On Level Set Formulations for Anisotropic Mean Curvature Flow and Surface Diffusion

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Abstract. Anisotropic mean curvature motion and in particular anisotropic surface diffusion play a crutial role in the evolution of material interfaces. This evolution interacts with conservations laws in the adjacent phases on both sides of the interface and are frequently expected to undergo topological chances. Thus, a level set formulation is an appropriate way to describe the propagation. Here we recall a general approach for the integration of geometric gradient flows over level set ensembles and apply it to derive a variational formulation for the level set representation of anisotropic mean curvature motion and anisotropic surface flow. The variational formulation leads to a semi-implicit discretization and enables the use of linear finite elements.

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1. Introduction

To capture the surface evolution in epitaxial growth on a large scale it is appropriate to assume the surface to be smooth and to describe its evolution by continuum equations. For a review of such approaches see [1]. Even if most of these models are heuristically introduced, most of them can be considered as small slope approximations of anisotropic geometric evolution laws, such as mean curvature flow or surface diffusion. These geometric nonlinear evolution laws can therefore be viewed as a prototype of a more general class of models to describe epitaxial growth. In this paper we present an approach to anisotropic second and fourth order geometric evolution laws in a level set formulation. For more details on the background and the derivation we refer to [2, 3]. A connection to the physical problems in epitaxial growth and a comparison with the models described in [1] will be given elsewhere. Level set formulations for isotropic geometric evolution laws have already been discussed and used numerically in the literature. In particular, isotropic surface diffusion is considered within a level set context i.e. in [4, 5, 6].

For a given initial surface $\mathcal{M}_0$, a geometric evolution law defines a family of surfaces $\mathcal{M}(t)$, $t \geq 0$ with $\mathcal{M}(0) = \mathcal{M}_0$. Now consider $\mathcal{M}(t)$ to be given implicitly as a specific level set of a corresponding function $\phi(t)$. Thus the evolution of $\mathcal{M}(t)$ can be described by an evolution of $\phi(t)$. Given a velocity field v the evolution of ϕ is described by the convection equation $\partial_t \phi + \|\nabla \phi\| v = 0$, which is called the level set equation. If the velocity v is determined through the geometric evolution law we can implicitly evolve the surface $\mathcal{M}$ by solving the level set equation, [7]. Let us emphasize that different from second order problems, such as mean curvature flow, no maximum principle is known for fourth order problems. Indeed two surfaces both undergoing an evolution by surface diffusion might intersect in finite time. Hence, a level set formulation in general will lead to singularities and we expect a blow up of the gradient of ϕ in finite time. If one is solely interested in the evolution of a single level set, one presumably can overcome this problem by a reinitialization with a signed distance function with respect to this evolving level set. We are here aiming to derive a suitable weak formulation for such evolution problem, which only makes use of first derivatives of unknown functions and test functions. In particular this will allow for a discretization based on a mixed formulation with piecewise affine finite elements, closely related to results by Rusu [8]. Hence we have to reformulate the problem in order to avoid curvature terms, i.e. derivatives of the normal. Here, we take advantage of a fairly general gradient flow perspective to geometric evolution problems. Indeed, given a gradient flow for parametric surfaces, we derive a level set formulation, which describes the simultaneous evolution of all level sets corresponding to this gradient flow. This approach is based on the co-area formula (cf. for example to the book of Ambrosio et al. [9]) and a proper identification of the temporal change of the level set function and the corresponding evolution speed of the level surfaces. Thereby, we are able to identify the natural dependent variables. This approach provides an insight into the geometry of evolution problems on the space of level set ensembles.

The paper is organized as follows: In Section 2 we present a level set formulation for a gradient flow with respect to the L^2 surface metric and recover the well known level set formulation for isotropic mean curvature flow, as given in [10]. In Section 3 this setting is adapted to derive a level set formulation for anisotropic mean curvature flow and anisotropic surface diffusion. Based on these results a numerical scheme for anisotropic surface diffusion is described in Section 4, and in Section 5 some numerical results in two and three dimensions are presented.

