The Wulff-shape minimizes an anisotropic Willmore functional

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May 27, 2003

Abstract

The aim of this paper is to find a fourth order energy having Wulff-shapes as minimizers. This question is motivated by surface restoration problems. In surface restoration a damaged region of a surface is replaced by a surface patch which restores the region in a suitable way. In particular one aims for C^1-continuity at the patch boundary. A fourth order energy is considered to measure fairness and to allow appropriate boundary conditions ensuring continuity of the normal field. Here, anisotropy comes into play if a surface is destroyed which contains edges and corners. In the present paper we define a generalization of the classical Willmore functional and prove that Wulff-shapes are the only minimizers.

1 Introduction

Consider a closed, immersed, oriented, and smooth surface $x : \mathcal{M} \to \mathbb{R}^3$ with a two-dimensional parameter manifold $\mathcal{M}$. The differential of the normal mapping $n : \mathcal{M} \to S^2$ induces the shape operator S via $Dx \circ S = Dn$. The classical Willmore functional is defined as

$$W[x] = \frac{1}{2} \int_{\mathcal{M}} h^2 \, dA \,, \tag{1}$$

where dA is the induced area element and $h = \operatorname{tr} S$ is the mean curvature. This functional is used e.g. for modelling elastic surfaces.

A geometric analysis concerning the structure of integrands $f(\kappa_1, \kappa_2)$ appearing in elasticity is due to Nitsche. Here, κ_1 and κ_2 are the principal curvatures of x. Nitsche considered integrands $f(\kappa_1, \kappa_2)$ which are symmetric, definite and of polynomial growth of order at most two. He shows that such integrands are of the form:

$$\alpha + \beta(h - h_0)^2 - \gamma k \,,$$

where α, β, γ and h_0 are constants fulfilling certain structural inequalities [11]. Furthermore, $k = \det S$ is the classical Gauß curvature. Nevertheless, the physical meaning of the pure Willmore functional ($\alpha = \gamma = h_0 = 0$, $\beta = 1$) is limited.

Any round sphere is a minimizer and the area of Willmore surfaces cannot be bounded.

On the other hand, it is a well known fact that spheres are the only minimizers of W. For the construction of minimizers in the classes of fixed genus we refer to [18] in the case of genus one and to [1] for arbitrary genus.

Recently, the corresponding L^2-gradient flow of W – the Willmore flow – was considered analytically as well as numerically. For initial data close to spheres w.r.t. the $C^{2,\alpha}$ topology, Simonett is able to show global existence of the flow and convergence to a sphere [19]. Kuwert and Schätzle [8] prove a lower bound on the maximal time of smooth existence of the Willmore flow in terms of the concentration of the curvature. In [7, 9] they are able to show that for surfaces of sphere type and initial energy less than or equal 8π, Willmore flow converges to a round sphere. (Note that in the present paper spheres have energy 8π in contrast to 4π which is the usual convention in geometry.)

Mayer and Simonett present a numerical scheme for axisymmetric solutions based on finite differences [10]. Their numerical experiments predict the appearance of singularities under Willmore flow. Furthermore, these experiments show that the result obtained in [9] is optimal.

The case of curves, moving in space w.r.t. Willmore flow (and also curve diffusion) is considered in [5] analytically and numerically. Here, results of Polden for planar curves [13, 14] are generalized and a semi-implicit discretization scheme based on a mixed formulation is given.

A discretization scheme for triangulated surfaces without boundary is obtained by Rusu [15]. A generalization of this to bounded surfaces is used in [2] for applications in surface restoration.

The present work is motivated by restoration problems. Restoring a surface, usually means replacing a damaged domain of a surface by a patch which restores the region in a suitable way which means in particular C^1-continuity at the boundary [22]. Therefore, it is crucial to use at least fourth order methods if one wants to obtain smoothness at the boundary. With the aid of Willmore flow, one is able to prescribe boundary conditions for the position vector and the normal. Often, real-world restoration problems are of anisotropic nature, e.g., if the edge of a surface is destroyed. In such cases, using the isotropic Willmore functional will not lead to results respecting this anisotropy, see [2]. Therefore, one is interested in anisotropic fourth order functionals with corresponding minimizers. In this paper, we propose to replace the integrand h^2 by a generalized mean curvature appearing as first variation of the functional

