_F[t(X)] \geq (n-1)\,\left[\lambda_1 - b\lambda_n - |H_F|\sqrt{\frac{x_1^2+\ldots+x_n^2}{(n-1)^2}+b^2x_{n+1}^2}\,\right]\,.$$

Thus we arrive at

Theorem 2.5 *Let $X : M \to \mathbb{R}^{n+1}$ be a C^2-immersion of F-mean curvature H_F and F be an elliptic, parametric integrand. Then we have for the quadratic polynomial $t(y) = y_1^2+\ldots+y_n^2-(n-1)b\,y_{n+1}^2$ the relation*

$$\Theta_F\,[t(X)] \geq 0\,,$$

if

$$b\lambda_n + |H_F|\sqrt{\frac{x_1^2+\ldots+x_n^2}{(n-1)^2}+b^2x_{n+1}^2} \leq \lambda_1.$$

We obtain the following corollaries. The suppositions of Theorem 2.5 are always assumed to be valid; moreover let

$$K_b := \{(y_1,\ldots,y_{n+1}) \in \mathbb{R}^{n+1}\,|\,y_1^2+\ldots+y_n^2 \leq (n-1)b\,y_{n+1}^2\}$$

and $K_b^{\pm}$ denote the two connected components of $K_b - \{0\}$.

Corollary 2.6 *Assume that for the surface X the inequalities*

$$\begin{aligned}
q := \sup_M |X|\,|H_F| &< \inf_M \lambda_1 \quad \textit{and} \\
b := \inf_M\,[\,(\lambda_1-q)/\lambda_n\,] &> 0
\end{aligned}$$

are true. Then for $t(y) := y_1^2+\ldots+y_n^2-(n-1)b\,y_{n+1}^2$ we have $\Theta_F\,[t(X)] \geq 0$. If $\partial\mathcal{M} \subset K_b$, $\partial\mathcal{M}\cap K_b^+$ and $\partial\mathcal{M}\cap K_b^-$ are nonempty, then M cannot be a connected manifold.

Corollary 2.7 *Suppose that the elliptic, parametric integrand F fulfills*

$$q := \sup_{(y,z)\in\mathbb{R}^{n+1}\times S^n} |y| \left| \sum_{i=1}^{n+1} F_{y_i z_i}(y,z) \right| < \inf_{(y,z)\in\mathbb{R}^{n+1}\times S^n} \lambda_1 \ ,$$

where λ_1 is the smallest eigenvalue of $F_{zz}(y,z)$. Then for any $\mathcal{F}$-extremal X we have the relation $\Theta_F[t(X)] \geq 0$ if

$$b := \inf_{(y,z)\in\mathbb{R}^{n+1}\times S^n} \frac{\lambda_1 - q}{\lambda_n}$$

is positive ($t(y) = y_1^2 + \ldots + y_n^2 - (n-1)b\, y_{n+1}^2$).
Furthermore we can state: If $\partial\mathcal{M}$ is contained in K_b and $\partial\mathcal{M}$ has parts in the two connected components $K_b^{\pm}$, then M is not connected.

In the following corollary, we focus on surfaces of vanishing F-mean curvature.

Corollary 2.8 *Let $X : M \to \mathbb{R}^{n+1}$ be a connected immersion of vanishing F-mean curvature and S_1, $S_2 \subset \mathbb{R}^{n+1}$ be compact sets. Assume $\partial\mathcal{M} \subset S_1 \cup S_2$, $\partial\mathcal{M} \cap S_1$ and $\partial\mathcal{M} \cap S_2$ are both nonempty. Under the condition*

$$b := \inf_{(y,z)\in\mathbb{R}^{n+1}\times S^n} \frac{\lambda_1}{\lambda_n} > 0 \ , \tag{16}$$

we have:

(i) If S_1, S_2 are closed balls, i.e. $S_1 := \overline{B_{r_1}(X_1)}$, $S_2 := \overline{B_{r_2}(X_2)}$ and $R := |X_1 - X_2|$ then R can be estimated as follows

$$R \leq \sqrt{\frac{(n-1)b\, +\, 1}{(n-1)b}}\,(r_1 + r_2) \ .$$

(ii) Let S_1, S_2 be arbitrary with diameter d_1, d_2 and assume that they are seperated by a slab of positive width r; this implies

$$r \leq \sqrt{\frac{2\,(1+(n-1)b)\,(n+1)}{(n-1)\,(n+2)\,b}}\,(d_1 + d_2) \ .$$

(iii) If S_1, S_2 are two n-dimensional discs with centers X_1, X_2 and radii r_1, r_2 respectively that are contained in parallel hyperplanes of distance r, then the sum of the radii is estimated by

$$r_1 + r_2 \geq \sqrt{(n-1)\,b\,\left(r^2 + \frac{d^2}{1+(n-1)b}\right)} \ ,$$

where $R := |X_1 - X_2|$ and $d := \sqrt{R^2 - r^2}$.

Proof. We want to examine translations and rotations of X. To this aim, consider $Q \in SO(n+1)$, $p \in \mathbb{R}^{n+1}$ and set $\tilde{X} := Q\,(X-p)$. In general it is not true that the F-mean curvature of $\tilde{X}$ vanishes. Therefore we look at the following integrand:

$$\tilde{F}(\tilde{y},\tilde{z}) := F(Q^T\tilde{y}+p, Q^T\tilde{z}) \;.$$

For $\tilde{F}$ we easily see:

$$\tilde{F}_{\tilde{z}\tilde{z}}(\tilde{y},\tilde{z}) = Q\,F_{zz}(Q^T\tilde{y}+p, Q^T\tilde{z})\,Q^T \;.$$

We obtain for the $\tilde{F}$-mean curvature of $\tilde{X}$:

$$\begin{aligned} H_{\tilde{F}}(\tilde{X}) &= \tilde{g}^{ij}\left\langle \tilde{F}_{\tilde{z}\tilde{z}}(\tilde{X},\tilde{N})\,\partial_i\tilde{N}, \partial_j\tilde{X}\right\rangle \\ &= g^{ij}\left\langle Q\,F_{zz}(Q^T\,Q(X-p)+p, Q^T\,QN)\,Q^T\,\partial_i(QN), \partial_j(Q(X-p))\right\rangle \\ &= g^{ij}\left\langle F_{zz}(X,N)\,\partial_iN, \partial_jX\right\rangle = H_F(X) = 0 \;. \end{aligned}$$

Note that the real number b in (16) is the same for F and $\tilde{F}$ respectively.
From the above considerations we infer that in case (i) we can assume $B_{r_1}(X_1)$ to be contained in the upper connected component of $K_b - \{0\}$ and $B_{r_2}(X_2)$ to be contained in the lower one, if

$$R > \sqrt{\frac{(n-1)b\,+\,1}{(n-1)b}}\,(r_1+r_2) \;.$$

Thus Corollary 2.6 shows that (i) is true. Using the theorem of Jung [4, p.200] we see that (ii) is nothing else than a special case of (i).
For a proof of (iii) we only have to apply the "disc-separating Theorem" cited in [3, p.211].

□

Remark: Note, that the eigenvalues $\lambda_1,\ldots,\lambda_n$ of F_{zz} coincide in the case of the area functional. That means $b = (1-q)$ for Corollary 2.6 and $b = 1$ for Corollary 2.8; thus we regain the results of Dierkes [3].

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