2. Parametric gradient flow

Consider a closed, immersed, oriented, and smooth surface $x : \mathcal{M} \to \mathbb{R}^3$ with a two-dimensional parameter manifold $\mathcal{M}$. Given an energy density $f : \mathcal{M} \to \mathbb{R}$, the surface energy is denoted by

$$e[\mathcal{M}] := \int_{\mathcal{M}} f \, \mathrm{d}A.$$

We consider the gradient flow with respect to a specific surface metric $g(\mathcal{M})$

$$\partial_t x = -\mathrm{grad}_{g(\mathcal{M})} e[\mathcal{M}].$$

To derive a level set formulation of this equation, let us first introducing some useful geometric notation. Furthermore, we derive representations for geometric quantities on level sets $\mathcal{M}$ in terms of the corresponding level set function ϕ. Let $\phi : \Omega \to \mathbb{R}$ be some smooth function on a domain $\Omega \subset \mathbb{R}^{d+1}$. Suppose $\mathcal{M}_c := \{x \in \Omega \,|\, \phi(x) = c\}$ is a level set of ϕ for the level value c. In what follows we write $\mathcal{M} = \mathcal{M}_c$ if no confusion is possible and allways assume that $\|\nabla\phi\| \neq 0$ on $\mathcal{M}$. Thus, $\mathcal{M}_c$ is a smooth hypersurface and the normal

$$n = \frac{\nabla\phi}{\|\nabla\phi\|}$$

on the tangent space $\mathcal{T}_x\mathcal{M}$ is defined for every x on $\mathcal{M}$. The projection onto $\mathcal{T}_x\mathcal{M}$ is given by

$$P := \mathbb{1} - n \otimes n \,,$$

where $\mathbb{1}$ denotes the identity on $\mathbb{R}^{d+1}$. In what follows we will make extensive use of the Einstein summation convention. Furthermore vectors $v \in \mathbb{R}^{d+1}$ and matrixes $A \in \mathbb{R}^{d+1,d+1}$ are written in index form $v = (v_i)_i$, $A = (A_{ij})_{ij}$. For a tangential vector field v on $\mathcal{M}$ and a scalar function u on $\mathbb{R}^{d+1}$ the tangential divergence and tangential gradient are defined as

$$\begin{aligned} \mathrm{div}_{\mathcal{M}} v &= \partial_i v_i - n_i n_j \partial_j v_j \\ \nabla_{\mathcal{M}} u &= (\partial_i u - n_i n_j \partial_j u)_i \,. \end{aligned}$$

Furthermore we use the notation $\partial_i u = u_{,i}$ and $\partial_i\partial_j u = u_{,ij}$. Furthermore ∂_n denotes the normal derivative and $h := \mathrm{div}\, n = n_{i,i}$ the mean curvature on $\mathcal{M}$. Finally, for the the shape operator S on $\mathcal{M}$ - which is defined as the restriction of Dn on the tangent space $\mathcal{T}_x\mathcal{M}$ - we obtain

$$S = Dn\,P = \frac{1}{\|\nabla\phi\|} P\, D^2\phi\, P\,. \tag{2.1}$$

2.1. The general procedure to derive a level set formulation

Let us assume that we simultaneously want to evolve all level sets $\mathcal{M}_c$ of a given level set function ϕ. We take into account the co-area formula [18, 9] and define the global energy

$$E[\phi] := \int_{\mathbb{R}} e[\mathcal{M}_c] \mathrm{d}c = \int_{\Omega} \|\nabla\phi\| \, f \, \mathrm{d}x.$$

Here, we set $e[\mathcal{M}_c] = 0$ if $\mathcal{M}_c = \emptyset$. We interpret a function ϕ or the corresponding set $\{\mathcal{M}_c\}_{c\in\mathbb{R}}$ as an element of the manifold $\mathcal{L}$ of level set ensembles. Here, we follow the exposition in [3, 11]. A tangent vector $s := \partial_t\phi$ on $\mathcal{L}$ can be identified with a motion velocity v of the corresponding level set $\mathcal{M}_c$ via the classical level set equation

$$s + \|\nabla\phi\|\, v = 0$$

(cf. the book of Osher and Paragios [12] for a detailed study). Thus, we are able to define the corresponding metric on $\mathcal{L}$. If the L^2 metric is used it follows

$$\begin{aligned} g_\phi(s_1, s_2) &:= \int_{\mathbb{R}} \int_{\mathcal{M}_c} v_1 \cdot v_2 \,\mathrm{d}A\,\mathrm{d}c \\ &= \int_\Omega \frac{s_1}{\|\nabla\phi\|} \frac{s_2}{\|\nabla\phi\|} \|\nabla\phi\| \,\mathrm{d}x = \int_\Omega s_1 s_2 \, \|\nabla\phi\|^{-1} \,\mathrm{d}x\,. \end{aligned}$$