$$A_\gamma[x] = \int_{\mathcal{M}} \gamma(n)\, dA\,.$$

Here γ is a smooth function

$$\begin{aligned} \gamma \ : \ & S^2 \to \mathbb{R}^+ \\ & z \mapsto \gamma(z)\,, \end{aligned} \tag{2}$$

and we may assume, that γ is given as a one-homogeneous function on $\mathbb{R}^3$, i.e., for $\lambda > 0$ we have $\gamma(\lambda z) = \lambda\gamma(z)$. In addition, let there be a positive constant

m such that for the second derivative we have

$$D^2(\gamma(z) - m|z|) \geq 0\,.$$

In this case, γ is called elliptic and the eigenvalues of $D^2\gamma(z)$ restricted to $z^\perp = \{x \in \mathbb{R}^3 | x \cdot z = 0\}$ are bounded from below by m. Let us mention here that the Euclidean scalar product of two vectors $a, b \in \mathbb{R}^3$ will always be denoted by $a \cdot b$.

Considering a surface $x : \mathcal{M} \to \mathbb{R}^3$, we can give a version of the second derivative of γ on its tangential space as follows

$$\begin{aligned} a_\gamma \quad : \quad & T_\xi\mathcal{M} \to T_\xi\mathcal{M} \\ & v \mapsto Dx^{-1}\gamma_{zz}(n) Dx(v)\,. \end{aligned} \tag{3}$$

The endomorphism field a_γ is well defined due to the fact that $D^2\gamma(z)\, z = 0$ for all $z \neq 0$. By ellipticity, a_γ is positive definite. The classical area functional is obtained for the function $\gamma(z) = |z|$. In this case, a_γ is the identity.

The first variation of A_γ in direction ϑ may be represented in the L^2-metric by a generalized mean curvature vector:

$$\langle A'_\gamma[x], \vartheta\rangle = \int_\mathcal{M} h_\gamma (n \cdot \vartheta)\, dA\,. \tag{4}$$

Here, $h_\gamma = \mathrm{tr}\,(a_\gamma S)$ will be called the γ-mean curvature.

It is known since the beginning of the last century [21], that solutions of the isoperimetric problem for A_γ exist and are given by the so called Wulff-shapes $\mathcal{W}_\gamma$ that may be obtained from γ via the following parametrization over the unit sphere:

$$\begin{aligned} \gamma_z \quad : \quad & S^2 \to \mathcal{W}_\gamma \\ & z \mapsto \gamma_z(z)\,. \end{aligned} \tag{5}$$

For a proof of the isoperimetric property of the Wulff-shape and more references to the literature see [6].

In the present paper we want to give a different characterization of Wulff-shapes. For the application in anisotropic restoration problems one seeks for a fourth order functional W_γ which has the Wulff-shape $\mathcal{W}_\gamma$ as minimizer. Here we show that one possible choice is

$$W_\gamma[x] = \frac{1}{2}\int_\mathcal{M} h_\gamma^2\, dA\,. \tag{6}$$

In this sense we have found a suitable fourth order functional which is well suited for anisotropic restoration problems. For a second order approach to surface fairing by locally prescribing Wulff-shapes we refer to [3].

The paper is organized as follows: Section 2 contains a proof of a formula for the linearization of a generalized resp. anisotropic mean curvature (Theorem

2.2). This formula may be also interesting in different applications because the anisotropy we consider there is not related to an integrand γ as in (2).

In section 3 we apply the result of section 2 in the special case of anisotropic mean curvature obtained by an integrand γ to derive the Euler equation of the functional (6). An important tool is a generalized Codazzi equation (12). Moreover, we show that the Wulff-shape is a solution of the Euler equation.

In section 4 we prove the main result of this paper. Here, it is shown that Wulff-shapes are not only stationary points of W_γ but also minimizers and essentially the only minimizers. The proof is based on a symmetrization argument (16) and the generalized Codazzi equation (12). For a related reasoning see the work on stable surfaces of constant γ-mean curvature [12].

2 Linearization of generalized mean curvature

In this section we will consider a family of surfaces $x : \mathcal{M} \times (-\eta, \eta) \to \mathbb{R}^3$, thus $x(t) = x(\cdot, t)$ is an immersion of a two-dimensional orientable manifold $\mathcal{M}$ and η is a small positive real number. This family is considered as a perturbation of a surface $x : \mathcal{M} \to \mathbb{R}^3$, i.e., $x(\cdot, 0) = x$. The evolution of $x(t)$ is assumed to be given by the equation

$$\partial_t x = \varphi(t) n(t) + Dx(v(t)) \,, \tag{7}$$

where $\varphi(t)$ is a smooth function and $v(t)$ is a smooth vector field on $\mathcal{M}$.

The following notion is essential in our considerations:

Definition 2.1 *Let $\alpha \in \mathbb{R}^{3\times 3}$ be a symmetric endomorphism depending on $z \in S^2$ with the property $\alpha(z)\, z = 0$ for all $z \in S^2$. Then α induces an endomorphism field on the surface $x : \mathcal{M} \to \mathbb{R}^3$*

$$\bar{a} : \mathcal{TM} \to \mathcal{TM} \,, \tag{8}$$

where $\bar{a}$ is given by $\bar{a} = Dx^{-1} \circ \alpha(n) \circ Dx$ and $n : \mathcal{M} \to S^2$ is the surface normal. The corresponding generalized mean curvature is defined as

$$h_{\bar{a}} = \mathrm{tr}\,(\bar{a} \circ S) \,.$$

In the next section we will apply this notion in the case of $\alpha(z) = \gamma_{zz}(z)$ for an elliptic integrand γ as in (2). Therefore we assume in the following that $\alpha(z) : z^\perp \to z^\perp$ is positive definite for all $z \in S^2$. The main result of this section will be

Theorem 2.2 *Let x be a family of surfaces evolving w.r.t. rule (7). The generalized mean curvature $h_{\bar{a}}$ fulfils the following evolution equation:*

$$\begin{aligned} -\partial_t h_{\bar{a}} &= \Delta_{\bar{a}} \varphi + |S|^2_{\bar{a}} \varphi \\ &\quad - g(\mathrm{div}\, \bar{a}, \mathrm{grad}\, \varphi) - \mathrm{tr}\,(\bar{a} \circ [(\nabla_{\bullet} S) v]) - g^{ij} (\partial_t \alpha\, \partial_i x) \cdot \partial_j n \,. \end{aligned} \tag{9}$$

Here we use: $\Delta_{\bar{a}} = \mathrm{div}\,(\bar{a}\, \mathrm{grad}\;)$ and $|S|^2_{\bar{a}} = \mathrm{tr}\,(\bar{a} \circ S^2)$.

Proof: Equation (7) implies the following relation for the normal n:

$$\partial_t n = -Dx(\operatorname{grad}\varphi) + Dn(v)\,. \tag{10}$$

With the classical notation for g_{ij}, g^{ij} and $h_{\bar{a},ij} = -g(\bar{a}\partial_i, S\partial_j) = -(\alpha\partial_i x)\cdot\partial_j n$, the mean curvature $h_{\bar{a}}$ can be written as

$$-h_{\bar{a}} = g^{ij}\, h_{\bar{a},ij}$$

and consequently $-\partial_t h_{\bar{a}} = \partial_t g^{ij}\, h_{\bar{a},ij} + g^{ij}\partial_t h_{\bar{a},ij} = I + II$. Here and in the following we will use Einstein summation convention. On account of the fact

$$\partial_t g^{ij} = -g^{ik}\partial_t g_{kl} g^{lj}$$

we can state ($h_{ij} = -\partial_i x \cdot \partial_j n$)

$$\begin{aligned}
\partial_t g^{ij} \;&=\; 2\varphi g^{ik} h_{kl} g^{lj} \\
&\quad -g^{ik}\, Dx(\nabla_k v)\cdot\partial_l x\, g^{lj} \\
&\quad -g^{ik}\,\partial_k x\cdot Dx(\nabla_l v)\, g^{lj}.
\end{aligned}$$