Finally, we are able to rewrite the simultaneous gradient flow of all level sets in terms of the level set function ϕ as

$$\partial_t \phi \;=\; -\mathrm{grad}_{g_\phi} E[\phi]\,,$$

which is equivalent to

$$g_\phi(\partial_t\phi, \vartheta) = \int_\Omega \partial_t\phi\, \vartheta \, \|\nabla\phi\|^{-1} \,\mathrm{d}x = -\langle E'[\phi], \vartheta\rangle \tag{2.2}$$

for all functions $\vartheta \in C_0^\infty(\Omega)$.

3. Anisotropic evolution laws

By considering the specific choice of the energy density $f = \gamma(n)$, with γ an anisotropy function, we obtain a generalization of the area functional in the above example.

3.1. Anisotropy function

The anisotropy function γ is a smooth function

$$\begin{aligned} \gamma \;:\; & S^2 \to \mathbb{R}^+ \\ & z \mapsto \gamma(z)\,, \end{aligned} \tag{3.1}$$

and we may assume, that γ is given as a one-homogeneous function on $\mathbb{R}^3$, i.e., for $\lambda > 0$ we have $\gamma(\lambda z) = \lambda\gamma(z)$. In addition, let there be a positive constant m such that for the second derivative we have

$$D^2(\gamma(z) - m|z|) \geq 0\,.$$

In this case, γ is called elliptic and the eigenvalues of $D^2\gamma(z)$ restricted to $z^\perp = \{x \in \mathbb{R}^3 | x \cdot z = 0\}$ are bounded from below by m.

By parameterizing γ over the unit sphere

$$\begin{aligned} \gamma_z \;:\; & S^2 \to \mathcal{W}_\gamma \\ & z \mapsto \gamma_z(z)\,, \end{aligned} \tag{3.2}$$

the so called Wulff-shapes $\mathcal{W}_\gamma$ are obtained. Solutions of the isoperimetric problem for e_γ are given by these Wulff-shapes [13]. For a proof of the isoperimetric property of the Wulff-shape and more references to the literature see [14].

Considering a surface $x : \mathcal{M} \to \mathbb{R}^3$, we can give a version of the second derivative of γ on its tangential space as follows

$$\begin{aligned} \mu_\gamma \ : \ & T_\xi\mathcal{M} \to T_\xi\mathcal{M} \\ & v \mapsto Dx^{-1}\gamma_{zz}(n)Dx(v)\,. \end{aligned} \tag{3.3}$$

The endomorphism field μ_γ is well defined due to the fact that $D^2\gamma(z)\,z = 0$ for all $z \neq 0$. By ellipticity, μ_γ is positive definite. The classical area functional is obtained for the function $\gamma(z) = |z|$. In this case, μ_γ is the identity

3.2. Anisotropic mean curvature flow

Anisotropic mean curvature flow is the L^2-gradient flow of $e_\gamma[\mathcal{M}] := \int_\mathcal{M} \gamma \, dA$. The first variation of e_γ in direction ϑ may be represented in the L^2-metric by a generalized mean curvature vector

$$\langle e'_\gamma[x], \vartheta\rangle = \int_\mathcal{M} h_\gamma (n \cdot \vartheta)\, dA\,. \tag{3.4}$$

Here $h_\gamma = \operatorname{tr}(\mu_\gamma S)$ will be called the γ-mean curvature. In level set formulation this can also be expressed as follows (where S_{ij} are the components of the symmetric shape operator defined in Section 2)

$$\begin{aligned} h_\gamma &= \gamma_{z_j z_i} S_{ij} && (3.5)\\ &= \frac{1}{||\nabla\phi||}\gamma_{z_j z_i}(\delta_{ik} - n_i n_k)\partial_{kl}\phi(\delta_{lj} - n_l n_j) && (3.6)\\ &= \frac{1}{||\nabla\phi||}\gamma_{z_j z_i}\partial_{ij}\phi = \operatorname{div}\gamma_z(n). && (3.7) \end{aligned}$$