Note that for the tangential part of $\partial_k Dx(v)$ we have $[\partial_k Dx(v)]^{\tan} = Dx(\nabla_k v)$. Thus for the term I we obtain:

$$\begin{aligned}
I \;&=\; 2\varphi g^{ik} h_{kl} g^{lj} h_{\bar{a},ij} \\
&\quad -g^{ik}\, Dx(\nabla_k v)\cdot\partial_l x\, g^{lj} h_{\bar{a},ij} \\
&\quad -g^{ik}\,\partial_k x\cdot Dx(\nabla_l v)\, g^{lj} h_{\bar{a},ij} \\
&=\; 2\varphi|S|^2_{\bar{a}} + g^{ik} g(S\bar{a}\,\partial_i, \nabla_k v) + g^{lj} g(\nabla_l v, \bar{a}S\partial_j) \\
&=\; 2\varphi|S|^2_{\bar{a}} + \operatorname{tr}\left(\,(\bar{a}S + S\bar{a})\nabla_{\bullet} v\,\right).
\end{aligned}$$

The computation of $\partial_t h_{\bar{a},ij}$ gives:

$$\begin{aligned}
\partial_t h_{\bar{a},ij} \;&=\; -\,\partial_t\alpha\partial_i x\cdot\partial_j n \\
&\quad -\,\alpha\,\partial_i(\varphi n + Dx(v))\cdot\partial_j n \\
&\quad -\,\alpha\,\partial_i x\cdot\partial_j(-Dx(\operatorname{grad}\varphi) + Dn(v)) \\
&=\; -\,\partial_t\alpha\partial_i x\cdot\partial_j n \\
&\quad -\varphi\,\alpha\partial_i n\cdot\partial_j n \,-\, \alpha Dx(\nabla_i v)\cdot\partial_j n \\
&\quad +\,\alpha\partial_i x\cdot\partial_j Dx(\operatorname{grad}\varphi) \,-\, \alpha\partial_i x\cdot\partial_j Dn(v) \\
&=\; -\,\partial_t\alpha\partial_i x\cdot\partial_j n \\
&\quad -\varphi\,\alpha\partial_i n\cdot\partial_j n \,-\, \alpha Dx(\nabla_i v)\cdot\partial_j n \\
&\quad +\,\alpha\partial_i x\cdot\partial_j Dx(\operatorname{grad}\varphi) \,-\, \alpha\partial_i x\cdot Dx(S\nabla_j v) \\
&\quad -\,\alpha\partial_i x\cdot Dx((\nabla_j S)\, v)\,.
\end{aligned}$$

Therefore, the term II can geometrically be interpreted as follows:

$$\begin{aligned}
II &= -g^{ij}\,\partial_t\alpha\partial_i x\cdot\partial_j n \;-\varphi g^{ij}\,\alpha\partial_i n\cdot\partial_j n\\
&\quad -g^{ij}\,\alpha Dx(\nabla_i v)\cdot\partial_j n\\
&\quad +g^{ij}\,\alpha\partial_i x\cdot\partial_j Dx(\operatorname{grad}\varphi)\\
&\quad -g^{ij}\,\alpha\partial_i x\cdot Dx(S\nabla_j v)\\
&\quad -g^{ij}\,\alpha\partial_i x\cdot Dx((\nabla_j S)\,v)\\
&= -g^{ij}\,\partial_t\alpha\partial_i x\cdot\partial_j n \;-\varphi|S|^2_{\bar a}-\operatorname{tr}(S\bar a\,\nabla_\bullet v)\\
&\quad +g^{ij}\,\alpha\partial_i x\cdot\partial_j Dx(\operatorname{grad}\varphi)\;-\operatorname{tr}(\bar a S\,\nabla_\bullet v)\\
&\quad -\operatorname{tr}(\bar a\circ[(\nabla_\bullet S)v])\,.
\end{aligned}$$

Now, we finish the proof. The linearization of $h_{\bar a}$ is the sum of I and II:

$$\begin{aligned}
-\partial_t h_{\bar a} &= \varphi|S|^2_{\bar a}+g^{ij}\,\partial_i x\cdot\alpha\partial_j Dx(\operatorname{grad}\varphi)\\
&\quad -g^{ij}\,\partial_t\alpha\partial_i x\cdot\partial_j n\\
&\quad -\operatorname{tr}(\bar a\circ[(\nabla_\bullet S)v])\\
&= \varphi|S|^2_{\bar a}+g^{ij}\,\partial_i x\cdot\partial_j(\alpha Dx(\operatorname{grad}\varphi))\\
&\quad -g^{ij}\,\partial_i x\cdot\partial_j\alpha\;Dx(\operatorname{grad}\varphi)\\
&\quad -g^{ij}\,\partial_t\alpha\partial_i x\cdot\partial_j n\;-\operatorname{tr}(\bar a\circ[(\nabla_\bullet S)v])\\
&= \Delta_{\bar a}\,\varphi+|S|^2_{\bar a}\varphi\\
&\quad -g^{ij}\,\partial_i x\cdot\partial_j\alpha\;Dx(\operatorname{grad}\varphi)\\
&\quad -\operatorname{tr}(\bar a\circ[(\nabla_\bullet S)v])-g^{ij}\,\partial_t\alpha\partial_i x\cdot\partial_j n\;.
\end{aligned}$$