The corresponding level set functional is similar to the isotropic case

$$E_\gamma[\phi] = \int_\mathbb{R} e_\gamma[\mathcal{M}_c]\, dc = \int_\Omega \gamma(n)||\nabla\phi||\, dx = \int_\Omega \gamma(\nabla\phi)\, dx\,. \tag{3.8}$$

From this we easily obtain the representation of the anisotropic mean curvature flow in level set formulation

$$\langle E'_\gamma[\phi], \vartheta\rangle = \int_\Omega \gamma_z(\nabla\phi)\nabla\vartheta\, dx\,. \tag{3.9}$$

Note that at this point we need a suitable regularization of $\nabla\phi/|\nabla\phi|$ due to the fact that γ_z is zero-homogeneous. This regularization, denoted by n^ϵ, will be chosen as

$$n^\epsilon = \frac{\nabla\phi}{\sqrt{\epsilon^2 + ||\nabla\phi||^2}} = \frac{\nabla\phi}{||\nabla\phi||_\epsilon}\,,$$

for $\epsilon > 0$, where obviously we use the definition $||\nabla\phi||_\epsilon = \sqrt{\epsilon^2 + ||\nabla\phi||^2}$. For more details on regularization see [3]. If we consider a regularized version of the

L^2-gradient flow in the level set case we obtain for any admissible test function ϑ

$$\int_\Omega \partial_t \phi \vartheta \frac{1}{\|\nabla \phi\|}\, dx = \int_\Omega \gamma_z(n^\epsilon) \nabla \vartheta \, dx\,, \tag{3.10}$$

which is the weak formulation of anisotropic curvature motion in level set form.

3.3. Anisotropic surface diffusion

Anisotropic surface diffusion is the H^{-1}-gradient flow of e_γ. The corresponding H^{-1} metric is

$$g(s_1, s_2) = -\int_\Omega (\Delta_{\mathcal{M}})^{-1} \left[\frac{s_1}{\|\nabla \phi\|}\right] \frac{s_2}{|\nabla \phi\|} \|\nabla \phi\| \, dx\,,$$

where $\mathcal{M}$ denotes the level set, the integration point x belongs to. The gradient flow w.r.t. the H^{-1} metric of e_γ in level set formulation is now given by

$$g(\partial_t \phi, \vartheta) = -\int_\Omega \gamma_z(\nabla \phi) \nabla \vartheta \, dx\,,$$

which is by the above representation of g

$$-\int_\Omega (\Delta_{\mathcal{M}})^{-1} \left[\frac{\vartheta}{\|\nabla \phi\|}\right] \partial_t \phi \, dx = -\int_\Omega \gamma_z(\nabla \phi) \nabla \vartheta \, dx\,.$$

Replacing ϑ by $\|\nabla \phi|\Delta_{\mathcal{M}} \vartheta$ we arrive at

$$\int_\Omega \partial_t \phi \, \vartheta \, dx = -\int_\Omega h_\gamma \, \Delta_{\mathcal{M}} \vartheta \, \|\nabla \phi\| \, dx\,.$$

From this representation of the H^{-1} gradient flow, one is lead to introduce the additional variable $y = h_\gamma$. Thus we just have to find a numerically suitable form of the expression $\int_\Omega y |\nabla \phi| \Delta_{\mathcal{M}} \, dx$. We observe

$$\begin{aligned}
\int_\Omega y \|\nabla \phi\| \, \Delta_{\mathcal{M}} \vartheta \, dx &= \int_{\mathbb{R}} \int_{\mathcal{M}_c} y \, \Delta_{\mathcal{M}_c} \vartheta \, dA \, dc = -\int_{\mathbb{R}} \int_{\mathcal{M}_c} \nabla_{\mathcal{M}_c} y \; \nabla_{\mathcal{M}_c} \vartheta \, dA \, dc \\
&= -\int_\Omega P \nabla y \cdot P \nabla \vartheta \, \|\nabla \phi\| \, dx
\end{aligned}$$

taking into account the properties $P^T = P$ and $P^2 = P$. Concluding, the weak formulation of anisotropic surface diffusion for level sets is given by

$$\begin{aligned}
\int_\Omega \partial_t \phi \, \vartheta \, dx &= \int_\Omega P \nabla y \cdot \nabla \vartheta \, \|\nabla \phi\| \, dx \\
\int_\Omega y \, \psi \, dx &= -\int_\Omega \gamma_z(\nabla \phi) \nabla \psi \, dx\,.
\end{aligned}$$

4. Numerical schemes

In this section we want to derive numerical schemes which can be used to discretize the anisotropic surface diffusion problem. We provide both, spatial and time discretization schemes.