It remains to show $g(\operatorname{div}\bar a, w)=g^{ij}\,\partial_i\alpha\;\partial_j x\cdot Dx(w)$ for all vector fields w:

$$\begin{aligned}
g(\operatorname{div}\bar a, w) &= g^{ij}\,g(\nabla_i\bar a\,\partial_j, w)\\
&= g^{ij}\;Dx((\nabla_i\bar a)\partial_j)\cdot Dx(w)\\
&= g^{ij}\;Dx(\nabla_i(\bar a\partial_j)-\bar a\nabla_i\partial_j)\cdot Dx(w)\\
&= g^{ij}\;[\partial_i\,Dx(\bar a\partial_j)-\alpha Dx(\nabla_i\partial_j)]\cdot Dx(w)\\
&= g^{ij}\;\partial_i\alpha\,\partial_j x\cdot Dx(w)\;. \qquad (11)
\end{aligned}$$

□

3 Anisotropic Willmore energies

Now we want to apply the result on the linearization of generalized mean curvature to compute the derivative of anisotropic Willmore-functionals.

We consider the energy W_γ defined as in (6). For a test function $\vartheta\in C^1(\mathcal{M},\mathbb{R}^3)$ we have

$$\langle W'_\gamma[x],\vartheta\rangle=\frac{d}{dt}W_\gamma[x_t]_{|t=0}\,,$$

where x_t fulfils $\partial_t x_{t|t=0}=\vartheta$. According to section 2 we split ϑ in a normal component φn and a tangential component $Dx(v)$, i.e., $\vartheta=\varphi n+Dx(v)$.

To derive the Euler equation we have to compute the derivative of $h^2_{\gamma,t}$ and of the area element dA_t. The latter is contained in [17] and the result is $\partial_t dA_{t|t=0} = \mathrm{div}\,\vartheta\, dA$.

In the special case where the endomorphism field $\bar{a}$ from section 2 is given by the second derivative of an elliptic integrand γ, Theorem 2.2 simplifies in the following way:

By (11) for $\alpha = \gamma_{zz}$ we have due to $\partial_t n_{t|t=0} = -Dx(\mathrm{grad}\,\varphi) + Dn(v)$:

$$\begin{aligned} g^{ij}\left(\partial_t \alpha_{t|t=0}\partial_i x\right)\cdot \partial_j n &= g^{ij}\gamma_{zzz}(n)\left[\partial_t n_{t|t=0}, \partial_i x, \partial_j n\right] \\ &= g^{ij}\partial_j(\gamma_{zz}(n))(-Dx(\mathrm{grad}\,\varphi) + Dn(v))\cdot \partial_i x \\ &= g(\mathrm{div}\, a_\gamma, -\mathrm{grad}\,\varphi + Sv)\,, \end{aligned}$$

where a_γ is defined as in (3). For the linearization of $h_\gamma := h_{a_\gamma}$ we therefore obtain with $\Delta_\gamma = \mathrm{div}\,(a_\gamma\,\mathrm{grad}\,)$ and $|S|^2_\gamma = \mathrm{tr}\,(a_\gamma S^2)$

$$-\partial_t h_\gamma = \Delta_\gamma\varphi + |S|^2_\gamma\varphi - \mathrm{tr}\,(a_\gamma \circ [(\nabla_\bullet S)v]) - g(\mathrm{div}\, a_\gamma, Sv)\,.$$

The Codazzi equation as well as the symmetry of S and a_γ imply the following

Lemma 3.1 *For the divergence of the endomorphism field Sa_γ and all vector fields v the identity*

$$g(\mathrm{div}\,(Sa_\gamma), v) = \mathrm{tr}\,(a_\gamma \circ [(\nabla_\bullet S)v]) + g(\mathrm{div}\, a_\gamma, Sv)$$

is valid. Furthermore, choosing $\varphi = 0$ in (7) the tangential part of the linearization $\partial_t h_\gamma$ is $g(\mathrm{grad}\, h_\gamma, v)$ from which one concludes

$$\mathrm{div}\,(Sa_\gamma) = \mathrm{grad}\, h_\gamma\,. \tag{12}$$

Proof: By definition of the divergence, one gets

$$\begin{aligned} g(\mathrm{div}\,(Sa_\gamma), v) &= g^{ik}g(\nabla_i(Sa_\gamma)\partial_k, v) \\ &= g^{ik}g((\nabla_i S)\, a_\gamma\partial_k, v) + g^{ik}g(S(\nabla_i a_\gamma)\partial_k, v) \\ &= g^{ik}g(a_\gamma\partial_k, (\nabla_i S)v) + g^{ik}g((\nabla_i a_\gamma)\partial_k, Sv) \\ &= g^{ik}g(\partial_i, (\nabla_{a_\gamma\partial_k} S)v) + g(\mathrm{div}\, a_\gamma, Sv) \\ &= \mathrm{tr}\,((\nabla_{a_\gamma\bullet} S)\, v) + g(\mathrm{div}\, a_\gamma, Sv)\,. \end{aligned}$$

Defining the endomorphism Σ via $\Sigma w = (\nabla_w S)\, v$ we have

$$\mathrm{tr}\,((\nabla_{a_\gamma\bullet} S)\, v) = \mathrm{tr}\,(\Sigma\circ a_\gamma) = \mathrm{tr}\,(a_\gamma\circ\Sigma)$$

and the result is shown.

□

The Euler equation:
From the above considerations we obtain the identity:

$$-\partial_t h_\gamma = \Delta_\gamma \varphi + |S|^2_\gamma \varphi - g(\operatorname{grad} h_\gamma, v)\,. \tag{13}$$

Using this result, we obtain

$$\begin{aligned}\langle W'_\gamma[x], \vartheta\rangle &= \int_{\mathcal{M}} h_\gamma\,(-\Delta_\gamma\varphi - |S|^2_\gamma\varphi + g(\operatorname{grad} h_\gamma, v)) + \frac{1}{2}h^2_\gamma \operatorname{div}\vartheta\, dA \\ &= \int_{\mathcal{M}} h_\gamma\,(-\Delta_\gamma\varphi - |S|^2_\gamma\varphi) + \frac{1}{2}hh^2_\gamma\varphi + \frac{1}{2}\operatorname{div}(h^2_\gamma\, v)\, dA\,. \qquad (14)\end{aligned}$$

This relation implies the Euler equation of the Willmore functional for surfaces without boundary:

$$-\Delta_\gamma h_\gamma - h_\gamma|S|^2_\gamma + \frac{1}{2}hh^2_\gamma = 0\,. \tag{15}$$

Especially we obtain the following

Proposition 3.2 *The Wulff-shape $\mathcal{W}_\gamma$ is a solution of the Euler-equation for the anisotropic Willmore functional.*

Proof: We use the parametrization of the Wulff-shape given in (5). The normal of $\mathcal{W}_\gamma$ at $D\gamma(z)$ is given by z because $D^2\gamma(z)z = 0$ for $z \neq 0$. Therefore, one gets $S = a_\gamma^{-1}$ for the shape operator. This especially implies the fact that on the Wulff-shape we have $h_\gamma = \text{const}$. The result is shown, if we can prove:

$$|S|^2_\gamma - \frac{1}{2}h_\gamma h = 0\,,$$

but this is a consequence of the above consideration:

$$\begin{aligned}|S|^2_\gamma &= \operatorname{tr}(a_\gamma S^2) = \operatorname{tr} a_\gamma^{-1} \\ \frac{1}{2}h_\gamma h &= \frac{1}{2}\operatorname{tr}(\mathrm{Id})\operatorname{tr} a_\gamma^{-1} = \operatorname{tr} a_\gamma^{-1}\,.\end{aligned}$$

□

This proposition clearly follows from the result of the next section. Nevertheless, the discussion of the Euler equation seems worth to be pointed out.