4.1. Spatial discretization

Let us consider a uniform mesh $\mathcal{C}$ covering the whole image domain Ω and consider the corresponding interpolation on cells $C \in \mathcal{C}$ to obtain discrete intensity functions in the accompanying finite element space V^h. Here, the superscript h indicates the grid size. Suppose $\{\Phi_i\}_{i\in I}$ is the standard basis of hat shaped base functions corresponding to nodes of the mesh indexed over an index set I. To clarify the notation we will denote spatially discrete quantities with upper case letters to distinguish them from the corresponding continuous quantities in lower case letters. Hence, we obtain

$$V^h = \{\Phi \in C^0(\Omega) \,|\, \Phi|_C \in \mathcal{P}_1 \,\forall C \in \mathcal{C}\}\ ,$$

where $\mathcal{P}_1$ denotes the space of $(d{+}1)$-linear functions. Suppose $\mathcal{I}_h$ is the Lagrangian interpolation onto V^h. Now, we formulate the semi discrete and regularized finite element problem

Problem 4.1. Find a function $\Phi : \mathbb{R}_0^+ \to V^h$ with $\Phi(0) = \mathcal{I}_h\phi_0$ and a corresponding weighted mean curvature function $Y : \mathbb{R}^+ \to V^h$, such that

$$\begin{aligned} \int_\Omega \partial_t\Phi(t)\Theta \,\mathrm{d}x &= \int_\Omega P_\epsilon[\Phi(t)]\nabla Y(t)\cdot\nabla\Theta \,\|\nabla\Phi(t)\|\,\mathrm{d}x \\ \int_\Omega Y(t)\Psi \,\mathrm{d}x &= \int_\Omega \gamma_z(N^\epsilon)\cdot\nabla\Psi \,\mathrm{d}x \end{aligned}$$

for all $t > 0$ and all test functions Θ, $\Psi \in V^h$.

Here, we use the notation

$$P_\epsilon[\Phi] = \left(\mathbb{1} - \frac{\nabla\Phi}{\|\nabla\Phi\|_\epsilon} \otimes \frac{\nabla\Phi}{\|\nabla\Phi\|_\epsilon}\right)\ ,\ \ N^\epsilon = \frac{\nabla\Phi}{\|\nabla\Phi\|_\epsilon}$$

and consider Neumann boundary conditions on $\partial\Omega$.

4.2. Time discretization

For a given time step $\tau > 0$ we aim to compute discrete functions $\Phi^k(\cdot) \in V^h$, which approximate $\phi(k\tau,\cdot)$ on Ω. Thus, we replace the time derivative $\partial_t\phi$ by a backward difference quotiont and evaluate all terms related to the metric on the previous time step. In particular in the $(k+1)$th time step the weight $\|\nabla\Phi\|$ and the projection P are taken from the kth time step. Explicit time discretizations are ruled out due to accompanying severe time step restrictions of the type $\tau \leq C(\epsilon)h^4$, where h is the spatial grid size (cf. results presented in [15, 16]). We are left to decide, which terms in each time step to consider explicitly and which implicitly. Taking all linear terms implicitly, leads to

Problem 4.2. Find a sequence of level set functions $(\Phi^k)_{k=0,\cdots} \subset V^h$ with $\Phi^0 = \mathcal{I}_h\phi_0$ and a corresponding sequence of weighted mean curvature functions $(Y^k)_{k=0,\cdots}$

$\subset V^h$ such that

$$\int_\Omega \frac{\Phi^{k+1}-\Phi^k}{\tau}\Theta\,\mathrm{d}x = \int_\Omega P_\epsilon[\Phi^k]\nabla Y^{k+1}\cdot\nabla\Theta\,\|\nabla\Phi^k\|_\epsilon\,\mathrm{d}x$$
$$\int_\Omega Y^{k+1}\,\Psi\,\mathrm{d}x = \int_\Omega \gamma_z\left(\frac{\nabla\Phi^k}{\|\nabla\Phi^k\|_\epsilon}\right)\cdot\nabla\Psi\,\mathrm{d}x$$

for all test functions Θ, $\Psi \in V^h$.

However, this algorithm leads to a completely explicit time-discretization. Therefore, we use the advice of Deckelnick and Dziuk [17] and add an implicit term to the second equation as follows.

Problem 4.3. Find a sequence of level set functions $(\Phi^k)_{k=0,\cdots} \subset V^h$ with $\Phi^0 = \mathcal{I}_h\phi_0$ and a corresponding sequence of weighted mean curvature functions $(Y^k)_{k=0,\cdots}$ $\subset V^h$ such that

$$\int_\Omega \frac{\Phi^{k+1}-\Phi^k}{\tau}\Theta\,\mathrm{d}x = \int_\Omega P_\epsilon[\Phi^k]\nabla Y^{k+1}\cdot\nabla\Theta\,\|\nabla\Phi^k\|_\epsilon\,\mathrm{d}x$$
$$\int_\Omega Y^{k+1}\,\Psi\,\mathrm{d}x = \int_\Omega \gamma_z\left(\frac{\nabla\Phi^k}{\|\nabla\Phi^k\|_\epsilon}\right)\cdot\nabla\Psi\,\mathrm{d}x + \lambda\int_\Omega \frac{\gamma\left(\frac{\nabla\Phi^k}{\|\nabla\Phi^k\|_\epsilon}\right)}{\|\nabla\Phi^k\|_\epsilon}\nabla(\Phi^{k+1}-\Phi^k)\nabla\Psi\ \mathrm{d}x$$

for all test functions Θ, $\Psi \in V^h$ and some parameter λ.

As a motivation for this additional term lets consider the isotropic case. Here $\gamma = |z|$ and $\gamma_z = id$. Thus the second equation in Problem 4.3 becomes

$$\int_\Omega Y^{k+1}\,\Psi\,\mathrm{d}x = \int_\Omega \frac{\nabla\Phi^k}{\|\nabla\Phi^k\|_\epsilon}\cdot\nabla\Psi\,\mathrm{d}x + \lambda\int_\Omega \frac{\nabla(\Phi^{k+1}-\Phi^k)}{\|\nabla\Phi^k\|_\epsilon}\nabla\Psi\ \mathrm{d}x$$

which leads for $\lambda = 1$ to the implicit formulation. Compare [17] for more details and an optimal choice of the parameter λ.

5. Implementation and numerical results

In this section we show some preliminary results of our implementation. All calculations were performed on a regular, uniform triangulation, where we used standard Courant finite elements (i.e. globaly continuous and piecewise affine). We employed a Schur complement approach and used left and right diagonal preconditioning for the resulting linear system. In Figures 1 and 2 we show results for the isotropic case in 2d and 3d respectively.

Neumann boundary conditions have been imposed and one sees that the bizarly shaped initial data become ever more ball shaped as the evolution proceeds, approximating the steady state solution of isotropic surface diffusion.

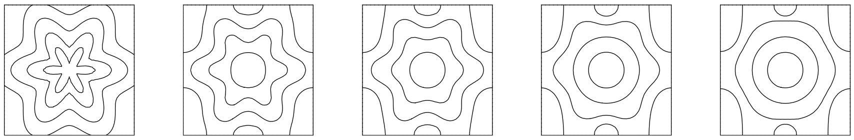

FIGURE 1. Isotropic surface diffusion in 2d. Level sets -0.5, 0.0, 0.5, 1.0 (from inner to outer curve). From left to right $t = 0.0, 0.01, 0.02, 0.05, 0.1$. Computational domain: 4×4 square; triangulation: 2.100 grid points; time step: 10^{-4}; $\lambda = 1.0$.