4 Main result

As was mentioned in the introduction, spheres are not only extremals but also minimizers of the classical Willmore functional. The aim of this section is to prove an anisotropic version of this result. We will show

Theorem 4.1 *Let $x : \mathcal{M} \to \mathbb{R}^3$ be an immersion of a compact surface $\mathcal{M}$ without boundary into $\mathbb{R}^3$. We can estimate the anisotropic Willmore-energy $W_\gamma[x]$ from below by*

$$W_\gamma[x] \geq 2|\mathcal{W}_\gamma|\,,$$

where $|\mathcal{W}_\gamma|$ is the area of the Wulff-shape $\mathcal{W}_\gamma$. The Wulff-shape itself is the unique minimizer of W_γ.

Proof: We may follow the classical proof given e.g. in [20, pp. 270]. At first we want to estimate W_γ by a total curvature term. In the case of elliptic integrands, the γ-mean curvature h_γ may also be written as

$$h_\gamma = \mathrm{tr}\,(a_\gamma S) = \mathrm{tr}\,(a_\gamma^{1/2} S a_\gamma^{1/2})\,. \tag{16}$$

The endomorphism field $a_\gamma^{1/2} S a_\gamma^{1/2}$ is symmetric and may be diagonalized with eigenvalues μ_1, μ_2. Related symmetrizations were also used in [16] and [4]. Introducing the corresponding anisotropic Gauß curvature $k_\gamma = \det(a_\gamma S) = \det(a_\gamma^{1/2} S a_\gamma^{1/2})$, one obtains the relation

$$h_\gamma^2 - 4k_\gamma = (\mu_1 + \mu_2)^2 - 4\mu_1\mu_2 = (\mu_1 - \mu_2)^2 \geq 0$$

and therefore we can give the following estimate:

$$W_\gamma[x] \geq \frac{1}{2} \int_{k_\gamma^+} h_\gamma^2 \, dA \geq 2 \int_{k_\gamma^+} k_\gamma \, dA\,,$$

where $k_\gamma^+ = \{\xi \in \mathcal{M} | k_\gamma(\xi) \geq 0\}$. By the area formula, the expression $\int_{k_\gamma^+} k_\gamma \, dA$ is the area (counted by multiplicity) of $D\gamma(n(k_\gamma^+))$. Due to the ellipticity of γ, we may conclude

$$k_\gamma^+ = \{\xi \in \mathcal{M} | k(\xi) \geq 0\}\,,$$

where k is the classical Gauß curvature. Therefore, and on account of the compactness of $\mathcal{M}$, we obtain $n(k_\gamma^+) = S^2$ and especially:

$$\int_{k_\gamma^+} k_\gamma \, dA \geq |\mathcal{W}_\gamma|\,.$$

Assume now, that equality holds in all of the above inequalities. Then we can conclude $\mu_1 = \mu_2 =: \mu$. Thus we have

$$a_\gamma^{1/2} S a_\gamma^{1/2} = \mu \, \mathrm{Id}$$

from which we get

$$S\, a_\gamma = \mu \, \mathrm{Id}$$

and therefore

$$\mathrm{div}\,(S a_\gamma) = \mathrm{grad}\, \mu\,.$$

On the other hand we know by Lemma 3.1

$$\mathrm{div}\,(S a_\gamma) = \mathrm{grad}\, h_\gamma\,.$$

This implies $h_\gamma - \mu = \text{const}$. On account of $h_\gamma = \mathrm{tr}\,(S a_\gamma) = 2\mu$ one gets $\mu = \text{const}$. Thus we obtain

$$Dx(a_\gamma S - \mu \, \mathrm{Id}) = D(\gamma_z(n) - \mu x) = 0$$

and integration leads to

$$x = x_0 + \frac{1}{\mu}\gamma_z(n) ,$$

where $x_0 \in \mathbb{R}^3$ is a constant vector and the result is shown. □

Acknowledgements: The author enjoyed the hospitality of the Mathematische Institut at the University of Freiburg in September 2002 and March 2003. He would like to thank Prof. Dr. Gerhard Dziuk for giving the possibility to be in Freiburg and Prof. Dr. Ernst Kuwert for discussing the problem of minimizers of anisotropic Willmore energies.

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