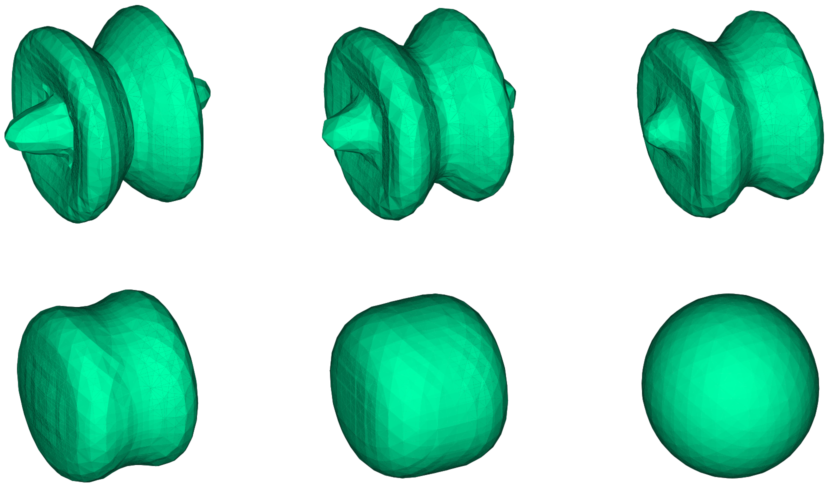

FIGURE 2. Isotropic surface diffusion in 3d. Level set 0 at $t =$ 0.0, 0.001, 0.002, 0.005, 0.01, 0.05 (from top left to bottom right). Computational domain: $4 \times 4 \times 4$ cube; triangulation: 36.000 grid points; time step: 10^{-5}; $\lambda = 1.0$.

In Figures 3 and 4 we present numerical results for anisotropic surface diffusion with anisotropy function

$$\gamma(x) = \left(|x| \sum_{k=0}^{d} \left(\epsilon |x|^2 + |x_k|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}, \quad \epsilon = 0.01; \quad d = 2, 3. \tag{5.1}$$

Here the limit configuration is clearly determined by the rectangular symmetry of the anisotropy.

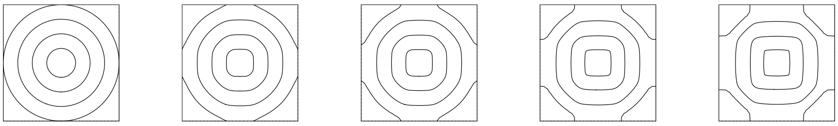

FIGURE 3. Anisotropic surface diffusion in 2d. Levesets -0.5, 0.0, 0.5, 1.0 (from inner to outer curve). From top left to bottom right $t = 0.0, 0.001, 0.002, 0.005, 0.01$. Computational domain: 4×4 square; triangulation: 2.100 grid points; time step: 10^{-4}, $\lambda = 10.0$.

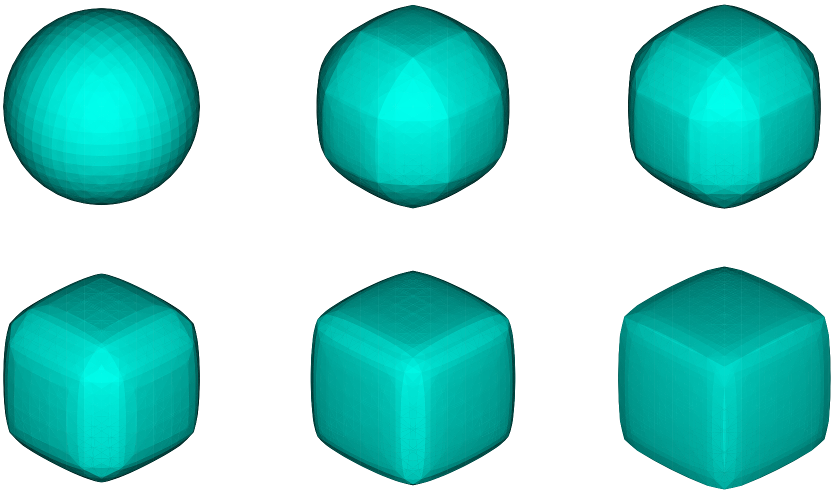

FIGURE 4. Anisotropic surface diffusion in 3d. Levelset 0 at $t = 0.0, 0.001, 0.002, 0.005, 0.01, 0.02$ (from top left to bottom right). Computational domain: $4 \times 4 \times 4$ cube; triangulation: 36.000 grid points; time step: 10^{-5}, $\lambda = 20.0$.

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